Math Methods 1
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Series Solutions of Ordinary Differential Equations – regular-singular point

Recall that $x = x_0$ is a **regular-singular point** of the equation

$$(x - x_0)^2 y'' + \overline{p}(x)(x - x_0)y' + \overline{q}(x)y = 0$$

if the functions $\overline{p}(x)$ and $\overline{q}(x)$ are analytic at an interval containing $x = x_0$.

For simplicity, we can assume that $x_0 = 0$. If that is not the case, the substitution $t = x - x_0$ can convert the equation with a regular-singular point $x = x_0$ to an equation with regular-singular point $t = 0$.

With the assumption that $x_0 = 0$, we search for a solution in the form

$$y = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r}$$

for some number r. This series is called **Frobenius series**. We can assume that $a_0 \neq 0$ since otherwise we can redefine a_1 or some higher coefficient as a_0.

For $y = \sum_{n=0}^{\infty} a_n x^{n+r}$, we have that

$$y' = \sum_{n=0}^{\infty}(n + r)a_n x^{n+r-1} \text{ and } y'' = \sum_{n=0}^{\infty}(n + r)(n + r - 1)a_n x^{n+r-2}.$$

The solution y can be obtained following the steps below.

1. Substitute y, y' and y'' in the equation.

2. Write the left hand side of the equation as a single power series.

3. Consider the coefficient with *the lowest power of* x (equivalently, cancel the smallest power of x and then plug $x = 0$ to obtain this coefficient). Equate this coefficient to zero. The quadratic equation in r that you obtained in this way is called an **indical equation**. Let r_1 and r_2 denote the two solutions of the indical equation, called the **indices**, and let $r_1 \geq r_2$. We distinguish three relevant cases.

   (a) The difference $r_1 - r_2$ is not an integer. In this case, the two linearly independent solutions y_1 and y_2 are given by

   $$y_1 = x^{r_1} \sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad y_2 = x^{r_2} \sum_{n=0}^{\infty} b_n x^n.$$

(b) The difference $r_1 - r_2$ is a nonzero integer. If r_1 is the larger root, the two linearly independent solutions y_1 and y_2 are given by

$$y_1 = x^{r_1} \sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad y_2 = c y_1 \ln x + x^{r_2} \sum_{n=0}^{\infty} b_n x^n.$$

(c) The difference $r_1 - r_2$ is zero. In this case, the two linearly independent solutions y_1 and y_2 are given by

$$y_1 = x^{r_1} \sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad y_2 = y_1 \ln x + x^{r_1+1} \sum_{n=0}^{\infty} b_n x^n.$$

The coefficients a_n and b_n can be determined from two recursive equations obtained in the same way as in the case of a regular point – by substituting the solution $y = x^r \sum_{n=0}^{\infty} a_n x^n$ in the equation for two values r_1 and r_2.

4. Just in the case of a regular point, if you can express these series as elementary functions, then the solution is said to be in the **closed form**. The close form will not always be possible to obtain. Also, the **radius of convergence** of this power series is the distance from the center $x = x_0$ to the next nearest singularity of $\overline{p}$ and $\overline{q}$.

In the second or third case, the form of the second solution listed above can be obtained using a method known as **the derivative method**. One of your project topics asks you to look into this method in more details.

The following three examples illustrate these three cases.

Example 1 – Case 1 Example. Consider the equation $4xy'' + 2y' + y = 0$. Show that $x = 0$ is a regular-singular point. Then find the closed form of the series solutions about $x = 0$ and determine the interval of convergence.

Solution. The point $x = 0$ is not regular since $p = \frac{1}{2x}$ and $q = \frac{1}{4x}$ are not defined at $x = 0$. However, $x = 0$ is a regular-singular point since the equation can be written as $x^2 y'' + \frac{x}{2} y' + \frac{x}{4} y = 0$ so $\overline{p} = \frac{1}{2} = \frac{1}{2} + 0x + 0x^2 + \ldots$ and $\overline{q} = \frac{x}{4} = 0 + \frac{1}{4}x + 0x^2 + 0x^3 + \ldots$ are analytic at $x = 0$ and converge for any point x. Thus,

$$y = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r} \Rightarrow y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} \text{ and } y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}.$$

Substitute into the equation to get

$$\sum_{n=0}^{\infty} 4(n+r)(n+r-1) a_n x^{n+r-1} + \sum_{n=0}^{\infty} 2(n+r) a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0. \qquad (*)$$

The smallest power of x appears in the first two sums for $n = 0$ and it is x^{r-1}. The coefficient with x^{r-1} is $4r(r-1)a_0 + 2ra_0$. This gives you the indical equation

$$4r(r-1)a_0 + 2ra_0 = 0 \Rightarrow 2a_0(2r^2 - 2r + r) = 0 \Rightarrow 2r^2 - r = 0 \Rightarrow r(2r-1) = 0 \Rightarrow r = 0 \text{ or } r = \frac{1}{2}.$$

Thus, this is the first case since the difference $r_1 - r_2$ is not an integer.

The case $r = 0$ produces the first fundamental solution $y_1 = x^0 \sum_{n=0}^{\infty} a_n x^n$. To obtain this solution, substitute $r = 0$ into the equation $(*)$.

$$\sum_{n=0}^{\infty} 4n(n-1)a_n x^{n-1} + \sum_{n=0}^{\infty} 2na_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

Note that the first term of the first two sums is zero so these two sums can be written as $\sum_{n=1}^{\infty} 4n(n-1)a_n x^{n-1}$ and $\sum_{n=1}^{\infty} 2na_n x^{n-1}$ respectively. Shifting the indices by one in order to obtain all three sums with the matching powers of x, we obtain $\sum_{n=0}^{\infty} 4(n+1)na_{n+1}x^n$ and $\sum_{n=0}^{\infty} 2(n+1)a_{n+1}x^n$ for these first two sums. Thus we have

$$\sum_{n=0}^{\infty} 4(n+1)na_{n+1}x^n + \sum_{n=0}^{\infty} 2(n+1)a_{n+1}x^n + \sum_{n=0}^{\infty} a_n x^n = 0 \Rightarrow$$

$$\sum_{n=0}^{\infty} \left[4(n+1)na_{n+1} + 2(n+1)a_{n+1} + a_n\right] x^n = 0$$

Equating each coefficient of the series on the left side with zero produces the recursive relation

$$4(n+1)na_{n+1} + 2(n+1)a_{n+1} + a_n = 0 \Rightarrow (4n(n+1) + 2(n+1))a_{n+1} = -a_n \Rightarrow$$

$$2(n+1)(2n+1)a_{n+1} = -a_n \Rightarrow a_{n+1} = \frac{-a_n}{(2n+2)(2n+1)}$$

for $n = 0, 1, 2, \ldots$ Choosing $a_0 = 1$ produces $a_1 = \frac{-1}{2}$, $a_2 = \frac{-a_1}{4\cdot 3} = \frac{1}{4\cdot 3\cdot 2} = \frac{1}{4!}$, $a_3 = \frac{-1}{6!}$, $\ldots$, $a_n = \frac{(-1)^n}{(2n)!}$. Thus, the first solution is

$$y_1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^n$$

Comparing this sum with the series expansion for $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$ we conclude that

$$y_1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (\sqrt{x})^{2n} = \cos\sqrt{x}.$$

Let us consider now the case $r = \frac{1}{2}$ which yields the second solution $y = x^{1/2} \sum_{n=0}^{\infty} a_n x^n$. Substituting $r = \frac{1}{2}$ in the equation $(*)$ produces

$$\sum_{n=0}^{\infty} 4(n+\frac{1}{2})(n+\frac{1}{2}-1)a_n x^{n-\frac{1}{2}} + \sum_{n=0}^{\infty} 2(n+\frac{1}{2})a_n x^{n-\frac{1}{2}} + \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}} = 0.$$

Divide by $x^{\frac{1}{2}}$ to obtain the integer powers. Also, note that the coefficients in the first two sums are

$$4(n+\frac{1}{2})(n+\frac{1}{2}-1)a_n = 2(n+\frac{1}{2})\,2(n+\frac{1}{2}-1)a_n = (2n+1)(2n-1)a_n \text{ and } 2(n+\frac{1}{2})a_n = (2n+1)a_n.$$

Hence we have

$$\sum_{n=0}^{\infty}(2n+1)(2n-1)a_n x^{n-1} + \sum_{n=0}^{\infty}(2n+1)a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0 \Rightarrow$$

$$\sum_{n=0}^{\infty}\left[(2n+1)(2n-1)+(2n+1)\right]a_nx^{n-1}+\sum_{n=0}^{\infty}a_nx^n=0\Rightarrow\sum_{n=0}^{\infty}(2n+1)(2n)a_nx^{n-1}+\sum_{n=0}^{\infty}a_nx^n=0$$

Shift the first sum to obtain both sums with the matching powers of x.

$$\sum_{n=0}^{\infty}(2n+3)(2n+2)a_{n+1}x^n+\sum_{n=0}^{\infty}a_nx^n=0\Rightarrow\sum_{n=0}^{\infty}\left[(2n+3)(2n+2)a_{n+1}+a_n\right]x^n=0\Rightarrow$$

$$(2n+3)(2n+2)a_{n+1}=-a_n\Rightarrow a_{n+1}=\frac{-a_n}{(2n+3)(2n+2)}\text{ for }n=1,2,\ldots$$

Choosing $a_0=1$ produces $a_1=\frac{-1}{2\cdot 3}=\frac{-1}{3!}$, $a_2=\frac{-a_1}{5\cdot 4}=\frac{1}{5\cdot 4\cdot 3\cdot 2}=\frac{1}{5!}$, $a_3=\frac{-1}{7!}$, $\ldots$, $a_n=\frac{(-1)^n}{(2n+1)!}$. Thus, the second solution is

$$y_2=x^{\frac{1}{2}}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^n$$

Comparing this sum with the series expansion for $\sin x=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1}$ we conclude that

$$y_1=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{n+\frac{1}{2}}=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}(x^{\frac{1}{2}})^{2n+1}=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}(\sqrt{x})^{2n+1}=\sin\sqrt{x}.$$

Thus, the general solution is $y=c_1\cos\sqrt{x}+c_2\sin\sqrt{x}$. The functions $\overline{p}$ and $\overline{q}$ converge on $(-\infty,\infty)$ and so the interval of convergence of the solutions is also $(-\infty,\infty)$.

Example 2 – Case 2 Example. Consider the equation $x^2y''+xy'-y=0$. Show that $x=0$ is a regular-singular point. Then find the closed form of the series solutions about $x=0$.

Solution. Note that $x=0$ is regular-singular point since $\overline{p}=1$ and $\overline{q}=-1$ are analytic (all derivatives are zero or, alternatively, constant functions have convergent power series expansion).

$$y=x^r\sum_{n=0}^{\infty}a_nx^n=\sum_{n=0}^{\infty}a_nx^{n+r}\Rightarrow y'=\sum_{n=0}^{\infty}(n+r)a_nx^{n+r-1}\text{ and }y''=\sum_{n=0}^{\infty}(n+r)(n+r-1)a_nx^{n+r-2}.$$

Substitute into the equation to get $\sum_{n=0}^{\infty}(n+r)(n+r-1)a_nx^{n+r}+\sum_{n=0}^{\infty}(n+r)a_nx^{n+r}-\sum_{n=0}^{\infty}a_nx^{n+r}=0\Rightarrow\sum_{n=0}^{\infty}[(n+r)(n+r-1)+(n+r)-1]a_nx^{n+r}=0$

Equating the first term with zero produces the indical equation

$$r(r-1)+r-1=0\Rightarrow r^2-1=0\Rightarrow(r-1)(r+1)=0\Rightarrow r_1=1,r_2=-1$$

so we are dealing with the case when the difference of the roots is a non-negative integer.

Plugging that $r=1$ in the equation gives us

$$x\sum_{n=0}^{\infty}[(n+1)n+(n+1)-1]a_nx^n=0\Rightarrow[n^2+2n]a_n=0\Rightarrow n(n+2)a_n=0,n=0,1,2,\ldots$$

Since the expression $n(n+2)$ is not zero for any positive value of n, we conclude that $a_n=0$ for all $n=1,2,\ldots$. Thus, all the coefficients a_n are zero except possibly the first one, a_0. Taking $a_0=1$ you obtain the first solution $y_1=x(1+0)=x$.

The second solution has the form $y = cy_1 \ln x + x^{r_2} \sum_{n=0}^{\infty} b_n x^n = cx \ln x + x^{-1} \sum_{n=0}^{\infty} b_n x^n = cx \ln x + \sum_{n=0}^{\infty} b_n x^{n-1}$. The derivatives are

$y' = c + c \ln x + \sum_{n=0}^{\infty} (n-1) b_n x^{n-2}$ and $y'' = \frac{c}{x} + \sum_{n=0}^{\infty} (n-1)(n-2) b_n x^{n-3}$.

Plug the function and the derivatives into the equation and obtain

$$cx + \sum_{n=0}^{\infty} (n-1)(n-2) b_n x^{n-1} + cx + cx \ln x + \sum_{n=0}^{\infty} (n-1) b_n x^{n-1} - cx \ln x - \sum_{n=0}^{\infty} b_n x^{n-1} = 0$$

The terms with $\ln x$ cancel and the remaining terms in $2cx + \sum_{n=0}^{\infty} [(n-1)(n-2) + (n-1) - 1] b_n x^{n-1}$ have to be equal to zero. Simplify to get $2cx + \sum_{n=0}^{\infty} (n^2 - 3n + 2 + n - 2) b_n x^{n-1} = 0 \Rightarrow$

$$2cx + \sum_{n=0}^{\infty} n(n-2) b_n x^{n-1} = 0 \Rightarrow 2cx - b_1 + 3b_3 x^2 + 4(2) b_4 x^3 + 5(3) b_5 x^4 + \ldots = 0$$

Considering the coefficients with each term we obtain that $c = 0$, $b_1 = 0$ and $b_3 = b_4 = b_5 \ldots = 0$. Thus, b_0 and b_2 are the only two possible nonzero coefficients. Thus, $y_2 = \frac{1}{x}(b_0 + b_2 x^2) = \frac{b_0}{x} + b_2 x$. Since the last term is a constant multiple of the first solution, we can take $b_2 = 0$. Taking $b_0 = 1$ for simplicity, we obtain the second solution $y_2 = \frac{1}{x}$.

Thus, the general solution is $y = c_1 x + c_2 \frac{1}{x}$.

Example 3 – Case 3 Example. Consider the equation $x^2 y'' - xy' + y = 0$. Show that $x = 0$ is a regular-singular point. Then find the closed form of the series solutions about $x = 0$.

Solution. Note that $x = 0$ is regular-singular point since $\overline{p} = -1$ and $\overline{q} = 1$ are analytic (all derivatives are zero or, alternatively, constant functions have convergent power series expansion).

$$y = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r} \Rightarrow y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} \text{ and } y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}.$$

Substitute into the equation to get $\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r} - \sum_{n=0}^{\infty} (n+r) a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0 \Rightarrow$

$$\sum_{n=0}^{\infty} [(n+r)(n+r-1) - (n+r) + 1] a_n x^{n+r} = 0$$

Equate the coefficient of the smallest power of x with zero to produce the indical equation

$$r(r-1) - r + 1 = 0 \Rightarrow (r^2 - 2r + 1) = 0 \Rightarrow (r-1)(r-1) = 0 \Rightarrow r_1 = r_2 = 1$$

so we are dealing with the case when the difference of roots is zero.

Plugging that $r = 1$ in the equation gives us

$$x \sum_{n=0}^{\infty} [(n+1)n - (n+1) + 1] a_n x^n = 0 \Rightarrow [n^2 + n - n - 1 + 1] a_n = 0 \Rightarrow n^2 a_n = 0$$

Thus, all the coefficients a_n are zero except the first one, a_0 when $n = 0$. Take $a_0 = 1$ and obtain the first solution $y_1 = x(1 + 0) = x$.

The second solution has the form $y = y_1 \ln x + x^{r_1+1}\sum_{n=0}^{\infty} b_n x^n = x\ln x + x^2 \sum_{n=0}^{\infty} b_n x^n = x\ln x + \sum_{n=0}^{\infty} b_n x^{n+2}$. The derivatives are
$y' = 1 + \ln x + \sum_{n=0}^{\infty}(n+2)b_n x^{n+1}$ and $y'' = \frac{1}{x} + \sum_{n=0}^{\infty}(n+2)(n+1)b_n x^n$.
Plug the function and the derivatives into the equation and obtain

$$x + \sum_{n=0}^{\infty}(n+2)(n+1)b_n x^{n+2} - x - x\ln x - \sum_{n=0}^{\infty}(n+2)b_n x^{n+2} + x\ln x + \sum_{n=0}^{\infty} b_n x^{n+2} = 0$$

All the "non-series" terms cancel and the remaining series $\sum_{n=0}^{\infty}[(n+2)(n+1)b_n - (n+2)b_n + b_n]x^{n+2}$ has to have zero terms. Thus $(n+2)(n+1)b_n - (n+2)b_n + b_n = 0 \Rightarrow$

$$(n^2 + 3n + 2 - n - 2 + 1)b_n = 0 \Rightarrow (n^2 + 2n + 1)b_n = 0 \Rightarrow (n+1)^2 b_n = 0 \Rightarrow b_n = 0$$

for all n. Thus, the second solution is $y_2 = x\ln x$ and the general solution is $y = c_1 x + c_2 x \ln x$.

The next example illustrate that in some special instances of the second case, the second solution can be obtained from the first one. In other words, both solutions can be obtained in the form $y = x^r \sum_{n=0}^{\infty} a_n x^n$.

Example 4. – "Lucky instance of the second case" Example. Consider the equation $x^2y'' - 2xy' + 2y = 0$. Show that $x = 0$ is a regular-singular point. Then find the closed form of the series solutions about $x = 0$.

Solutions. For this equation, $\overline{p} = -2$ and $\overline{q} = 2$ and these two functions are analytic (all derivatives are zero, alternatively, constant functions have convergent power series expansion). So $x = 0$ is a regular-singular point. Plugging the solution $y = \sum_{n=0}^{\infty} a_n x^{n+r}$ and its derivatives yield the equation

$$\sum_{n=0}^{\infty}[(n+r)(n+r-1) - 2(n+r) + 2]a_n x^{n+r} = 0$$

Equating the first term with zero produces the indical equation $r(r-1) - 2r + 2 = 0 \Rightarrow r^2 - 3r + 2 = 0 \Rightarrow (r-1)(r-2) = 0$. So, the difference $r_1 - r_2$ is an integer. The first solution can be obtained by taking the larger of the two r-values, $r = 2$. In this case $[(n+2)(n+1) - 2(n+2) + 2]a_n = 0 \Rightarrow (n^2 + 3n + 2 - 2n - 4 + 2)a_n = 0 \Rightarrow (n^2 + n)a_n = 0 \Rightarrow n(n+1)a_n = 0$. In this case all coefficients a_n are zero except possibly a_0. By taking $a_0 = 1$, we obtain the solution $y_1 = x^2(1+0) = x^2$.
Consider now $r = 1$. In this case $(n+1)na_n - 2(n+1)a_n + 2a_n = 0 \Rightarrow (n^2 + n - 2n - 2 + 2)a_n = 0 \Rightarrow (n^2 - n)a_n = 0 \Rightarrow n(n-1)a_n = 0$. In this case all coefficients a_n are zero except possibly a_0 (when $n = 0$) and a_1 (when $n - 1 = 0$). Thus, $y_2 = x(a_0 + a_1 x) = a_0 x + a_1 x^2$. Since the second part has the form of the first solution, you can take $a_1 = 0$. Taking $a_0 = 1$ for simplicity, you obtain the solution $y_2 = x$ that is linearly independent from $y_1 = x^2$.
Thus, the general solution is $y = c_1 x^2 + c_2 x$.

Practice Problems.

1. Consider the equation $3x^2y'' - 4xy' + 2y = 0$. Show that $x = 0$ is a regular-singular point. Then find the closed form of the series solutions about $x = 0$ and determine the interval of convergence.

2. Consider the equation $xy'' - xy' + y = 0$. Show that $x = 0$ is a regular-singular point. Then find the series solutions about $x = 0$. Find the closed form of one solution and list first few terms of the second solution.

3. Consider the equation $x^2y'' + 2xy' - x^2y = 0$. Show that $x = 0$ is a regular-singular point. Then find the series solutions about $x = 0$. Find the series form of one solution and write down the form of the second solution (do not need to solve for coefficients of the second solution).

4. Consider the equation $x(1-x)y'' + (1-x)y' + y = 0$. Show that $x = 0$ is a regular-singular point. Then find the series solutions about $x = 0$. Find the closed form of one solution and write down the form of the second solution (do not need to solve for coefficients of the second solution). Determine the interval of convergence.

Solutions.

1. For this equation, $\overline{p} = -\frac{4}{3}$ and $\overline{q} = \frac{2}{3}$. Constant functions have power series expansions which converges at every point. So $x = 0$ is a regular-singular point and the series solution is convergent on $(-\infty, \infty)$.

   Plugging the solution $y = \sum_{n=0}^{\infty} a_n x^{n+r}$ and its derivatives yield the equation

   $$\sum_{n=0}^{\infty} 3(n+r)(n+r-1)a_n x^{n+r} - \sum_{n=0}^{\infty} 4(n+r)a_n x^{n+r} + \sum_{n=0}^{\infty} 2a_n x^{n+r} = 0$$

   Equating the coefficient of the smallest power of x, x^r in this case, with zero produces the indical equation $3r(r-1) - 4r + 2 = 0 \Rightarrow 3r^2 - 7r + 2 = 0 \Rightarrow r_1 = 2$ and $r_2 = \frac{1}{3}$. So, the difference $r_1 - r_2$ is not an integer.

   When $r = 2$ the equation becomes

   $$\sum_{n=0}^{\infty}[3(n+2)(n+1)a_n - 4(n+2)a_n + 2a_n]x^{n+2} = 0 \Rightarrow [3(n+2)(n+1) - 4(n+2) + 2]a_n = 0 \Rightarrow$$

   $(3n+5)na_n = 0 \Rightarrow a_n = 0$ for all $n > 0$. By taking $a_0 = 1$ we obtain the solution $y_1 = x^2(1+0) = x^2$.

   When $r = \frac{1}{3}$ the equation becomes

   $$\sum_{n=0}^{\infty}[3(n+\frac{1}{3})(n-\frac{2}{3})a_n - 4(n+\frac{1}{3})a_n + 2a_n]x^{n+2} = 0 \Rightarrow [3(n+\frac{1}{3})(n-\frac{2}{3}) - 4(n+\frac{1}{3}) + 2]a_n = 0 \Rightarrow$$

   $(3n-5)na_n = 0 \Rightarrow a_n = 0$ for all $n > 0$. By taking $a_0 = 1$ we obtain the solution $y_2 = x^{1/3}(1+0) = x^{1/3}$.

   So, the general solution is $y = c_1x^2 + c_2x^{1/3}$.

2. For this equation, $\overline{p} = -x$ and $\overline{q} = x$ are analytic (power series expansions are $\overline{p} = 0 - 1x + 0x^2 + 0x^3 + \ldots$ and $\overline{q} = 0 + 1x + 0x^2 + 0x^3 + \ldots$ and are convergent for any x). So $x = 0$ is

a regular-singular point. Plugging the solution $y = \sum_{n=0}^{\infty} a_n x^{n+r}$ and its derivatives yield the equation

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-1} - \sum_{n=0}^{\infty}(n+r)a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

The smallest power of x is x^{r-1}. Its coefficient produces the indical equation $r(r-1) = 0 \Rightarrow r_1 = 1$ and $r_2 = 0$. So, the difference $r_1 - r_2$ is an integer.

One solution can be obtained by considering $r = 1$. In this case

$$\sum_{n=0}^{\infty} n(n+1)a_n x^n - \sum_{n=0}^{\infty}(n+1)a_n x^{n+1} + \sum_{n=0}^{\infty} a_n x^{n+1} = 0 \Rightarrow$$

$$\sum_{n=0}^{\infty}[(n+1)(n+2)a_{n+1} - (n+1)a_n + a_n]x^{n+1} = 0 \Rightarrow$$

$2a_1 = 0$ and $a_{n+1} = \frac{na_n}{(n+2)(n+1)}$ for $n = 1, 2\ldots$ Thus $a_1 = 0$ so $a_2 = a_3 = \ldots = 0$. Taking $a_0 = 1$ we obtain the first solution $y_1 = x(1+0) = x$.

The other solution can be found in the form $y = cx\ln x + \sum_{n=0}^{\infty} b_n x^n$. Plugging this and its derivatives in the equation produces

$$c + \sum_{n=0}^{\infty} n(n-1)b_n x^{n-1} - cx\ln x - cx - \sum_{n=0}^{\infty} nb_n x^n + cx\ln x + \sum_{n=0}^{\infty} b_n x^n = 0 \Rightarrow$$

$$c - cx + \sum_{n=0}^{\infty}[(n+1)nb_{n+1} - nb_n + b_n]x^n = 0 \Rightarrow$$

Equate the left-hand side terms with x^n for any n with zero.

For $n = 0$, we have that $c + b_0 = 0$. So, $b_0 = -c$.

For $n = 1$, we have that $-c + 2b_2 = 0$. So, $b_2 = \frac{c}{2}$

For $n = 2, 3, \ldots$, we have that $[(n+1)nb_{n+1} - nb_n + b_n \Rightarrow b_{n+1} = \frac{(n-1)b_n}{n(n+1)}$.

Thus $b_3 = \frac{b_2}{3\cdot 2} = \frac{c}{3\cdot 2\cdot 2} = \frac{c}{2\cdot 3!}$, $b_4 = \frac{2b_3}{4\cdot 3} = \frac{c}{4\cdot 3\cdot 3\cdot 2} = \frac{c}{3\cdot 4!}$, $b_5 = \frac{3b_4}{5\cdot 4} = \frac{c}{5\cdot 4\cdot 4\cdot 3\cdot 2} = \frac{c}{4\cdot 5!}\ldots$ and so $b_n = \frac{c}{(n-1)n!}$ for $n = 2, 3, \ldots$. We can take $b_1 = 0$ and $c = 1$ for simplicity. Thus $y_2 = x\ln x - 1 + \sum_{n=2}^{\infty} \frac{x^n}{(n-1)\cdot n!}$ and the general solution is $y = c_1 x + c_2 y_2$.

3. For this equation, $\overline{p} = 2$ and $\overline{q} = -x^2$. These functions are analytic with convergent power series expansions are $\overline{p} = 2 + 0x + 0x^2 + \ldots$ and $\overline{q} = 0 + 0x - 1x^2 + 0x^3 + 0x^4 \ldots$. So $x = 0$ is a regular-singular point. Plugging the solution $y = \sum_{n=0}^{\infty} a_n x^{n+r}$ and its derivatives yield the equation

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r} + 2\sum_{n=0}^{\infty}(n+r)a_n x^{n+r} - \sum_{n=0}^{\infty} a_n x^{n+r+2} = 0$$

Equating the coefficient of the smallest power of x (the first term in the first two sums in this case) with zero produces the indical equation $r(r-1)+2r=0 \Rightarrow r(r+1)=0 \Rightarrow r_1=0$ and $r_2=-1$ So, the difference r_1-r_2 is an integer.

One solution can be obtained by considering $r=0$. In this case
$\sum_{n=0}^{\infty} n(n-1)a_n x^n + 2\sum_{n=0}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^{n+2} = 0 \Rightarrow$
$2a_1 x + \sum_{n=0}^{\infty}[(n+2)(n+1)a_{n+2} + 2(n+2)a_{n+2} - a_n]x^{n+2} = 0 \Rightarrow$
$2a_1 = 0$ and $a_{n+2} = \frac{a_n}{(n+2)(n+3)}$ for $n=0,1,\ldots$ Thus $a_1=0$ and so $a_3=a_5=\ldots=0$. The even terms are $a_2=\frac{a_0}{3\cdot 2}=\frac{a_0}{3!}, a_4=\frac{a_0}{5!}, \ldots\ a_{2n}=\frac{a_0}{(2n+1)!}$. Taking $a_0=1$ we obtain the first solution $y_1=\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n+1)!}$. The second solution has the form $y_2 = cx\ln x + x^{-1}\sum_{n=0}^{\infty} b_n x^n$. The general solution is $y=c_1y_1+c_2y_2$.

4. Multiplying the equation by x and dividing by $1-x$ we obtain the form $x^2y''+xy'+\frac{x}{1-x}y=0$. For this form, we can see that $\overline{p}=1$ and $\overline{q}=\frac{x}{1-x}$. The function $\overline{p}$ is analytic with the power series $1+0x+0x^2+\ldots$ convergent for any n. The function $\overline{q}=x\frac{1}{1-x}=x(1+x+x^2+\ldots)=x+x^2+x^3+\ldots$ is convergent on interval $(-1,1)$ since the expansion $1+x+x^2+\ldots$ of $\frac{1}{1-x}$ is convergent on $(-1,1)$. So, $x=0$ is a regular-singular point and the solutions are convergent on interval $(-1,1)$. You can reach the same conclusion regarding the interval of convergence by noting that $x=1$ is a singularity of $\overline{q}$ so the radius of convergence is 1 (which is the distance from the center 0 to the singularity 1) and, hence, the interval of convergence is $(-1,1)$.

   Plugging the solution $y=\sum_{n=0}^{\infty} a_n x^{n+r}$ and its derivatives into the equation $xy''-x^2y''+y'-xy'+y=0$ produces

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-1} - \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r} + \sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} -$$

$$\sum_{n=0}^{\infty}(n+r)a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

   Equating the coefficient of the smallest power of x, x^{r-1} in this case, with zero produces the indical equation $r(r-1)+r=0 \Rightarrow r^2=0 \Rightarrow r_1=r_2=0$. So, this is the third case.

   For $r=0$, the equation becomes $\sum_{n=0}^{\infty} n(n-1)a_n x^{n-1} - \sum_{n=0}^{\infty} n(n-1)a_n x^n + \sum_{n=0}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0 \Rightarrow \sum_{n=0}^{\infty}[(n+1)na_{n+1} - n(n-1)a_n + (n+1)a_{n+1} - na_n + a_n]x^n = 0 \Rightarrow a_{n+1} = \frac{(n^2-1)a_n}{(n+1)^2} = \frac{(n-1)a_n}{n+1} \Rightarrow a_1=-a_0,\ a_2=0 \Rightarrow a_3=a_4=a_5=\ldots 0$. With $a_0=1$, we obtain $y_1=1-x$.

   The second solution has the form $y_2=(1-x)\ln x + x\sum_{n=0}^{\infty} b_n x^n$ and the general solution is $y=c_1(1-x)+c_2y_2$.