Math Methods 1
Lia Vas

Series Solutions of Ordinary Differential Equations

Let us consider a linear differential equation of the second order

$$a(x)y'' + b(x)y' + c(x)y = g(x)$$

where $a, b, c,$ and g are some real-valued functions.

Recall that the equation $a(x)y''+b(x)y'+c(x)y = 0$ is the **homogeneous part** of the above relation. If y_h is the solution of the homogeneous part, the particular solution y_p of the non-homogeneous equation $a(x)y''+b(x)y'+c(x)y = g(x)$ can be found using the variation of parameters method (or the undetermined coefficients method) and the general solution of the equation has the form $y = y_h + y_p$. Since these methods are covered extensively in a differential equations course, we shall concentrate on methods for finding the solution of the *homogeneous* equation

$$a(x)y'' + b(x)y' + c(x)y = 0.$$

Also, since the case when $a(x), b(x)$ and $c(x)$ are all constant is covered in a differential equations course, we shall concentrate on the case when the functions $a(x), b(x)$ or $c(x)$ are *not constant.*

To produce the general solution, recall that one needs to find two independent (i.e. not constant multiple of each other) solutions y_1 and y_2, called the **fundamental solutions.** Then, the general solution is of the form

$$y = c_1y_1 + c_2y_2$$

where c_1 and c_2 are two constants.

There are certain methods for solving some non-constant coefficient equations in some particular cases as the following examples illustrate.

- **Legendre's linear equation** $a(px + q)^2y'' + b(px + q)y' + cy = 0$ where a, b, c, p and q are constants. This equation can be reduced to an equation with constant coefficients using the substitution $px + q = e^t$. If $p = 1$ and $q = 0$, the equation $ax^2y'' + bxy' + cy = 0$ is called **Euler's equation**.

- **Exact equations**. If the left hand side of the equation $a(x)y'' + b(x)y' + c(x)y = 0$ is a derivative of another equation, it is said to be **exact**. It can be shown that an equation is exact if $c(x) - b'(x) + a''(x) = 0$. Note that if $a(x) = 0$, we can write the equation in the form $b(x)dy = -c(x)ydx$ and denote $Q = b$ and $P = -cy$, then the condition $c(x) - b'(x) = 0$ is equivalent to $P_y = Q_x$ which is the criterion of exactness for the first order differential equation.

- **Partially known complementary functions**. If one solution y_1 of the homogeneous equation $a(x)y''+b(x)y'+c(x)y = 0$ is known, to find the second solution y_2 and the general solution $y = c_1y_1 + c_2y_2$ as well, assume that it has the form $y_2 = vy_1$ where v is an unknown function. Substituting the derivatives of y_2 in the equation yields a separable differential equation in v'. Solving for v' first and then finding v produces the second complementary function y_2 of the general solution.

However, none of the above methods are general. Our next goal is to present a general method that produces the solution of the equation $a(x)y'' + b(x)y' + c(x)y = 0$ in a form of a **power series** $y = \sum_{n=0}^{\infty} a_n(x - x_0)^n$. Having a solution in the form of a power series should not be considered as a downside – most commonly occurring functions such as polynomials, exponential and trigonometric functions have power series extension.

Let us start by assuming that $a(x) \neq 0$ (so that the equation is really of the second order) and by dividing by $a(x)$ to obtain the form

$$y'' + p(x)y' + q(x)y = 0$$

Although we denote the independent variable as x suggesting it is a real variable, our following discussion and the methods presented are valid for **functions of complex variables** as well.

Regular, Singular and Regular-Singular Points. Let us consider the equation $y'' + p(x)y' + q(x)y = 0$ and let us assume that the function $p(x)$ and $q(x)$ are **analytic** that is, that they have power series expansions

$$p(x) = \sum_{n=0}^{\infty} p_n(x - x_0)^n \text{ and } q(x) = \sum_{n=0}^{\infty} q_n(x - x_0)^n$$

centered at a point $x = x_0$ which are convergent on some interval containing x_0. In this case, the point $x = x_0$ is said to be a **regular (or ordinary) point**.

If the functions $p(x)$ or $q(x)$ are not analytic at $x = x_0$, then the point $x = x_0$ is said to be a **singular point**. In this case when it is possible to write the equation in the form

$$(x - x_0)^2 y'' + \overline{p}(x)(x - x_0)y' + \overline{q}(x)y = 0$$

where the functions $\overline{p}(x)$ and $\overline{q}(x)$ are analytic at $x = x_0$, the point $x = x_0$ is said to be a **regular-singular point**. If it is not possible to represent the equation in this way, the point $x = x_0$ is said to be **irregular or essential singularity.**

The following examples showcase some equations that appear in applications in physics, and the classifications of their relevant singularities.

Equation		singular points	type
Legendre	$(1 - x^2)y'' - 2xy' + k(k + 1)y = 0$	± 1	regular-singular
Chebyshev (type 1)	$(1 - x^2)y'' - xy' + p^2 y = 0$	± 1	regular-singular
Chebyshev (type 2)	$(1 - x^2)y'' - 3xy' + k(k + 2)y = 0$	± 1	regular-singular
Bessel	$x^2 y'' + xy' + (x^2 - a^2)y = 0$	0	regular-singular
Laguerre	$xy'' + (1 - x)y' + ay = 0$	0	regular-singular
Hermite	$y'' - 2xy' + 2ay = 0$	none	all pts regular
Simple harmonic oscillator	$y'' + a^2 y = 0$	none	all pts regular

The interval of convergence of solutions. The **radius of convergence** of the power series of y is the distance from the center $x = x_0$ to the next nearest singularity of p and q. If R is this radius (possibly infinite), then the **interval of convergence** is $(x_0 - R, x_0 + R)$.

Example 1. Consider the equation $y'' - \frac{1}{x-1}y = 0$. Show that $x = 0$ is a regular point of this equation and determine the interval of convergence of the solution.

Solutions. For this equation, $p = 0$ and $q = \frac{-1}{x-1} = \frac{1}{1-x}$. Both of these functions are analytic: the first because it is a polynomial and the second since it is a rational function, so both functions have continuous derivatives of any order, p at any point and q at any point different than 1. Thus, the point $x = 0$ is regular. The distance from the center $x = 0$ to the first singularity $x = 1$ is 1. So, the solution converges on the interval $(-1, 1)$.

Alternatively, the functions p and q are analytic since they have convergent power series expansions at $x = 0$. The expansion of p is $0 + 0x + 0x^2 + \ldots$. The expansion of q is $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$. This last series is convergent on $(-1, 1)$ and so the power series solution of this equation is convergent on $(-1, 1)$.

The method of finding series solutions about a regular point. If $x = x_0$ is a regular point of $y'' + p(x)y' + q(x)y = 0$, then every solution of this equation can be represented as a power series centered at x_0.

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

Finding derivatives, one obtains

$$y' = \sum_{n=0}^{\infty} n a_n (x - x_0)^{n-1} \quad \text{and} \quad y'' = \sum_{n=0}^{\infty} n(n-1) a_n (x - x_0)^{n-2}.$$

In some cases, these derivatives can be plugged into the equation as they are. For some equations, they may need to be written in a form so the n-th term has n as the power of $(x - x_0)$. To achieve this for y', note first that the first term of y' (term with $n = 0$) is $0a_0(x - x_0)^{-1} = 0$, so that the sum can be started with $n = 1$ so that $y' = \sum_{n=1}^{\infty} n a_n (x - x_0)^{n-1}$. Then, one can replace every n by $n+1$ and get

$$y' = \sum_{n=1}^{\infty} n a_n (x - x_0)^{n-1} = \sum_{n+1=1}^{\infty} (n+1) a_{n+1} (x - x_0)^{n+1-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} (x - x_0)^n.$$

Similarly, since the first two terms of y'' are zero, $y'' = \sum_{n=2}^{\infty} n(n-1) a_n (x - x_0)^{n-2}$. Shifting this last series from n to $n + 2$ produces the following.

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n (x - x_0)^{n-2} = \sum_{n+2=2}^{\infty} (n+2)(n+2-1) a_{n+2} (x - x_0)^{n+2-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} (x - x_0)^n$$

The solution y can be obtained following the steps below.

1. Substitute y, y' and y'' in the equation and shift the series you obtain if necessary so that all the sums have the same power of $(x - x_0)$.

2. Combine all the series into single one so that the left hand side of the equation is a single power series.

3. Equate the coefficients of the series you obtained to zero. This produces a **recursive equation** that computes the terms of the sequence a_n.

4. If possible, obtain the sequence a_n as a function of n. Then obtain power series for two fundamental solutions y_1 and y_2 from the recursive sequence, usually by choosing some convenient values of a_0 and a_1. The "standard" choice of a_0 and a_1 is that $a_0 = 1$ and $a_1 = 0$ for one and $a_0 = 0$ and $a_1 = 1$ for another solution. Any "nonstandard" choice of a_0 and a_1 would be given to you in problems in this course.

   In each case, once you determine the sequence $a_0, a_1, a_2 \ldots$, you can substitute these values back in
   $$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$$
   One choice of a_0 and a_1 produces one fundamental solution y_1 and another produces y_2. The general solution is $y = c_1 y_1 + c_2 y_2$ where c_1, c_2 are two constants.

5. If you can find sums of those series and express y_1 and y_2 as an elementary function, then it is said that the solution is in the **closed form**. The closed form will not always be possible to obtain.

   When finding the closed form of the solutions, the power series expansions of the following elementary functions may be useful:
   $$e^x = \sum_{n=0}^{\infty} \frac{1}{n!}\, x^n \qquad\qquad \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$
   $$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\, x^{2n+1} \qquad\qquad \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\, x^{2n}$$

Let us review the process of finding the sum of a given series: for a given power series $\sum_{n=0}^{\infty} a_n (x-a)^n$, we are finding an elementary function $f(x)$ equal to its sum. The equality $f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$ is valid when the series is convergent and the domain of the function $f(x)$ is equal to the convergence interval of the series.

Example 2. Find the sum of the following series.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n$ (b) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{n+2}$ (c) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!} x^{2n}$

(d) $\displaystyle\sum_{n=0}^{\infty} 2^n x^n$ (e) $\displaystyle\sum_{n=0}^{\infty} (-1)^n x^{2n}$ (f) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^n$

Solution. Have the four "basic" expansions (e^x, $\frac{1}{1-x}$, $\sin x$, and $\cos x$) in front when you are finding the sum.

(a) This series matches the expansion for e^x pretty closely: only $(-1)^n$ term is "off" but you can group it with x^n to have $(-x)^n$. Thus, the series is $\sum_{n=0}^{\infty} \frac{1}{n!}(-x)^n$. Note that this series is exactly the series obtained by replacing x by $-x$ in the formula for e^x. Hence,

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n = \sum_{n=0}^{\infty} \frac{1}{n!}(-x)^n = e^{-x}.$$

(b) Write x^{n+2} as $x^n\ x^2$ so that x^2 can be factored in front of the series. The remaining part matches the previous sum. Hence,

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{n+2} = x^2 \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n = x^2 \sum_{n=0}^{\infty} \frac{1}{n!}(-x)^n = x^2 e^{-x}.$$

(c) Since $x^{2n} = (x^2)^n$, this series is exactly the series obtained by replacing x by x^2 in the expansion of e^x. Hence,

$$\sum_{n=0}^{\infty} \frac{1}{n!} x^{2n} = \sum_{n=0}^{\infty} \frac{1}{n!}(x^2)^n = e^{x^2}.$$

Note that each of the three series in (a),(b), and (c) converges for every x and so their three closed forms are equal to the initial series for every value of x.

(d) Since $\sum_{n=0}^{\infty} 2^n x^n = \sum_{n=0}^{\infty} (2x)^n$, you can use the formula for $\frac{1}{1-x}$ but replace x by $2x$. Hence,

$$\sum_{n=0}^{\infty} 2^n x^n = \sum_{n=0}^{\infty} (2x)^n = \frac{1}{1-2x}.$$

Note that the interval of convergence corresponds to $|2x| < 1 \Rightarrow -1 < x < 1 \Rightarrow \frac{-1}{2} < x < \frac{1}{2}$.

(e) Since $(-1)^n x^{2n} = (-1)^n (x^2)^n = (-x^2)^n$,

$$\sum_{n=0}^{\infty} (-1)^n x^{2n} = \sum_{n=0}^{\infty} (-x^2)^n = \frac{1}{1-(-x^2)} = \frac{1}{1+x^2}.$$

Note that the interval of convergence corresponds to $|-x^2| < 1 \Rightarrow |x|^2 < 1 \Rightarrow -1 < x < 1$.

(f) The given series matches the expansion for $\cos x$ up to the power of x which is even for the expansion of cosine. Hence, write x^n as $(\sqrt{x}^2)^n = (\sqrt{x})^{2n}$ and replace x by $\sqrt{x}$ in the expansion for cosine to obtain that

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (\sqrt{x})^{2n} = \cos \sqrt{x}.$$

The series converges for every x so the interval of convergence is $(-\infty, \infty)$.

To illustrate the entire method, we start by an example that can be solved using much simpler approach. Thus, we stress that this example is here just to illustrate how the method works, not to illustrate any real need for the method.

Example 3 ("fake" example). Consider the equation $y'' - 2y' + y = 0$. Show that $x = 0$ is a regular point of this equation and determine the interval of convergence of the solution. Then find the series solution at $x = 0$ and express your answers in the closed form by choosing $a_0 = a_1 = 1$ for one and $a_0 = 0, a_1 = 1$ for the other fundamental solution.

Solution. For this equation, $p = -2$ and $q = 1$. The functions p and q are analytic since they are polynomials and polynomials have continuous derivatives of any order. In particular, $p' = p'' = p''' = \ldots 0$ and $q' = q'' = \ldots = 0$. The interval of convergence is $(-\infty, \infty)$ because polynomials have no singularities.

Alternatively, you can say that p and q are analytic since their power series expansions at $x = 0$ are $p = -2 + 0x + 0x^2 + \ldots$ and $q = 1 + 0x + 0x^2 + \ldots$ and they convergent at every value of x (including $x = 0$) since just finitely many terms are nonzero. Thus, the point $x = 0$ is regular and the solutions are convergent on $(-\infty, \infty)$.

The solution of the equation can be found in the form

$$y = \sum_{n=0}^{\infty} a_n x^n.$$

Find the derivatives to be

$$y' = \sum_{n=0}^{\infty} n a_n (x - x_0)^{n-1} = \sum_{n=1}^{\infty} n a_n (x - x_0)^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} (x - x_0)^n \text{ and}$$

$$y'' = \sum_{n=0}^{\infty} n(n-1) a_n (x - x_0)^{n-2} = \sum_{n=2}^{\infty} n(n-1) a_n (x - x_0)^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} (x - x_0)^n$$

and substitute them into the equation to get $\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - 2\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n + \sum_{n=0}^{\infty} a_n x^n = 0 \Rightarrow$

$$\sum_{n=0}^{\infty} [(n+2)(n+1)a_{n+2} - 2(n+1)a_{n+1} + a_n] x^n = 0$$

Since the series is identically equal to zero, all its terms have to be equal to zero. Thus,

$$(n+2)(n+1)a_{n+2} - 2(n+1)a_{n+1} + a_n = 0 \Rightarrow a_{n+2} = \frac{2(n+1)a_{n+1} - a_n}{(n+1)(n+2)} \text{ for all } n = 0, 1, \ldots$$

The above formula represents a recursive equation which computes the coefficients of the sequence a_n. Note that it depends on two initial conditions a_0 and a_1. By choosing $a_0 = a_1 = 1$ as instructed, you have that

$$a_2 = \frac{2-1}{1 \cdot 2} = \frac{1}{2} = \frac{1}{2!}, \quad a_3 = \frac{2(2)\frac{1}{2} - 1}{3 \cdot 2} = \frac{1}{3!}, \quad a_4 = \frac{2(3)\frac{1}{3!} - \frac{1}{2}}{4 \cdot 3} = \frac{2-1}{4 \cdot 3 \cdot 2} = \frac{1}{4!}, \ldots, \quad a_n = \frac{1}{n!}.$$

This produces the solution

$$y_1 = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \ldots = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = e^x.$$

By taking $a_0 = 0$ and $a_1 = 1$, you obtain that

$$a_2 = \frac{2}{2} = 1, \quad a_3 = \frac{2(2) - 1}{3 \cdot 2} = \frac{1}{2}, \quad a_4 = \frac{2(3)\frac{1}{2} - 1}{4 \cdot 3} = \frac{2}{4 \cdot 3} = \frac{1}{2 \cdot 3} = \frac{1}{3!}, \quad a_5 = \frac{8\frac{1}{6} - \frac{1}{2}}{5 \cdot 4} = \frac{5}{5!} = \frac{1}{4!}.$$

This indicates that $a_n = \frac{1}{(n-1)!}$ so $a_{n+1} = \frac{1}{n!}$. Thus, the second fundamental solution is

$$y_2 = x + x^2 + \frac{1}{2!}x^3 + \frac{1}{3!}x^4 + \ldots = \sum_{n=0}^{\infty} \frac{1}{n!}x^{n+1} = x\sum_{n=0}^{\infty} \frac{1}{n!}x^n = xe^x.$$

Hence, the general solution is $y = c_1 xe^x + c_2 e^x$.

As we pointed out, the equation $y'' - 2y' + y = 0$ has constant coefficients so it could be solved by considering the characteristic equation $r^2 - 2r + 1 = 0 \Rightarrow (r-1)(r-1) = 0$ and finding the solution in the form $y = c_1 e^x + c_2 xe^x$. So, we turn to more relevant examples – examples of equations that do not have constant coefficients.

Example 3 ("real" example). Consider the equation $(1-x)^2 y'' - 2y = 0$. Show that $x = 0$ is a regular point of this equation and determine the radius of the convergence of the solution. Find the series solution at $x = 0$. Express your solution in the closed form by choosing $a_0 = 1, a_1 = -2$ for one and $a_0 = a_1 = 1$ for the other fundamental solution.

Solution. Dividing by $(1-x)^2$ we obtain the form $y'' - \frac{2}{(1-x)^2}y = 0$ so that $p = 0$ and $q = \frac{-2}{(1-x)^2}$. Both of these functions are analytic at $x = 0$: the first because it is a polynomial and the second since it is a rational function with $x = 1$ as the only singularity. So, both have continuous derivatives of any order, p at any point and q at any point different than 1. Thus, the point $x = 0$ is regular.

Alternatively, you can argue that p and q are analytic since they have convergent power series expansions at $x = 0$. The expansion of p is $0 + 0x + 0x^2 + \ldots$. The expansion of q can be obtained differentiating $-2\frac{1}{1-x} = -2\sum_{n=0}^{\infty} x^n$. So, $\frac{-2}{(1-x)^2} = -2\sum_{n=0}^{\infty} nx^{n-1}$ and this series is convergent at $x = 0$.

Since 1 is a singularity of q and p has no singularity, the solutions converge in an interval of length 1 around $x = 0$. So, the interval of convergence is $(-1, 1)$.

The solution can be found in the form

$$y = \sum_{n=0}^{\infty} a_n x^n \quad \text{so that} \quad y' = \sum_{n=0}^{\infty} na_n x^{n-1} \quad \text{and} \quad y'' = \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2}$$

Substitute the function and its derivatives into the equation.

$$(1 - 2x + x^2)\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} - 2\sum_{n=0}^{\infty} a_n x^n = 0 \Rightarrow$$

$$\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} - 2\sum_{n=0}^{\infty} n(n-1)a_n x^{n-1} \sum_{n=0}^{\infty} n(n-1)a_n x^n - 2\sum_{n=0}^{\infty} a_n x^n = 0.$$

Note that the first two sums are equal to $\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$ and $\sum_{n=1}^{\infty} n(n-1)a_n x^{n-1}$ so that the first one can be shifted by 2 (replace each n by $n+2$) and the second one by 1 (replace each n by $n+1$) to obtain $\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$ and $\sum_{n=0}^{\infty}(n+1)na_{n+1}x^n$. Hence, we have the following.

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - 2\sum_{n=0}^{\infty}(n+1)na_{n+1}x^n + \sum_{n=0}^{\infty} n(n-1)a_n x^n - 2\sum_{n=0}^{\infty} a_n x^n = 0$$

Combine all four sums into a single one as follows.

$$\sum_{n=0}^{\infty}[(n+2)(n+1)a_{n+2} - 2(n+1)na_{n+1} + n(n-1)a_n - 2a_n]x^n = 0.$$

Since the series is identically equal to zero, all its terms have to be equal to zero. Thus,

$$(n+2)(n+1)a_{n+2} - 2(n+1)na_{n+1} + n(n-1)a_n - 2a_n = 0 \text{ for all } n = 0, 1, \ldots$$

The equation can be simplified as follows.

$$(n+2)(n+1)a_{n+2} - 2(n+1)na_{n+1} + (n^2 - n - 2)a_n = 0 \Rightarrow$$

$$(n+2)(n+1)a_{n+2} - 2(n+1)na_{n+1} + (n-2)(n+1)a_n = 0$$

Note that every term has $(n+1)$ factor, so you can simplify this further as follows and then solve for a_{n+2} in terms of a_n and a_{n+1}.

$$(n+2)a_{n+2} - 2na_{n+1} + (n-2)a_n = 0 \Rightarrow a_{n+2} = \frac{2na_{n+1} - (n-2)a_n}{(n+2)}$$

Note that $a_2 = \frac{2a_0}{2} = a_0,\ \ a_3 = \frac{2a_0+a_1}{3},\ \ a_4 = \frac{4a_3}{4} = a_3,\ \ a_5 = \frac{6a_3-a_3}{5} = a_3,\ \ a_6 = \frac{8a_3-2a_3}{6} = a_3 \ldots$

Choosing $a_0 = 1$ and $a_1 = -2$ produces $a_2 = 1$ and $0 = a_3 = a_4 = a_5 = \ldots$. So the only non-zero coefficients are $a_0 = 1$, $a_1 = -2$ and $a_2 = 1$. This yields a **polynomial solution**. Substitute these values back to $y_1 = \sum_{n=0}^{\infty} a_n x^n$ to get your first solution

$$y_1 = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots = 1 - 2x + 1x^2 + 0x^3 + 0x^4 + \ldots = 1 - 2x + x^2 = (1-x)^2.$$

Choosing $a_1 = a_0 = 1$, we obtain $a_2 = 1, a_3 = \frac{2a_0+a_0}{3} = 1, a_4 = a_3 = 1, \ldots a_n = 1$. This produces

$$y_2 = \sum_{n=0}^{\infty} a_n x^n = 1 + x + x^2 + x^3 + x^4 + \ldots = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

Note that the radius of convergence of this series is 1 so it converges for $-1 < x < 1$ just as we concluded earlier. The two solutions y_1 and y_2 are obviously independent (since one is a polynomial and the other is not) so a general solution is

$$y = c_1(1-x)^2 + c_2\frac{1}{1-x}.$$

Practice Problems.

1. Consider the differential equation $y'' - y = 0$. Show that $x = 0$ is a regular point of this equation and determine the interval of convergence of the solution. Find the series solutions of the given equation about $x = 0$. Express your answers in the closed form by choosing $a_0 = a_1 = 1$ for one and $a_0 = -a_1 = 1$ for the other fundamental solution.

2. Consider Hermite equation $y'' - 2xy' + 4y = 0$. Show that $x = 0$ is a regular point of this equation and determine the interval of convergence of the solution. Find the series solutions of the given equation about $x = 0$. Find the closed form of one solution and list a few terms of the second solution.

Solutions.

1. For this equation, $p = 0$ and $q = -1$. The functions p and q are analytic since they are polynomials and polynomials have continuous derivatives of any order. In particular, $p' = p'' = p''' = q' = q'' = \ldots 0$. Thus, $x = 0$ is a regular point. The interval of convergence is $(-\infty, \infty)$ because polynomials have no singularities.

   Alternatively, you can say that p and q are analytic since their power series expansions at $x = 0$ are $p = 0 + 0x + 0x^2 + \ldots$ and $q = -1 + 0x + 0x^2 + \ldots$ and they convergent at every value of x (including $x = 0$) since just finitely many terms are nonzero. Thus, the point $x = 0$ is regular and the solutions are convergent on $(-\infty, \infty)$.

   The solution can be found in the form $y = \sum_{n=0}^{\infty} a_n x^n$. Since the series of p and q converge for all values of x, the interval of convergence of solutions is $(-\infty, \infty)$.

   Find the derivatives to be

   $$y' = \sum_{n=0}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n \text{ and } y'' = \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

   and substitute them into the equation.

   $$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=0}^{\infty} a_n x^n = 0 \Rightarrow \sum_{n=0}^{\infty}[(n+2)(n+1)a_{n+2} - a_n]x^n = 0$$

   Since the series is identically equal to zero, all its terms have to be equal to zero. Thus,

   $$(n+2)(n+1)a_{n+2} - a_n = 0 \text{ for all } n = 0, 1, \ldots \Rightarrow a_{n+2} = \frac{a_n}{(n+1)(n+2)} \text{ for all } n = 0, 1, \ldots$$

   Choosing $a_0 = a_1 = 1$, we have that $a_2 = \frac{1}{1\cdot 2} = \frac{1}{2!}$, $a_3 = \frac{a_1}{2\cdot 3} = \frac{1}{3!}$, $a_4 = \frac{a_2}{3\cdot 4} = \frac{1}{4!} \Rightarrow \quad a_n = \frac{1}{n!}$. Hence, $y_1 = \sum_{n=0}^{\infty} \frac{1}{n!}x^n = e^x$.

   Choosing $a_0 = -a_1 = 1$, we obtain $a_2 = \frac{1}{1\cdot 2} = \frac{1}{2!}$, $a_3 = \frac{a_1}{2\cdot 3} = \frac{-1}{3!}$, $a_4 = \frac{a_2}{3\cdot 4} = \frac{1}{4!}$, $a_5 = \frac{a_3}{4\cdot 5} = \frac{-1}{5!} \Rightarrow \quad a_n = \frac{(-1)^n}{n!}$. Hence, $y_2 = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}x^n = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = e^{-x}$ and $y = c_1 e^x + c_2 e^{-x}$ is a general solution. Note that the radius of the convergence is infinity because the equation has no singular points.

2. For this equation, $p = -2x$ and $q = 4$. The functions p and q are analytic since they are polynomials and polynomials have continuous derivatives of any order. In particular, $p' = -2$ and $p'' = p''' = q' = q'' = \ldots 0$. Thus, $x = 0$ is a regular point. The interval of convergence is $(-\infty, \infty)$ because polynomials have no singularities.

   Alternatively, you can say that p and q are analytic since their power series expansions at $x = 0$ are $p = 0 - 2x + 0x^2 + 0x^3 + \ldots$ and $q = 4 + 0x + 0x^2 + \ldots$ and they convergent at every value of x (including $x = 0$) since just finitely many terms are nonzero. Thus, the point $x = 0$ is regular and the solutions are convergent on $(-\infty, \infty)$.

The solution can be found in the form $y = \sum_{n=0}^{\infty} a_n x^n$. Since the series of p and q converge for all values of x, the interval of convergence of solutions is $(-\infty, \infty)$.

$$y = \sum_{n=0}^{\infty} a_n x^n \Rightarrow y' = \sum_{n=0}^{\infty} n a_n x^{n-1} \text{ and } y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

Substitute these into the equation and get

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - 2 \sum_{n=0}^{\infty} n a_n x^n + 4 \sum_{n=0}^{\infty} a_n x^n = 0 \ \Rightarrow$$

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} - 2n a_n + 4 a_n \right] x^n = 0.$$

Since the series is identically equal to zero, all its terms have to be equal to zero. Thus,

$$(n+2)(n+1) a_{n+2} - 2n a_n + 4 a_n \text{ for all } n = 0, 1, \ldots \Rightarrow a_{n+2} = \frac{(2n-4) a_n}{(n+1)(n+2)} \text{ for all } n = 0, 1, \ldots$$

Thus the even-indexed coefficients depends on a_0 and the odd-indexed coefficients on a_1. We can obtain two linearly independent solutions by taking the "standard" choice of a_0 and a_1 : $a_0 = 1$ and $a_1 = 0$ for y_1 and $a_0 = 0$ and $a_1 = 1$ for y_2.

If $a_0 = 1$ and $a_1 = 0$, then $a_1 = a_3 = a_5 = a_7 \ldots = 0$. Since $a_0 = 1$, $a_2 = \frac{-4}{2} = -2$ and $a_4 = 0$. Thus, $a_4 = a_6 = a_8 = a_{10} = \ldots = 0$. So, the only two nonzero coefficients are $a_0 = 1$ and $a_2 = -2$. Hence, y_1 is a polynomial

$$y_1 = a_0 + a_2 x^2 = 1 - 2x^2.$$

If $a_0 = 0$ and $a_1 = 1$, then $a_0 = a_2 = a_4 = a_6 \ldots = 0$. Since $a_1 = 1$, $a_3 = \frac{-2}{2 \cdot 3} = \frac{-2}{3!}$, $a_5 = \frac{-2 \cdot 2}{2 \cdot 3 \cdot 4 \cdot 5} = \frac{-2^2}{5!}$, $a_7 = \frac{-6 \cdot 2^2}{5! \cdot 6 \cdot 7} = \frac{-3 \cdot 2^3}{7!} \ldots$ Thus

$$y_2 = x - \frac{2}{3!} x^3 - \frac{2^2}{5!} x^5 - \frac{3 \cdot 2^3}{7!} x^7 \ldots.$$

The general solution $y = c_1 y_1 + c_2 y_2$ is convergent on $(-\infty, \infty)$.