Math Methods 1
Lia Vas

Review for Exam 4

1. **Series Solutions**.

   (a) Find the sum of the following series.

   $$\text{(i)} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n \qquad \text{(ii)} \sum_{n=0}^{\infty} 2^n x^n \qquad \text{(iii)} \sum_{n=0}^{\infty} (-1)^n x^{2n} \qquad \text{(iv)} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^n$$

   (b) Consider the equation $(1-x)^2 y'' - 2y = 0$. Show that $x = 0$ is a regular point of this equation and determine the interval of convergence of the solution. Then, find the series solution at $x = 0$. Express your solution in the closed form by choosing $a_0 = 1, a_1 = -2$ for one and $a_0 = a_1 = 1$ for the other fundamental solution.

   (c) Consider Hermite equation $y'' - 2xy' + 4y = 0$. Show that $x = 0$ is a regular point of this equation and determine the interval of convergence of the solution. Find the series solutions of the given equation about $x = 0$. Find the closed form of one solution and list a few terms of the second solution.

   (d) Consider the equation $y'' - 2y' + y = 0$. Show that $x = 0$ is a regular point of this equation and determine the interval of convergence of the solution. Then find the series solution at $x = 0$ and express your answers in the closed form by choosing $a_0 = a_1 = 1$ for one and $a_0 = 0, a_1 = 1$ for the other fundamental solution.

2. **Groups.**

   (a) Show that the set of real numbers different from $-\frac{1}{3}$ is a group under the following operation

   $$a * b = a + b + 3ab.$$

   Determine whether it a group if $-\frac{1}{3}$ is included in the set.

   (b) Consider 2×2 matrices of the form $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ where $a \neq 0$ and $c \neq 0$. Show that the set of such matrices with matrix multiplication is a group.

   (c) Write down the Cayley tables for the following. (i) $C_2 \times C_3$, (ii) $C_2 \times C_2 \times C_2$, (iii) D_5.

   (d) Determine which of the following groups are isomorphic.

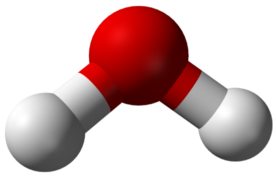
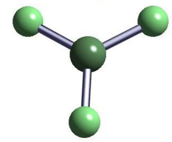
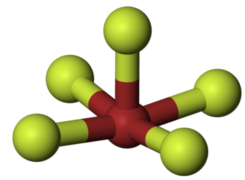
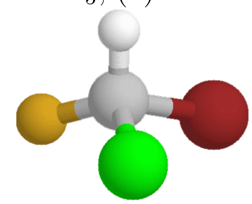
   (i) C_{24}, $C_{12} \times C_2$, $C_8 \times C_3$, $C_6 \times C_4$, $C_3 \times C_4 \times C_2$, $C_6 \times C_2 \times C_2$, $C_3 \times C_2 \times C_2 \times C_2$

   (ii) C_{36}, $C_{18} \times C_2$, $C_{12} \times C_3$, $C_9 \times C_4$, $C_6 \times C_6$, $C_9 \times C_2 \times C_2$, $C_6 \times C_3 \times C_2$, $C_3 \times C_3 \times C_4$, $C_3 \times C_3 \times C_2 \times C_2$

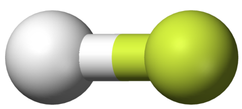
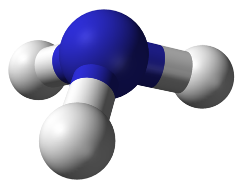
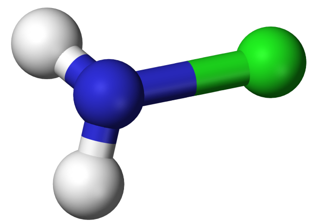
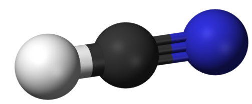
   (e) Determine if the following pairs of groups are isomorphic. If they are, produce the isomorphism. If they are not, explain why.

   (i) C_3 and D_3. (ii) C_6 and D_3.
   (iii) S_3 and D_3. (iv) S_n and D_n for $n > 3$.

(f) Describe the point groups of the following molecules. Write down a presentation of the point group of each molecule and identify each group element as a symmetry operation. (1) Water H_2O, (2) Boron trifluoride BF_3, (3) Bromine Pentafluoride BrF_5, (4) CHFClBr,

(5) Hydrogen chloride HCl, (6) Ammonia NH_3, (7) Chloramine NH_2Cl, (8) Hydrogen cyanide HCN.

Solutions.

1. Series solutions. More detailed solutions can be found on the class handouts.

(a) (i) $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n = \sum_{n=0}^{\infty} \frac{1}{n!}(-x)^n = e^{-x}$ (ii) $\sum_{n=0}^{\infty} 2^n x^n = \sum_{n=0}^{\infty}(2x)^n = \frac{1}{1-2x}$
(iii) $\sum_{n=0}^{\infty}(-1)^n x^{2n} = \sum_{n=0}^{\infty}(-x^2)^n = \frac{1}{1-(-x^2)} = \frac{1}{1+x^2}$
(iv) $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(\sqrt{x})^{2n} = \cos\sqrt{x}$

(b) Dividing by $(1-x)^2$ we have that $y'' - \frac{2}{(1-x)^2}y = 0$ so that $p = 0$ and $q = \frac{-2}{(1-x)^2}$. These functions are analytic at $x = 0$: p is a polynomial and q is a rational function with $x = 1 \neq 0$ as the only singularity. So, p and q have continuous derivatives of any order, p at any point and q at any point different than 1. Thus, the point $x = 0$ is regular. See the handout for an alternative way to argue that p and q are analytic. Since 1 is a singularity of q and p has no singularity, the solutions converge in an interval of length 1 around $x = 0$. So, the interval of convergence is $(-1, 1)$.

Plugging $y = \sum_{n=0}^{\infty} a_n x^n$ and its derivatives in the equation produces the recursive equation $a_{n+2} = \frac{2na_{n+1}-(n-2)a_n}{(n+2)}$ for $n = 0, 1, \ldots$. The two suggested choices for a_0 and a_1 produce two fundamental solutions $y_1 = 1-2x+x^2 = (1-x)^2$ and $y_2 = 1+x+x^2+x^3+x^4+\ldots = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$. Thus, the general solution is $y = c_1(1-x)^2 + c_2\frac{1}{1-x}$.

(c) For this equation, $p = -2x$ and $q = 4$. The functions p and q are analytic since they are polynomials and polynomials have continuous derivatives of any order. In particular, $p' = -2$ and $p'' = p''' = q' = q'' = \ldots 0$. Thus, $x = 0$ is a regular point. The interval of convergence is $(-\infty, \infty)$ because polynomials have no singularities. Alternatively, you can say that p and q are analytic since their power series expansions at $x = 0$ are $p = 0 - 2x + 0x^2 + 0x^3 + \ldots$ and $q = 4 + 0x + 0x^2 + \ldots$ and they convergent at every value of x (including $x = 0$) since just finitely many terms are nonzero. Thus, the point $x = 0$ is regular and the solutions are convergent on $(-\infty, \infty)$.

Plugging $y = \sum_{n=0}^{\infty} a_n x^n$ and its derivatives into the equation produces a recursive equation $a_{n+2} = \frac{(2n-4)a_n}{(n+1)(n+2)}$ for $n = 0, 1, \ldots$ Use the "standard" choices for a_0 and a_1.

With $a_0 = 1$ and $a_1 = 0$, obtain that $y_1 = 1 - 2x^2$. With $a_0 = 0$ and $a_1 = 1$, obtain that $y_2 = x - \frac{2}{3!}x^3 - \frac{2 \cdot 2}{5!}x^5 - 2\frac{\cdot 2 \cdot 4}{7!}x^7 \ldots$ The general solution is $y = c_1 y_1 + c_2 y_2$.

(d) For this equation, $p = -2$ and $q = 1$. The functions p and q are analytic since they are polynomials and polynomials have continuous derivatives of any order. In particular, $p' = p'' = p''' = \ldots 0$ and $q' = q'' = \ldots = 0$. The interval of convergence is $(-\infty, \infty)$ because polynomials have no singularities. Alternatively, you can say that p and q are analytic since their power series expansions at $x = 0$ are $p = -2 + 0x + 0x^2 + \ldots$ and $q = 1 + 0x + 0x^2 + \ldots$ and they convergent at every value of x (including $x = 0$) since just finitely many terms are nonzero.

Plugging $y = \sum_{n=0}^{\infty} a_n x^n$ and its derivatives into the equation produces a recursive equation $a_{n+2} = \frac{2(n+1)a_{n+1} - a_n}{(n+1)(n+2)}$ for $n = 0, 1, \ldots$. The two suggested choices for a_0 and a_1 produce two fundamental solutions $y_1 = xe^x$ and $y_2 = e^x$. The general solution is $y = c_1 xe^x + c_2 e^x$.

2. Groups. More detailed solutions can be found on the class handouts.

(a) Check that the axioms A1–A4 hold. For A1, show that if a and b are real numbers different from $-\frac{1}{3}$, the product $a * b = a + b + 3ab$ is a real number different from $-\frac{1}{3}$ as well since

$$a * b = a + b + 3ab = -\tfrac{1}{3} \Rightarrow a + b + 3ab + \tfrac{1}{3} = 0 \Rightarrow a + \tfrac{1}{3} + b(1 + 3a) = 0 \Rightarrow$$
$$a + \tfrac{1}{3} + 3b(\tfrac{1}{3} + a) = 0 \Rightarrow (a + \tfrac{1}{3})(1 + 3b) = 0 \Rightarrow a + \tfrac{1}{3} = 0 \text{ or } 1 + 3b = 0 \Rightarrow a = -\tfrac{1}{3} \text{ or } b = -\tfrac{1}{3}.$$

For A2, show that both $(a*b)*c$ and $a*(b*c)$ are equal to $a+b+c+3ab+3ac+3bc+9abc$. For A3, show that $x = 0$ is a solution of $a*x = x*a = a$. Thus, the group identity element is 0. For A4, show that $x = \frac{-a}{1+3a}$ is a solution of $a * x = x * a = 0$. x is well defined since $a \neq \frac{-1}{3}$ and $x \neq \frac{-1}{3}$ since otherwise you would have $1 = 0$.

If all real numbers are considered instead of all numbers different from $\frac{-1}{3}$, the axiom A4 would fail because the equation $\frac{-1}{3} * x = 0$ has no solution (show why that is so).

(b) Show that A1–A4 hold. For A1, show that the product of two upper triangular matrices is again an upper triangular matrix and that if the diagonal entries of both matrices are nonzero, then the diagonal entries of the product are nonzero also. For A2, show that associativity holds. For A3, show that $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the solution of $AX = XA = A$ for any upper triangular matrix A with nonzero entries on the diagonal. For A4, show that $X = \begin{bmatrix} \frac{1}{a} & \frac{-b}{ac} \\ 0 & \frac{1}{c} \end{bmatrix}$ is the solution of $AX = XA = I$ for $A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$. Note that the diagonal entries of X are defined since a and c are nonzero.

(c) $C_3 \times C_2 = \langle a, b | a^3 = b^2 = 1, ab = ba \rangle$, $C_2 \times C_2 \times C_2 = \langle a, b, c | a^2 = b^2 = c^2 = 1, ab = ba, ac = ca, bc = cb \rangle$, and $D_5 = \langle a, b | a^5 = 1, b^2 = 1, a^4 b = ba \rangle$. For Cayley's tables, see the solutions of matching problems on the handout on groups.

(d) There are three classes of abelian groups of order 24 such that the groups in the same class are isomorphic and the groups in different class are not isomorphic to each other.

1.	$C_3 \times C_2 \times C_2 \times C_2$	$\cong C_6 \times C_2 \times C_2$	
2.	$C_3 \times C_4 \times C_2$	$\cong C_{12} \times C_2$	$\cong C_6 \times C_4$
3.	$C_3 \times C_8 \cong C_{24}$		

There are four classes of abelian groups of order 36 such that the groups in the same class are isomorphic and the groups in different class are not isomorphic to each other.

1.	$C_3 \times C_3 \times C_2 \times C_2$	$\cong C_6 \times C_3 \times C_2$	$\cong C_6 \times C_6$
2.	$C_3 \times C_3 \times C_4$	$\cong C_3 \times C_{12}$	
3.	$C_9 \times C_2 \times C_2$	$\cong C_{18} \times C_2$	
4.	$C_9 \times C_4$	$\cong C_{36}$	

(e) (i) C_3 and D_3 are not isomorphic because one has 3 elements and the other has 6 elements.

(ii) C_6 and D_3 are not isomorphic because one is abelian and the other is not.

(iii) S_3 and D_3 are isomorphic. If 6 elements of S_3 are represented by maps f_1 to f_6 mapping (1, 2, 3) to (1, 2, 3), (2, 3, 1), (3, 1, 2), (1, 3, 2), (3, 2, 1), and (2, 1, 3), respectively, then f_1 is the identity, the order of f_2 and f_3 is 3 and the order of f_4, f_5 and f_6 is 2. If we denote $f_1 = 1$, $f_2 = a$ and $f_4 = b$, then $f_3 = f_2^2 = a^2$, $ab = f_2 f_4 = f_5$, $a^2 b = f_2^2 f_4 = f_6$, and $ba = f_4 f_2 = f_6 = a^2 b$. Thus, S_3 can be presented by $\langle a, b | a^3 = 1, b^2 = 1, ba = a^2 b\rangle$ which is the presentation of D_3 as well. So, the groups are isomorphic.

(iv) $S_n \not\cong D_n$ for $n > 3$ because $D_,$ has $2n$ and S_n has $n!$ elements and $n! > 2n$ for $n > 3$.

(f) (1) Water H_2O. There is a rotation a for 180 degrees and reflections with respect to vertical planes. Since $a^2 = 1$, if b is a reflection with respect to a vertical plane, then $ba = a^{2-1}b = ab$ so the group is abelian group $C_2 \times C_2 = \langle a, b | a^2 = b^2 = 1, ba = ab\rangle$. This group has four elements $1, a, b$, and ab.

(2) Boron trifluoride BF_3. If a denotes the rotation for 120, b the reflection with respect to a vertical plane and c the reflection with respect to the horizontal plane, then we have that $a^3 = 1, b^2 = c^2 = 1$, $ba = a^2 b$, $ac = ca$, Thus, the group is $D_3 \times C_2 = \langle a, b, c \mid a^3 = 1, b^2 = 1, c^2 = 1, ba = a^2 b, ca = ac, cb = bc\rangle$. It has 12 elements: 3 rotations $1, a, a^2$, three reflections with respect to the vertical planes $b, ab, a^2 b$ and six more elements $c, ac, a^2 c, bc, abc, a^2 bc$ which are compositions of these six with c.

(3) Bromine Pentafluoride BrF_5. The rotation a for 90 degrees and a reflection b with respect to a vertical plane generate the group. There is no symmetry with respect to the horizontal plane. Since a and b do not commute, the group is $D_4 = \langle a, b \mid a^4 = 1, b^2 = 1, ba = a^3 b\rangle$ It has eight elements, four rotations $1, a, a^2, a^3$ for 0, 90, 180 and 270 degrees respectively, and four reflections $b, ab, a^2 b, a^3 b$ with respect to four vertical planes (xz and yz planes and two vertical planes passing the diagonals of the square).

(4) CHFClBr has five different atoms so there is just the trivial symmetry. Thus, the point group is the trivial (one element) group C_1.

(5) Hydrogen chloride HCl. If x is a rotation for any angle between 0 and 2π and b is the reflection with respect to a vertical plane (recall that we imagine this molecule standing "upright"), then x and b do not commute and the relations $b^2 = 1$ and $bx = x^{-1}b$ hold in this group. Thus, it is D_∞ ($C_{\infty v}$ using the chemistry notation).

(6) Ammonia. If a is the rotation for 120 degrees and b is the symmetry with respect to a vertical plane, then a and b do not commute and generate the entire group. Thus, the group is $D_3 = \langle a, b \mid a^3 = 1, b^2 = 1, ba = a^2 b\rangle$ (C_{3v} for chemists). It has six elements, three rotations $1, a, a^2$ for 0, 120 and 240 degrees respectively, and three reflections with respect to three vertical planes.

(7) Chloramine NH_2Cl. The only non-identity group element is the reflection a switching the two white atoms. Since $a^2 = 1$, the point group is $C_2 = \langle a \mid a^2 = 1\rangle$.

(8) Hydrogen cyanide HCN. If x is a rotation for any angle between 0 and 2π and b is the reflection with respect to a vertical plane, then x and b do not commute and the relations $b^2 = 1$ and $bx = x^{-1}b$ hold in this group. Thus, it is D_∞.