Math Methods 1
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Series Solutions of Ordinary Differential Equations

Let us consider a linear differential equation of the second order

$$a(x)y'' + b(x)y' + c(x)y = g(x)$$

where a, b, c, and g are some real-valued functions.

Recall that the equation $a(x)y'' + b(x)y' + c(x)y = 0$ is the **homogeneous part** of the above relation. If y_h is the solution of the homogeneous part of the non-homogeneous equation $a(x)y'' + b(x)y' + c(x)y = g(x)$, the particular solution y_p can be found using the variation of parameters method and the solution of the homogeneous part. In this case, the general solution of the equation has the form $y = y_h + y_p$.

Knowing the variation of parameters method, the problem boils down to finding the solution of the homogeneous part. If the functions $a(x), b(x)$ and $c(x)$ are constant, methods of Differential Equations course can be used. However, if these functions are not constant, another method is needed.

So, we shall concentrate on methods for finding the solution of the homogeneous equation $a(x)y''+b(x)y' + c(x)y = 0$ if the functions $a(x), b(x)$ or $c(x)$ are not constant.

There are certain methods for finding this solution in some particular cases. For example,

- **Legendre's linear equation** $a(px + q)^2y'' + b(px + q)y' + cy = 0$ where a, b, c, p and q are constants. This equation can be reduced to an equation with constant coefficients using the substitution $px + q = e^t$. If $p = 1$ and $q = 0$, the equation $ax^2y'' + bxy' + cy = 0$ is called **Euler's equation**.

- **Exact equations**. If the left hand side of the equation $a(x)y'' + b(x)y' + c(x)y = 0$ is a derivative of another equation, it is said to be **exact**. It can be shown that an equation is exact if $c(x) - b'(x) + a''(x) = 0$. Note that if $a(x) = 0$, we can write the equation in the form $b(x)dy = -c(x)ydx$ and denote $Q = b$ and $P = -cy$, then the condition $c(x) - b'(x) = 0$ is equivalent to $P_y = Q_x$ which is the criterion of exactness for the first order differential equation.

- **Partially known complementary functions**. If one solution y_1 of the homogeneous equation $a(x)y''+b(x)y'+c(x)y = 0$ is known, to find the second solution y_2 and the general solution $y = c_1y_1 + c_2y_2$ as well, assume that it has the form $y_2 = vy_1$ where v is an unknown function. Substituting the derivatives of y_2 in the equation yields a separable differential equation in v'. Solving for v' first and then finding v produces the second complementary function y_2 of the general solution.

However, none of the above methods are general. The goal of this section is to present a general method that produces the solution of the equation $a(x)y'' + b(x)y' + c(x)y = 0$ in a form of a **power series**. Having a solution in the form of a power series should not be considered as a downside –

most commonly occurring functions such as polynomials, exponential and trigonometric functions have power series extension.

Let us start by assuming that $a(x) \neq 0$ (so that the equation is really of the second order) and by dividing by $a(x)$ to obtain the form

$$y'' + p(x)y' + q(x)y = 0$$

Although we shall denote the independent variable as x suggesting it is a real variable, our following discussion and the methods presented are valid for **functions of complex variables** as well.

Regular, Singular and Regular-Singular Points

Let us consider the equation $y'' + p(x)y' + q(x)y = 0$ and let us assume that the function $p(x)$ and $q(x)$ are **analytic** that is, that they have power series expansions

$$p(x) = \sum_{n=0}^{\infty} p_n (x - x_0)^n \text{ and } q(x) = \sum_{n=0}^{\infty} q_n (x - x_0)^n$$

centered at a point $x = x_0$ which are convergent on some interval containing x_0. In this case, the point $x = x_0$ is said to be a **regular (or ordinary) point**.

If the functions $p(x)$ or $q(x)$ are not analytic at $x = x_0$, then the point $x = x_0$ is said to be a **singular point**. In this case when it is possible to write the equation in the form

$$(x - x_0)^2 y'' + \overline{p}(x)(x - x_0)y' + \overline{q}(x)y = 0$$

where the functions $\overline{p}(x)$ and $\overline{q}(x)$ are analytic at $x = x_0$, the point $x = x_0$ is said to be a **regular-singular point**. If it is not possible to represent the equation in this way, the point $x = x_0$ is said to be **irregular or essential singularity.**

The following examples showcase some equations that appear in applications in physics, and the classifications of their relevant singularities.

Equation	singular points	type
Legendre $(1 - x^2)y'' - 2xy' + l(l + 1)y = 0$	± 1	regular-singular
Chebyshev $(1 - x^2)y'' - xy' + p^2 y = 0$	± 1	regular-singular
Bessel $x^2 y'' + xy' + (x^2 - a^2)y = 0$	0	regular-singular
Laguerre $xy'' + (1 - x)y' + ay = 0$	0	regular-singular
Hermite $y'' - 2xy' + 2ay = 0$	none	all pts regular
Simple harmonic oscillator $y'' + a^2 y = 0$	none	all pts regular

Solutions about an ordinary point

If $x = x_0$ is a regular point of $y'' + p(x)y' + q(x)y = 0$, then every solution of this equation can be represented as a power series centered at x_0.

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

Note that in this case

$$y' = \sum_{n=0}^{\infty} n a_n (x - x_0)^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} (x - x_0)^n \text{ and}$$

$$y'' = \sum_{n=0}^{\infty} n(n-1) a_n (x - x_0)^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} (x - x_0)^n$$

The solution y can be obtained following the steps below.

1. Substitute y, y' and y'' in the equation.
2. Write the left hand side of the equation as a single power series.
3. Equate the coefficients of the series you obtained to zero. This produces a **recursive equation** that computes the terms of the sequence a_n.
4. If possible, obtain the sequence a_n as a function of n. Then obtain the solution y as a power series. If you can express this series as an elementary function, then it is said that the solution is in **closed form**. The close form will not always be possible to obtain.

   When finding the closed form of the solutions, the power series expansions of the following elementary functions may be useful:

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n \qquad\qquad \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \qquad\qquad \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

5. The **radius of convergence** of this power series is the distance from the center $x = x_0$ to the next nearest singularity of p and q.

To illustrate this last point, consider the equation $y'' - \frac{1}{x-1} y = 0$ about the regular point $x = 0$. For this equation $p = 0$ and $q = \frac{-1}{x-1} = \frac{1}{1-x}$. Both of these functions are analytic: first because it is a polynomial and the second since it is a rational function so both have continuous derivatives of any order, p at any point and q at any point different than 1. Thus, the point $x = 0$ is regular. Alternatively, the functions p and q are analytic since they have convergent power series expansions at $x = 0$. The expansion of p is $0 + 0x + 0x^2 + \ldots$. The expansion of q is $\frac{1}{1-x} = 1 + x + x^2 + \ldots$.

This series is convergent on $(-1,1)$ and so the power series solution of this equation is convergent on $(-1,1)$ too. Note that the radius of convergence is 1 corresponds exactly the distance from the center $x=0$ to the first singularity $x=1$.

To illustrate the method, we will start by an example that can be solved using much simpler approach. Thus, we stress that this example is here just to illustrate how the method works, not to illustrate the usefulness of the method.

Example 1 – "Fake" example. Consider the equation $y''-2y'+y=0$. Show that $x=0$ is a regular point of this equation. Then find the series solution at $x=0$ and express your answers in the closed form.

Solution. In this case $p=-2$ and $q=1$. The functions p and q are analytic since the derivatives of these functions (the constant functions 0) are continuous. Alternatively, you can say that p and q are analytic since their power series expansions at $x=0$ are $p=2+0x+0x^2+\ldots$ and $q=1+0x+0x^2+\ldots$ which are convergent for every value of x. Thus, the point $x=0$ is regular and the radius of the convergence of the solutions is ∞ (that is, the solution converges on $(-\infty,\infty)$).

The solution of the equation can be found in the form

$$y=\sum_{n=0}^{\infty}a_nx^n.$$

Find the derivatives to be

$$y'=\sum_{n=0}^{\infty}na_nx^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n \text{ and}$$

$$y''=\sum_{n=0}^{\infty}n(n-1)a_nx^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

and substitute them into the equation to get $\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n-2\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n+\sum_{n=0}^{\infty}a_nx^n=0\Rightarrow$

$$\sum_{n=0}^{\infty}[(n+2)(n+1)a_{n+2}-2(n+1)a_{n+1}+a_n]x^n=0$$

Since the series is identically equal to zero, all its terms have to be equal to zero. Thus,

$$(n+2)(n+1)a_{n+2}-2(n+1)a_{n+1}+a_n=0\Rightarrow a_{n+2}=\frac{2(n+1)a_{n+1}-a_n}{(n+1)(n+2)} \text{ for all } n=0,1,\ldots$$

The above formula represents a recursive equation which computes the coefficients of the sequence a_n. Note that it depends on two initial conditions a_0 and a_1. Now, you are looking for convenient choices of a_0 and a_1 to obtain two solutions. For example, take that $a_0=0$ and $a_1=1$. In this case, you obtain that $a_2=\frac{2}{2}=1$. Continuing computing the terms, you obtain $a_3=\frac{2(2)-1}{3\cdot2}=\frac{1}{2}$, $a_4=\frac{2(3)\frac{1}{2}-1}{4\cdot3}=\frac{2}{4\cdot3}=\frac{1}{2\cdot3}=\frac{1}{3!}$, $a_5=\frac{8\frac{1}{6}-\frac{1}{2}}{5\cdot4}=\frac{\frac{8-3}{6}}{5\cdot4}=\frac{5}{5!}=\frac{1}{4!},\ldots,a_n=\frac{1}{(n-1)!}$ or $a_{n+1}=\frac{1}{n!}$ This produces the solution

$$y_1=x+x^2+\frac{1}{2!}x^3+\frac{1}{3!}x^4+\ldots=\sum_{n=0}^{\infty}\frac{1}{n!}x^{n+1}=x\sum_{n=0}^{\infty}\frac{1}{n!}x^n=xe^x.$$

You can check that the choice $a_0 = 1$ and $a_1 = 0$ does not produce an easy summable series so you can try $a + 0 = a_1 = 1$. In this case, $a_2 = \frac{2-1}{1\cdot 2} = \frac{1}{2}$ and with $a_0 = a_1$ this simplifies to $a_2 = \frac{1}{2}$. Compute that $a_3 = \frac{2(2)\frac{1}{2}-1}{3\cdot 2} = \frac{1}{3!}$, $a_4 = \frac{2(3)\frac{1}{3!}-\frac{1}{2}}{4\cdot 3} = \frac{2-1}{4\cdot 3\cdot 2} = \frac{1}{4!}, \ldots, a_n = \frac{a_0}{n!}$. This produces the solution

$$y_2 = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \ldots = \sum_{n=0}^{\infty} \frac{1}{n!}x^n = e^x.$$

Hence, the general solution is $y = c_1 x e^x + c_2 e^x$.

As we pointed out, the equation $y'' - 2y' + y = 0$ has constant coefficients so it could be solved by considering the characteristic equation $r^2 - 2r + 1 = 0 \Rightarrow (r-1)(r-1) = 0$ and finding the solution in the form $y = c_1 e^x + c_2 x e^x$. So, we turn to more relevant examples – examples of equations that do not have constant coefficients.

Example 2 – "Real" Example. Consider the equation $(1-x)^2 y'' - 2y = 0$. Show that $x = 0$ is a regular point of this equation. Then find the series solution at $x = 0$. Write your solution in the closed form and determine the radius of the convergence of the solution.

Solution. Dividing by $(1-x)^2$ we obtain the form $y'' - \frac{2}{(1-x)^2}y = 0$ so that $p = 0$ and $q = \frac{-2}{(1-x)^2}$. Both of these functions are analytic: first because it is a polynomial and the second since it is a rational function so both have continuous derivatives of any order, p at any point and q at any point different than 1. Thus, the point $x = 0$ is regular. Since 1 is a singularity of q, the radius of convergence of the solutions is 1 (which is the distance from the center 0 to the singularity 1). So, the interval of convergence of the solutions is $(-1, 1)$.

Alternatively, you can argue that p and q are analytic since they have convergent power series expansions at $x = 0$. The expansion of p is $0 + 0x + 0x^2 + \ldots$. The expansion of q can be obtained differentiating $-2\frac{1}{1-x} = -2(1 + x + x^2 + \ldots)$. This series is convergent on $(-1, 1)$ and so the expansion of q is convergent on $(-1, 1)$. Thus, the solutions are convergent on $(-1, 1)$, too.

The solution can be found in the form $y = \sum_{n=0}^{\infty} a_n x^n$ and the derivatives are

$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n \text{ and } y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

Substitute the function and its derivatives into the equation.

$$(1 - 2x + x^2) \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - 2 \sum_{n=0}^{\infty} a_n x^n = 0 \Rightarrow$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - 2 \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^{n+1} + \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^{n+2} - 2 \sum_{n=0}^{\infty} a_n x^n = 0 \Rightarrow$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - 2 \sum_{n=0}^{\infty} (n+1) n a_{n+1} x^n + \sum_{n=0}^{\infty} n(n-1) a_n x^n - 2 \sum_{n=0}^{\infty} a_n x^n = 0 \Rightarrow$$

$$\sum_{n=0}^{\infty} [(n+2)(n+1) a_{n+2} - 2(n+1) n a_{n+1} + n(n-1) a_n - 2a_n] x^n = 0.$$

Combine the terms with a_n and factor out $n+1$ to get

$$\sum_{n=0}^{\infty}[(n+2)(n+1)a_{n+2} - 2(n+1)na_{n+1} + (n^2-n-2)a_n]x^n = 0 \Rightarrow$$

$$\sum_{n=0}^{\infty}[(n+2)(n+1)a_{n+2} - 2(n+1)na_{n+1} + (n+1)(n-2)a_n]x^n = 0 \Rightarrow$$

$$\sum_{n=0}^{\infty}(n+1)[(n+2)a_{n+2} - 2na_{n+1} + (n-2)a_n]x^n = 0$$

Since the series is identically equal to zero, all its terms have to be equal to zero. Thus,

$$(n+2)a_{n+2} - 2na_{n+1} + (n-2)a_n = 0 \ \Rightarrow a_{n+2} = \frac{2na_{n+1} - (n-2)a_n}{(n+2)} \text{ for all } n = 0, 1, \ldots$$

Note that $a_2 = \frac{2a_0}{2} = a_0$, $\quad a_3 = \frac{2a_0+a_1}{3}$, $\quad a_4 = \frac{4a_3}{4} = a_3$, $\quad a_5 = \frac{6a_3-a_3}{5} = a_3$, $\quad a_6 = \frac{8a_3-2a_3}{6} = a_3 \ldots$ Thus, choosing $2a_0 + a_1 = 0$ would produce $0 = a_3 = a_4 = a_5 = \ldots$. So the only non-zero coefficients are a_0, $a_1 = -2a_0$ and $a_2 = a_0$. This yields a **polynomial solution**. Choosing $a_0 = 1$ we obtain the first solution

$$y_1 = 1 - 2x + x^2 = (1-x)^2.$$

Choosing $a_1 = a_0 = 1$ we obtain $a_2 = 1, a_3 = \frac{2a_0+a_0}{3} = 1, a_4 = a_3 = 1, \ldots a_n = 1$. This produces

$$y_2 = 1 + x + x^2 + x^3 + x^4 + \ldots = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

Note that the radius of convergence of this series is 1 so it converges for $-1 < x < 1$ just as we concluded earlier.

The two solutions y_1 and y_2 are obviously independent (since one is a polynomial and the other is not) so the general solution is

$$y = c_1(1-x)^2 + c_2\frac{1}{1-x}.$$

Practice Problems.

1. Consider the differential equation $y'' - y = 0$. Show that $x = 0$ is a regular point of this equation. Find the series solutions of the given equation about $x = 0$. Express your answers in closed form and determine the interval of convergence.

2. Consider Hermite equation $y'' - 2xy' + 4y = 0$. Show that $x = 0$ is a regular point of this equation. Find the series solutions of the given equation about $x = 0$. Find the closed form of one solution and list first few terms of the second solution. Determine the interval of convergence.

Solutions.

1. For this equation $p = 0$ and $q = -1$ so both of these functions are analytic (all derivatives are constant or, alternatively, the convergent power series expansions are $p = 0 + 0x + 0x^2 + \ldots$ and $q = -1 + 0x + 0x^2 + \ldots$). Thus, $x = 0$ is a regular point and the solution can be found in

the form $y = \sum_{n=0}^{\infty} a_n x^n$. Since the series of p and q converge for all values of x, the interval of convergence of solutions is $(-\infty, \infty)$.

Find the derivatives to be

$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n \text{ and } y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

and substitute them into the equation.

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} a_n x^n = 0 \Rightarrow \sum_{n=0}^{\infty} [(n+2)(n+1) a_{n+2} - a_n] x^n = 0$$

Since the series is identically equal to zero, all its terms have to be equal to zero. Thus,

$$(n+2)(n+1) a_{n+2} - a_n = 0 \text{ for all } n = 0, 1, \ldots \Rightarrow a_{n+2} = \frac{a_n}{(n+1)(n+2)} \text{ for all } n = 0, 1, \ldots$$

The above formula represents the recursive equation that computes the coefficients of the sequence a_n. In order to find the solution of the recursive equation, consider the first couple of terms.

$$a_2 = \frac{a_0}{1 \cdot 2}, \quad a_4 = \frac{a_2}{3 \cdot 4} = \frac{a_0}{1 \cdot 2 \cdot 3 \cdot 4} \quad \Rightarrow \quad a_{2n} = \frac{a_0}{(2n)!}$$

and

$$a_3 = \frac{a_1}{2 \cdot 3}, \quad a_5 = \frac{a_3}{4 \cdot 5} = \frac{a_1}{2 \cdot 3 \cdot 4 \cdot 5} \quad \Rightarrow \quad a_{2n+1} = \frac{a_1}{(2n+1)!}$$

Recall that we are looking for two linearly independent solutions y_1 and y_2 to be produced by these formulas. For example,
if we take $a_0 = a_1 = 1$, we obtain $a_n = \frac{1}{n!} \quad \Rightarrow \quad y_1 = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = e^x$ and
If we take $a_0 = -a_1 = 1$, we obtain $a_n = \frac{(-1)^n}{n!} \quad \Rightarrow \quad y_2 = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = e^{-x}$.
Alternatively, by taking $a_0 = 1$ and $a_1 = 0$, you obtain $\cosh(x)$ and by taking $a_0 = 0$ and $a_1 = 1$, you obtain $\sinh(x)$.

Note that the radius of the convergence is infinity because the equation has no singular points.

Thus, the general solution is $y = c_1 e^x + c_2 e^{-x}$ (alternatively $y = c_1 \sinh(x) + c_2 \cosh(x)$).

2. Here $p = -2x$ and $q = 4$ are analytic since their derivatives are constant functions. Alternatively, you can deduce that p and q are analytic since their power series expansions are $p = 0 - 2x + 0x^2 + \ldots$ and $q = 4 + 0x + 0x^2 + \ldots$. So the point $x = 0$ is regular. Both expansions for p and q are convergent for every x and so the series solution is convergent for every x too.

   The solution can be found in the form

$$y = \sum_{n=0}^{\infty} a_n x^n \Rightarrow y' = \sum_{n=0}^{\infty} n a_n x^{n-1} \text{ and } y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

Substitute these into the equation and get

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - 2\sum_{n=0}^{\infty} na_nx^n + 4\sum_{n=0}^{\infty} a_nx^n = 0 \;\Rightarrow$$

$$\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2} - 2na_n + 4a_n\right]x^n = 0.$$

Since the series is identically equal to zero, all its terms have to be equal to zero. Thus,

$$(n+2)(n+1)a_{n+2} - 2na_n + 4a_n \text{ for all } n = 0, 1, \ldots \Rightarrow a_{n+2} = \frac{(2n-4)a_n}{(n+1)(n+2)} \text{ for all } n = 0, 1, \ldots$$

Thus the even-indexed coefficients depends on a_0 and the odd-indexed coefficients on a_1. We can obtain two linearly independent solutions by taking $a_0 = 1$ and $a_1 = 0$ for y_1 and $a_0 = 0$ and $a_1 = 1$ for y_2.

In the first case, note that $a_2 = \frac{-4}{2} = -2$ and $a_4 = 0$. Thus $a_6 = a_8 = a_{10} = \ldots = 0$. So y_1 is a polynomial $y_1 = 1 - 2x^2$.

In the second case, $a_3 = \frac{-2}{2\cdot 3}$, $a_5 = \frac{-2\cdot 2}{2\cdot 3\cdot 4\cdot 5} = \frac{-2\cdot 2}{5!}$, $a_7 = \frac{-2\cdot 2\cdot 4}{7!} \ldots$ Thus $y_2 = x - \frac{2}{3!}x^3 - \frac{2\cdot 2}{5!}x^5 - 2\frac{\cdot 2\cdot 4}{7!}x^7 \ldots$. The general solution $y = c_1y_1 + c_2y_2$ is convergent on $(-\infty, \infty)$.