**Math Methods 1**
**Lia Vas**

# Review for Exam 4

1. **Series Solutions.**

   (a) Consider the equation $(1-x)^2y'' - 2y = 0$. Show that $x = 0$ is a regular point of this equation. Then find the series solution at $x = 0$. Write your solution in the closed form and determine the radius of the convergence of the solution. Determine the interval of convergence.

   (b) Consider Hermite equation $y'' - 2xy' + 4y = 0$. Show that $x = 0$ is a regular point of this equation. Find the series solutions of the given equation about $x = 0$. Find the closed form of one solution and list first few terms of the second solution. Determine the interval of convergence.

   (c) Consider the equation $y'' - 2y' + y = 0$. Show that $x = 0$ is a regular point of this equation. Find the series solution at $x = 0$ and express your answers in the closed form.

2. **Groups.**

   (a) Prove that the set of real numbers different from $-\frac{1}{3}$ is a group under the following operation
   $$a * b = a + b + 3ab.$$
   Determine whether it a group if $-\frac{1}{3}$ is included in the set.

   (b) Consider $2 \times 2$ matrices of the form $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ where $a \neq 0$ and $c \neq 0$. These matrices are called **upper triangular invertible** matrices. Show that the set of such matrices with matrix multiplication is a group.

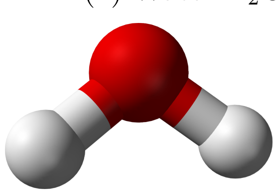
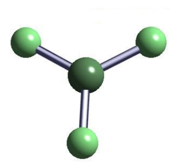
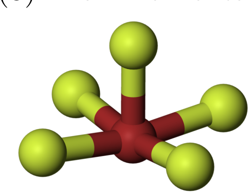
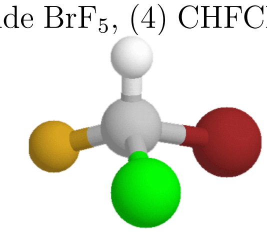
   (c) Write down the Cayley tables for the following. (i) $C_2 \times C_3$, (ii) $C_2 \times C_2 \times C_2$, (iii) $D_5$.

   (d) Produce all isomorphism classes of abelian groups of order 24. Do the same for groups of order 36.

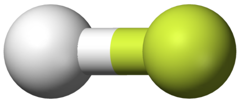
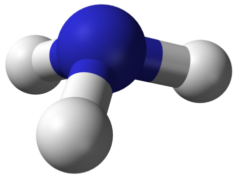
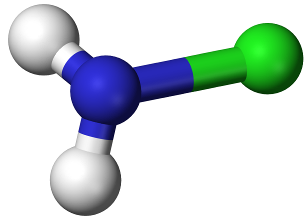
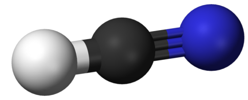
   (e) Determine if the following pairs of groups are isomorphic. If they are, produce the isomorphism. If they are not, explain why.

   (i) $C_3$ and $D_3$. (ii) $C_6$ and $D_3$.
   (iii) $S_3$ and $D_3$. (iv) $S_n$ and $D_n$ for $n > 3$.

   (f) Describe the point groups of the following molecules. Write down the presentations of the point groups. Identify each group element as a symmetry operation.
   (1) Water $H_2O$, (2) Boron trifluoride $BF_3$, (3) Bromine Pentafluoride $BrF_5$, (4) CHFClBr,

(5) Hydrogen chloride HCl, (6) Ammonia $NH_3$, (7) Chloramine $NH_2Cl$, (8) Hydrogen cyanide HCN.

**Solutions.**

1. Series solutions. More detailed solutions can be found on the class handouts.

   (a) Dividing by $(1-x)^2$ obtain $y'' - \frac{2}{(1-x)^2}y = 0$. Thus $p = 0$ and $q = \frac{-2}{(1-x)^2}$. Both of these functions are analytic: first because it is a polynomial and the second since it is a rational function so both have continuous derivatives of any order, $p$ at any point and $q$ at any point different than 1. Thus, the point $x = 0$ is regular. Since 1 is a singularity of $q$, the radius of convergence of the solutions is 1 (which is the distance from center 0 to singularity 1). So, the interval of convergence of the solutions is $(-1, 1)$.

   Alternatively, you can argue that $p$ and $q$ are analytic since they have convergent power series expansions at $x = 0$. The expansion of $p$ is $0 + 0x + 0x^2 + \ldots$. The expansion of $q$ can be obtained differentiating $-2\frac{1}{1-x} = -2(1 + x + x^2 + \ldots)$. This series is convergent on $(-1, 1)$ and so the expansion of $q$ is convergent on $(-1, 1)$. Thus, the solutions are convergent on $(-1, 1)$, too.

   Plugging $y = \sum_{n=0}^{\infty} a_n x^n$ and its derivatives in the equation produces the recursive equation $a_{n+2} = \frac{2na_{n+1}-(n-2)a_n}{(n+2)}$ for $n = 0, 1, \ldots$ which produces fundamental solutions $y_1 = 1 - 2x + x^2 = (1-x)^2$ and $y_2 = 1 + x + x^2 + x^3 + x^4 + \ldots = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$. Thus, the general solution is $y = c_1(1-x)^2 + c_2\frac{1}{1-x}$.

   (b) For this equation $p = -2x$ and $q = 4$. Both of these are analytic since their derivatives are constant functions. Alternatively, you can deduce that $p$ and $q$ are analytic since their power series expansions are $p = 0 - 2x + 0x^2 + \ldots$ and $q = 4 + 0x + 0x^2 + \ldots$. So the point $x = 0$ is regular. Both expansions for $p$ and $q$ are convergent for every $x$ and so the series solution is convergent for every $x$ too.

   Plugging $y = \sum_{n=0}^{\infty} a_n x^n$ and its derivatives into the equation produces a recursive equation $a_{n+2} = \frac{(2n-4)a_n}{(n+1)(n+2)}$ for $n = 0, 1, \ldots$ Considering even-indexed coefficients obtain one fundamental solution $y_1 = 1 - 2x^2$. Considering odd-indexed coefficients obtain $y_2 = x - \frac{2}{3!}x^3 - \frac{2\cdot 2}{5!}x^5 - 2\frac{\cdot 2\cdot 4}{7!}x^7 \ldots$ The general solution is $y = c_1y_1 + c_2y_2$.

   (c) In this case $p = -2$ and $q = 1$. The functions $p$ and $q$ are analytic since the derivatives of these functions (the constant functions 0) are continuous. Alternatively, you can say that $p$ and $q$ are analytic since their power series expansions at $x = 0$ are $p = 2 + 0x + 0x^2 + \ldots$ and $q = 1 + 0x + 0x^2 + \ldots$ which are convergent for every value of $x$. Thus, the point $x = 0$ is regular and the radius of the convergence of the solutions is $\infty$ (that is, the solution converges on $(-\infty, \infty)$).

   Plugging $y = \sum_{n=0}^{\infty} a_n x^n$ and its derivatives into the equation produces a recursive equation $a_{n+2} = \frac{2(n+1)a_{n+1} - a_n}{(n+1)(n+2)}$ for $n = 0, 1, \ldots$ which produces solutions $y_1 = xe^x$ and $y_2 = e^x$. Hence, the general solution is $y = c_1xe^x + c_2e^x$.

2. Groups. More detailed solutions can be found on the class handouts.

   (a) Check that the axioms A1–A4 hold. For A1, show that if $a$ and $b$ are real numbers different from $-\frac{1}{3}$, the product $a * b = a + b + 3ab$ is a real number different from $-\frac{1}{3}$ as well since

$$a * b = a + b + 3ab = -\tfrac{1}{3} \Rightarrow a + b + 3ab + \tfrac{1}{3} = 0 \Rightarrow a + \tfrac{1}{3} + b(1 + 3a) = 0 \Rightarrow$$
$$a+\tfrac{1}{3}+3b(\tfrac{1}{3}+a) = 0 \Rightarrow (a+\tfrac{1}{3})(1+3b) = 0 \Rightarrow a+\tfrac{1}{3} = 0 \text{ or } 1+3b = 0 \Rightarrow a = -\tfrac{1}{3} \text{ or } b = -\tfrac{1}{3}.$$

   For A2, show that both $(a*b)*c$ and $a*(b*c)$ are equal to $a+b+c+3ab+3ac+3bc+9abc$. For A3, show that $x = 0$ is a solution of $a*x = x*a = a$. Thus, the group identity element is 0. For A4, show that $x = \frac{-a}{1+3a}$ is a solution of $a * x = x * a = 0$. $x$ is well defined since $a \neq \frac{-1}{3}$ and $x \neq \frac{-1}{3}$ since otherwise you would have $1 = 0$.
   If all real numbers are considered instead of all numbers different from $\frac{-1}{3}$, the axiom A4 would fail because the equation $\frac{-1}{3} * x = 0$ has no solution (show why that is so).

   (b) Show that A1–A4 hold. For A1, show that the product of two upper triangular invertible matrices is again an upper triangular and invertible. For A2, show that associativity holds. For A3, show that $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the solution of $AX = XA = A$ for any $A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$.
   For A4, show that $X = \begin{bmatrix} \frac{1}{a} & \frac{-b}{ac} \\ 0 & \frac{1}{c} \end{bmatrix}$ is the solution of $AX = XA = I$ for $A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$.

   (c) $C_3 \times C_2 = \langle a, b | a^3 = b^2 = 1, ab = ba\rangle$, $C_2 \times C_2 \times C_2 = \langle a, b, c | a^2 = b^2 = c^2 = 1, ab = ba, ac = ca, bc = cb\rangle$, and $D_5 = \langle a, b | a^5 = 1, b^2 = 1, a^4b = ba\rangle$. For Cayley's tables, see pages 11 and page 24 of the handout "Groups".

   (d) $24 = 8 \cdot 3 = 2^3 \cdot 3$. There are 3 non-isomorphic abelian groups of order 24.

| | | | |
|---|---|---|---|
| 1. | $C_3 \times C_2 \times C_2 \times C_2$ | $\cong C_6 \times C_2 \times C_2$ | |
| 2. | $C_3 \times C_4 \times C_2$ | $\cong C_{12} \times C_2$ | $\cong C_6 \times C_4$ |
| 3. | $C_3 \times C_8 \cong C_{24}$ | | |

   $36 = 4 \cdot 9 = 2^2 \cdot 3^2$. There are 4 non-isomorphic abelian groups of order 36.

| | | | |
|---|---|---|---|
| 1. | $C_3 \times C_3 \times C_2 \times C_2$ | $\cong C_6 \times C_3 \times C_2$ | $\cong C_6 \times C_6$ |
| 2. | $C_3 \times C_3 \times C_4$ | $\cong C_3 \times C_{12}$ | |
| 3. | $C_9 \times C_2 \times C_2$ | $\cong C_{18} \times C_2$ | |
| 4. | $C_9 \times C_4$ | $\cong C_{36}$ | |

   (e) (i) $C_3$ and $D_3$ are not isomorphic because one has 3 elements and the other has 6 elements.
   (ii) $C_6$ and $D_3$ are not isomorphic because one is abelian and the other is not.
   (iii) $S_3$ and $D_3$ are isomorphic. If 6 elements of $S_3$ are represented by maps $f_1$ to $f_6$ mapping (1, 2, 3) to (1, 2, 3), (2, 3, 1), (3, 1, 2), (1, 3, 2), (3, 2, 1), and (2, 1, 3), respectively, then $f_1$ is the identity, the order of $f_2$ and $f_3$ is 3 and the order of $f_4$, $f_5$ and $f_6$ is 2. If we denote $f_1 = 1$, $f_2 = a$ and $f_4 = b$, then $f_3 = f_2^2 = a^2$, $ab = f_2f_4 = f_5$, $a^2b = f_2^2f_4 = f_6$, and $ba = f_4f_2 = f_6 = a^2b$. Thus, $S_3$ can be presented by $\langle a, b | a^3 = 1, b^2 = 1, ba = a^2b\rangle$ which is the presentation of $D_3$ as well. So, the groups are isomorphic.
   (iv) $S_n$ and $D_n$ are not isomorphic for $n > 3$ because one has $2n$ and the other $n!$ elements. $n!$ is larger than $2n$ for $n > 3$.

(f) (1) Water $H_2O$. This molecule has the following symmetries: identity $E$, rotation for 180 degrees $a$ reflections with the respect to the vertical plane $b$ and their product $ab$. Thus, this group is isomorphic to $C_2 \times C_2 = \langle a, b | a^2 = b^2 = 1, ba = ab \rangle$.

(2) Boron trifluoride $BF_3$. There are two nontrivial rotations: $a$ rotation for 120 and $a^2$ rotation for 240 degrees and the symmetries with respect to vertical plane $b$ and horizontal plane $c$. Since $a$ and $c$ commute and $ba = a^2 b$, we have $D_3 \times C_2 = \langle a, b, c | a^3 = 1, b^2 = 1, c^2 = 1, ca = ac, cb = bc, ba = a^2 b \rangle$.

(3) Bromine Pentafluoride $BrF_5$. Four fluor atoms line in the same plane forming the vertices of a square. Bromine atom is in the center of that square and the remaining fluor atom is directly above the bromine. Because of that fifth fluor, there are no symmetries with respect to horizontal plane and the symmetry group corresponds to the group of symmetries of a square is $D_4 = \langle a, b | a^4 = 1, b^2 = 1, ba = a^3 b \rangle$.

(4) CHFClBr has 5 different atoms so there is just the trivial symmetry. Thus, the point group is the trivial (one element) group $C_1 = \{1\}$.

(5) Hydrogen chloride HCl. All the rotations for any angle between 0 and $2\pi$ with 0 and $2\pi$ identified are the elements of the point group of this molecule. These rotations constitute the group denoted $C_\infty = SO(2, R)$. There is also the symmetry with respect to the vertical plane $b$. Since $b$ does not commute with the rotations, the group is $D_\infty$.

(6) Ammonia molecule has the same symmetries as the equilateral triangle. It has no symmetries with respect to the horizontal plane. Thus, the point group is $D_3 = \langle a, b | a^3 = 1, b^2 = 1, ba = a^2 b \rangle$.

(7) Chloramine $NH_2Cl$. The only non-identity group element is a single symmetry of order 2. So, the point group has two elements and so it is $C_2 = \langle a | a^2 = 1 \rangle$.

(8) Hydrogen cyanide HCN is a linear molecule so all the rotations for any angle between 0 and $2\pi$ are in its point group. There is also symmetry $b$ with respect to the vertical plane (vertical if the molecule stands "upright"). Since $b$ does not commute with the rotations, the group is $D_\infty$.