Math Methods 1
Lia Vas

Line and Surface Integrals. Flux. Stokes’ and Divergence Theorems

Review of Curves. Intuitively, we think of a curve as a path traced by a moving particle in space. Thus, a curve is a function of a parameter, say \( t \). Using the standard vector representations of points in the three-dimensional space as \( \mathbf{r} = (x, y, z) \), we can represent a curve as a vector function:

\[
\mathbf{r}(t) = (x(t), y(t), z(t))
\]

or using the parametric equations \( x = x(t), y = y(t), \) and \( z = z(t) \). The variable \( t \) is called the parameter.

Example 1.

1. **Line.** A line in space is given by the equations

\[
x = x_0 + at \quad y = y_0 + bt \quad z = z_0 + ct
\]

where \((x_0, y_0, z_0)\) is a point on the line and \((a, b, c)\) is a vector parallel to it. Note that in the vector form the equation \( \mathbf{r} = \mathbf{r}(0) + m t \) for \( \mathbf{r}(0) = (x_0, y_0, z_0) \) and \( m = (a, b, c) \), has exactly the same form as the well known \( y = b + m x \).

2. **Circle in horizontal plane.** Consider the parametric equations \( x = a \cos t \quad y = a \sin t \quad z = b \). Recall that the parametric equation of a circle of radius \( a \) centered in the origin of the \( xy \)-plane are \( x = a \cos t, y = a \sin t \). Recall also that \( z = b \) represents the horizontal plane passing \( b \) in the \( z \)-axis. Thus, the equations

\[
x = a \cos t \quad y = a \sin t \quad z = b
\]

represent the circle of radius \( a \) in the horizontal plane passing \( z = b \) on \( z \)-axis.

3. **Ellipse in a plane.** Consider the intersection of a cylinder and a plane. The intersection is an ellipse. For example, if we consider a cylinder with circular base \( x = a \cos t, y = a \sin t \) and the equation of the plane is
\[ mx + ny + kz = l \] with \( k \neq 0 \), the parametric equations of ellipse can be obtained by solving the equation of plane for \( z \) and using the equations for \( x \) and \( y \) to obtain the equation of \( z \) in parametric form. Thus \( z = \frac{1}{k}(l - mx - ny) \) and so \( x = a \cos t \quad y = a \sin t \quad z = \frac{1}{k}(l - ma \cos t - na \sin t) \).

4. **Circular helix.** A curve with equations \( x = a \cos t \quad y = a \sin t \quad z = bt \) is the curve spiraling around the cylinder with base circle \( x = a \cos t, y = a \sin t \).

5. **Plane curves.** All the concepts we develop for space curves correspond to plane curves simply considering that \( z = 0 \).

**Review of line integrals of scalar functions.** Suppose that \( C \) is a curve given by \( r(t) = (x(t), y(t), z(t)) \) on the interval \( a \leq t \leq b \). Recall that the length of \( C \) is

\[
L = \int_C ds = \int_C |r'(t)| dt = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} \, dt
\]

This integral can be considered to be a special case of the situation when we integrate a scalar (real-valued) function \( f(x, y, z) \) over the curve \( C \). In the general case, we consider the line integral of \( C \) with respect to arc length as

\[
\int_C f(x, y, z) \, ds = \int_C f(r(t)) \, |r'(t)| \, dt = \int_a^b f(x(t), y(t), z(t)) \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} \, dt
\]

In Calculus 3, you may have seen the application of the this type of line integrals: finding the mass \( m \) and the center of mass \((\bar{x}, \bar{y}, \bar{z})\) of a wire \( C \) with density \( \rho(x, y, z) \). In particular, the mass can be calculated as

\[
m = \int_C \rho(x, y, z) ds.
\]

Using this example, you can think of the line integral as the total mass of the line density function over the curve \( C \).

**Example 2.** Evaluate the integral \( \int_C x \, y^3 \, ds \) where \( C \) is the circular helix \( x = 4 \sin t, y = 4 \cos t, z = 3t \), for \( 0 \leq t \leq \pi/2 \).

**Solution.** \( x' = 4 \cos t, y' = -4 \sin t, z' = 3 \Rightarrow ds = \sqrt{16 \cos^2 t + 16 \sin^2 t + 9} = \sqrt{25} = 5 \). Thus \( \int_0^{\pi/2} 4 \sin t \, 4^3 \cos^3 t \, 5 \, dt = (5)4^4 - \cos^4 t |_0^{\pi/2} = (5)4^3 = 320 \).

**Review of Line Integrals of vector functions.** Another type of line integrals includes integrating a vector function over a curve. Suppose now that \( \mathbf{f} \) is a function that assigns to each point
\((x, y, z)\) a three dimensional vector
\[
f(x, y, z) = (P(x, y, z), Q(x, y, z), R(x, y, z))
\]

Such function is said to be a vector field.

An example of a space vector field is the gradient vector \(\nabla f\) of a scalar function \(f(x, y, z)\). The gradient is
\[
\nabla f = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right).
\]

**Line integrals of vector fields.** If \(f(x, y, z) = (P(x, y, z), Q(x, y, z), R(x, y, z))\) is a vector field, and the curve \(C\) is given by \(r(t) = (x(t), y(t), z(t))\), then the vector differential of the length element \(dr\) is the product \(r'(t)dt\). The line integral of \(f\) along \(C\) is defined as
\[
\int_C f \cdot dr = \int_C f(r(t)) \cdot r'(t) \, dt = \int_C Pdx + Qdy + Rdz.
\]

This type of integrals measures the total effect of a given field along a given curve. In particular, many basic (non-continuous, one dimensional) formulas in physics such as \(s = vt\) can be represented in terms of line integrals in continuous and multi-dimensional cases, for example, \(s = \int vdt\).

Another example includes the formula for calculating the work done by the force \(\overrightarrow{F}\) (possibly an electric or gravitational field) in moving the particle along the curve \(C\)
\[
W = \int_C \overrightarrow{F} \cdot dr.
\]

The length element \(ds\) and the vector differential of the length element \(dr\) are related by
\[
dr = r'(t)dt \quad \text{and} \quad ds = |r'(t)|dt.
\]

**Example 3.** Find the work done by the force field \(f = (-y, x, x^2 + y^2)\) when a particle moves under its influence along the positively oriented boundary of the part of the paraboloid \(z = 4 - x^2 - y^2\) in the first octant.

**Solution.** The boundary of the part of the paraboloid \(z = 4 - x^2 - y^2\) in the first octant consists of three curves, \(C_1\) in \(xy\)-plane, \(C_2\) in \(yz\)-plane, and \(C_3\) in \(xz\)-plane. Each of the three curves has a different set of parametric equations. The parametrizations can be obtained by considering intersections with three coordinate planes \(z = 0, x = 0,\) and \(y = 0\) respectively. The positive orientation means that the particle traversed the curves in counter-clockwise direction.

The work can be found as
\[
\int_C f \cdot dr = \int_C (-y, x, x^2 + y^2) \cdot (dx, dy, dz) = \int_C -ydx + xdy + (x^2 + y^2)dz.
\]

\(C_1\) The intersection \(C_1\) of the paraboloid and the \(xy\)-plane \(z = 0\) is a circle \(0 = 4 - x^2 - y^2 \Rightarrow x^2 + y^2 = 4\) which has parametric equations \(x = 2 \cos t, y = 2 \sin t\). Since we are considering
just the part with $x \geq 0$ and $y \geq 0$, we have that $0 \leq t \leq \frac{\pi}{2}$. Thus, this curve has parametric equations

$$x = 2 \cos t, \; y = 2 \sin t, \; z = 0$$

with $t : 0 \to \frac{\pi}{2}$ and $dx = -2 \sin t \, dt$, $dy = 2 \cos t \, dt$, $dz = 0$. Thus,

$$\int_{C_1} -y \, dx + x \, dy + (x^2 + y^2) \, dz = \int_0^{\pi/2} (4 \sin^2 t + 4 \cos^2 t + 4(0)) \, dt = \int_0^{\pi/2} 4 \, dt = 2\pi$$

C_2 The intersection $C_2$ of the paraboloid and the $yz$-plane $x = 0$ is a parabola $z = 4 - 0^2 - y^2 \Rightarrow z = 4 - y^2$. Using $y$ as a parameter produces parametric equations $x = 0, y = y, z = 4 - y^2$. As the particle travels upwards on $C_2$, the $y$-values decrease from 2 to 0. From the parametric equations, $dx = 0, dy = dy, dz = -2y \, dy$. Thus,

$$\int_{C_2} -y \, dx + x \, dy + (x^2 + y^2) \, dz = \int_2^0 (0 + 0 + (0 + y^2)(-2y)) \, dy = \int_2^0 -2y^3 \, dy = \left. -\frac{y^4}{4} \right|_2^0 = 8$$

C_3 The intersection $C_3$ of the paraboloid and the $xz$-plane $y = 0$ is a parabola $z = 4 - x^2 - 0^2 \Rightarrow z = 4 - x^2$. Using $x$ as a parameter produces parametric equations $x = x, y = 0, z = 4 - x^2$. As the particle travels downwards on $C_3$, the $x$-values increase from 0 to 2. From the parametric equations, $dx = dx, dy = 0, dz = -2x \, dx$. Thus,

$$\int_{C_3} -y \, dx + x \, dy + (x^2 + y^2) \, dz = \int_0^2 (0 + 0 + (x^2 + 0^2)(-2x)) \, dx = \int_0^2 -2x^3 \, dx = \left. -\frac{x^4}{4} \right|_0^2 = -8$$

The total work is the sum of work done along $C_1$, $C_2$, and $C_3$. Hence $W = \int_{C_1} + \int_{C_2} + \int_{C_3} \text{f} \cdot \text{dr} = 2\pi + 8 - 8 = 2\pi \approx 6.28$.

**Example 4.** Evaluate the integral $\int_C z^2 \, dx + y \, dy + 2y \, dz$ where $C$ consists of two parts $C_1$ and $C_2$. $C_1$ is the intersection of the cylinder $x^2 + y^2 = 16$ and the plane $z = 3$ from $(0,4,3)$ to $(-4,0,3)$. $C_2$ is a line segment from $(-4,0,3)$ to $(0,1,5)$.

**Solutions.** $C_1$ is on $x^2 + y^2 = 16$ thus $x = 4 \cos t$ and $y = 4 \sin t$. $C_1$ is also on $z = 3$ so $x = 4 \cos t, \; y = 4 \sin t, \; z = 3$

are parametric equations of $C_1$. On $C_1$, $dx = -4 \sin t \, dt$, $dy = 4 \cos t \, dt$ and $dz = 0$. The point $(0,4,3)$ corresponds to $t = \frac{\pi}{2}$ and the point $(-4,0,3)$ to $t = \pi$. Thus, $\int_{C_1} z^2 \, dx + y \, dy + 2y \, dz = \int_{\pi/2}^{\pi} 3^2(-4 \sin t) \, dt + 4 \sin t \, 4 \cos t \, dt + 8 \sin t(0) = (36 \cos t + 8 \sin^2 t)|_{\pi/2}^{\pi} = -36 - 8 = -44$.

The line segment $C_2$ is passing $(-4,0,3)$ in the direction of the vector $\overrightarrow{PQ} = (0,1,5) - (-4,0,3) = (4,1,2)$. So $C_2$ has equations $x = -4 + 4t, \; y = t$ and $z = 3 + 2t$ for $0 \leq t \leq 1$. So, on this segment $dx = 4 \, dt$, $dy = dt$ and $dz = 2 \, dt$. $\int_{C_2} z^2 \, dx + y \, dy + 2y \, dz = \int_0^1 (3 + 2t)^2 \, 4 \, dt + 2 \, dt + 2 \, t \, 2 \, dt = \int_0^1 (36 + 53t + 16t^2) \, dt = 36 + \frac{53}{2} + \frac{16}{3} = \frac{407}{6} = 67.83$.

So, the final answer is $\int_C = 67.83 - 44 = 23.83$.

The following table summarizes the two types of line integrals.

| Line integral of a scalar function $f(x,y,z)$ | $\int_C f \cdot ds = \int_C f(r(t)) \cdot |r'(t)| \, dt$ |
| Line integral of a vector function $f(x,y,z)$ | $\int_C f \cdot dr = \int_C f(r(t)) \cdot r'(t) \, dt$ |
In the next section, we review some basic facts about surfaces and then we present analogous two types of integrals over a surface: the surface integral of a scalar function and the flux integral of a vector function.

**Review of Surfaces**

Adding one more independent variable to a vector function describing a curve \( x = x(t), y = y(t), z = z(t) \), we arrive to equations that describe a surface. Thus, a surface in space is a vector function of two variables:

\[
\mathbf{r}(u, v) = (x(u, v), y(u, v), z(u, v)).
\]

These equations are called *parametric equations* of the surface and the surface given via parametric equations is called a *parametric surface*.

If \( x \) and \( y \) are used as parameters, the equations \( x = x, y = y, z = z(x, y) \) are frequently shortened to just \( z = z(x, y) \) and \( \mathbf{r}(x, y) = (x, y, z(x, y)) \) is also written shortly as \( z = z(x, y) \).

In some cases, a surface can be given by an *implicit function* \( F(x, y, z) = 0 \). In this case it is often needed to find parametric equations \( \mathbf{r}(u, v) = (x(u, v), y(u, v), z(u, v)) \). For example, a unit sphere can be described by \( x^2 + y^2 + z^2 = 1 \) can be parametrized as \( \mathbf{r} = (\cos \theta \sin \phi, \sin \theta \sin \phi, \cos \phi) \).

We recall the cylindrical and spherical coordinates which are frequently used to obtain parametric equations of some common surfaces.

**Cylindrical coordinates.**

\[
\begin{align*}
  x &= r \cos \theta \\
  y &= r \sin \theta \\
  z &= z
\end{align*}
\]

Here \( x \) and \( y \) are converted using polar coordinates and the only change in \( z \) may come just from changes in \( x \) and \( y \). The Jacobian determinant can be computed to be \( J = r \). Thus, \( dxdydz = rdrd\theta dz \).

**Spherical coordinates.** Let \( P = (x, y, z) \) be a point in space and \( O \) denote the origin.

- Let \( r \) denote the distance from the origin \( O \) to the point \( P = (x, y, z) \). Thus, \( x^2 + y^2 + z^2 = r^2 \).
- Let \( \theta \) be the angle between the projection of vector \( \overrightarrow{OP} = (x, y, z) \) on the \( xy \)-plane and the vector \( \mathbf{i} \) (positive \( x \)-axis).
- Let \( \phi \) be the angle between the vector \( \overrightarrow{OP} \) and the vector \( \mathbf{k} \) (positive \( z \)-axis).
The conversion equations are

\[ x = r \cos \theta \sin \phi \quad y = r \sin \theta \sin \phi \quad z = r \cos \phi. \]

The Jacobian determinant can be computed to be \( J = r^2 \sin \phi \). Thus, \( dx \, dy \, dz = r^2 \sin \phi \, dr \, d\phi \, d\theta \).

**Example 5.** The following are examples of parametric surfaces.

1. The cone \( z = \sqrt{x^2 + y^2} \) has representation using cylindrical coordinates as \( x = r \cos \theta, \ y = r \sin \theta, \ z = r \).

2. The paraboloid \( z = x^2 + y^2 \) has representation using cylindrical coordinates as \( x = r \cos \theta, \ y = r \sin \theta, \ z = r^2 \).

3. The sphere \( x^2 + y^2 + z^2 = 9 \) has representation using spherical coordinates as \( x = 3 \cos \theta \sin \phi, \ y = 3 \sin \theta \sin \phi, \ z = 3 \cos \phi \).

4. The cylinder \( x^2 + y^2 = 4 \) has representation using cylindrical coordinates as \( x = 2 \cos \theta, \ y = 2 \sin \theta, \ z = z \). The parameters here are \( \theta \) and \( z \).

5. The cylinder \( y^2 + z^2 = 4 \) has representation using cylindrical coordinates as \( x = x, \ y = 2 \cos \theta, \ z = 2 \sin \theta \). The parameters here are \( \theta \) and \( x \).
The Tangent Plane. For parametric surface \( r = (x(u,v), y(u,v), z(u,v)) \), the derivatives \( r_u \) and \( r_v \) are vectors in the tangent plane. Thus, their cross product

\[
\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} = (x_u, y_u, z_u) \times (x_v, y_v, z_v)
\]

is perpendicular to the tangent plane and, thus, to the surface as well.

If a surface is given by implicit function \( F(x, y, z) = 0 \), then this cross product also corresponds to the gradient \( \nabla F \) of \( F \),

\[
\nabla F = (F_x, F_y, F_z).
\]

Example 6. Find an equation of the plane tangent to the cylinder \( x^2 + z^2 = 4 \) at \((0, 3, 2)\).

Solutions. The cylinder can be parametrized as \( x = 2 \cos t, y = y, z = 2 \sin t \). We find \((x_t, y_t, z_t) = (-2 \sin t, 0, 2 \cos t)\) and \((x_y, y_y, z_y) = (0, 1, 0)\). The cross product is \((-2 \cos t, 0, 2 \sin t)\).

The \( t \)-value that corresponds to \((0, 3, 2)\) can be obtained from \( x = 2 \cos t = 0 \) and \( z = 2 \sin t = 2 \). Thus \( t = \frac{\pi}{2} \) and plugging this value in the equation of the vector we obtained gives us \((0, 0, 2)\). So the tangent plane passes \((0, 3, 2)\) and it is perpendicular to \((0, 0, 2)\). An equation of this plane can be obtained as \(0(x - 0) + 0(y - 3) + 2(z - 2) = 0 \Rightarrow z = 2\). Hence, the tangent plane is the horizontal plane passing \( 2 \) on \( z \)-axis.

Surface Integrals of scalar functions

Similarly as for line integrals, we can integrate a scalar or a vector function over a surface. Thus, we distinguish two types of surface integrals. The surface integrals of scalar functions are two-dimensional analogue of the line integrals of scalar functions.

\[
\begin{align*}
\text{Line integral of a scalar function} & \quad \longleftrightarrow \quad \text{Length} \\
\text{Surface integral of a scalar function} & \quad \longleftrightarrow \quad \text{Area}
\end{align*}
\]

The surface area of the surface \( r(u, v) \) over the region \( S \) in \( uv \)-plane can be obtained by integrating surface area elements \( dS \) over sub-rectangles of region \( S \). The area of each element \( dS \) can be approximated with the area of the parallelogram in the tangent plane. The area of a parallelogram formed by two vectors is the length of their cross product.

Thus,

\[
dS = |r_u \times r_v|\,dudv
\]

and so

\[
\text{Surface area} = \iint_S dS = \iint_S |r_u \times r_v|\,dudv
\]

This integral can be considered as a special case of the situation when we integrate the scalar function \( f = 1 \) over the surface \( r(u, v) \).
Let \( f(x, y, z) \) be a scalar (real-valued) function. Integrating \( f \) over the surface \( \mathbf{r}(u, v) \) we obtain the surface integral

\[
\int \int_S f(x, y, z) \, dS = \int \int_S f(\mathbf{r}(u, v)) \left| \mathbf{r}_u \times \mathbf{r}_v \right| \, du \, dv
\]

This integral computes the total effect of function \( f \) over the surface \( S \).

To evaluate a surface integral, you can follow the steps below.

1. **Parametrize the surface:** determine the parameters \( u \) and \( v \) and the equations \( \mathbf{r}(u, v) = (x(u, v), y(u, v), z(u, v)) \). This step is very important because without it you cannot proceed to compute \( dS \).

   For example, if \( S \) is the plane \( 4x + 2y + z = 8 \) or some part of it, you can use \( x \) and \( y \) for your parameters and parametrize the plane as \( \mathbf{r} = (x, y, 8 - 4x - 2y) \). If \( S \) is the cone \( z = \sqrt{x^2 + y^2} \) or some part of it, you can take \( r \) and \( \theta \) for the parameters and parametrize the cone as \( \mathbf{r} = (r \cos \theta, r \sin \theta, r) \). If \( S \) is the cylinder \( x^2 + y^2 = 1 \), you can take \( t \) and \( h \) (think of \( h \) as height) and parametrize the cylinder as \( \mathbf{r} = (\cos t, \sin t, h) \). If \( S \) is a sphere you can parametrize it using spherical coordinates.

2. **Calculate** \( dS \) using the formula \( dS = \left| \mathbf{r}_u \times \mathbf{r}_v \right| \, du \, dv \).

3. **Determine the bounds of integration.** Use the description of the surface and any constraints given in the problem to determine the bounds for \( u \) and \( v \).

   For example, if \( S \) is the part of the plane \( 4x + 2y + z = 8 \) in the first octant and the plane is parametrized by \( x \) and \( y \), then \( 0 \leq x \leq 2 \) and \( 0 \leq y \leq 4 - 2x \). If \( S \) is part of the cone \( z = \sqrt{x^2 + y^2} \) below \( z = 5 \), and the cone is parametrized as \( \mathbf{r} = (r \cos \theta, r \sin \theta, r) \), then \( 0 \leq \theta \) and \( 0 \leq r \leq 5 \). If \( S \) is the part of the cylinder \( x^2 + y^2 = 1 \) between the planes \( z = 0 \) and \( z = 3 \), and the cylinder is parametrized as \( \mathbf{r} = (\cos t, \sin t, h) \), then \( 0 \leq t \leq 2\pi \) and \( 0 \leq h \leq 3 \).

4. **Determine the integrand** in terms of \( u \) and \( v \). Substitute the parametrization \( x = x(u, v), y = y(u, v), z = z(u, v) \) in the integrand \( f(x, y, z) \).

5. **Evaluate the double integral** of the integrand from step 4 in the bounds from step 3.

   The applications of this type of line integrals include finding the mass \( m \) of a thin sheet \( S \) with the density function \( \rho(x, y, z) \). The mass \( m \) is given by

   \[
m = \int \int_S \rho(x, y, z) \, dS
\]

   Using this example, you can think of the surface integral as the total mass of the surface density function over the surface \( S \).

In the special case when the surface is parametrized by the parameters \( x \) and \( y \) as \( z = z(x, y) \) (thus \( \mathbf{r} = (x, y, z(x, y)) \)), the surface area element can be computed to be \( dS = \sqrt{1 + z_x^2 + z_y^2} \, dx \, dy \) so the surface integral of \( f \) over region \( S \) is

\[
\int \int_S f(x, y, z) \, dS = \int \int_S f(x, y, z(x, y)) \sqrt{1 + z_x^2 + z_y^2} \, dx \, dy
\]
Practice Problems.

1. Find the area of the following surfaces by using their parametric equations.

   (a) Part of of $z = y^2 + x^2$ between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.
   (b) Part of the cone $z = \sqrt{x^2 + y^2}$ between the cylinders $x^2 + y^2 = 4$ and $x^2 + y^2 = 9$.

2. Evaluate the surface integral where $S$ is the given surface.

   (a) $\int \int_S xz \, dS$, $S$ is the part of the plane $4x + 2y + z = 8$ that lies in the first octant.
   (b) $\int \int_S yz \, dS$, $S$ is the part of the plane $z = y + 3$ that lies inside the cylinder $x^2 + y^2 = 1$.
   (c) $\int \int_S z \, dS$, $S$ is the hemisphere $x^2 + y^2 + z^2 = 4$, $z \geq 0$.

3. Find the mass of the hemisphere $x^2 + y^2 + z^2 = 4$, $z \geq 0$ if it has constant density $\rho = a$.

Solutions. In all the problems, follow the five steps from page 8.

1. (a) The paraboloid can be parametrized by $x = r \cos t$, $y = r \sin t$, $z = x^2 + y^2 = r^2$. Thus $\mathbf{r}(r \cos t, r \sin t, r^2) \Rightarrow \mathbf{r}_r = (x, y, z) = (\cos t, \sin t, 2r)$. $\mathbf{r}_t = (x, y, z) = (-r \sin t, r \cos t, 0) \Rightarrow \mathbf{r}_r \times \mathbf{r}_t = (-2r \cos^2 t, -2r \sin^2 t, r) \Rightarrow$ The length $|\mathbf{r}_r \times \mathbf{r}_t| = \sqrt{4r^4 \cos^2 t + 4r^4 \sin^2 t + r^2} = \sqrt{4r^4 + r^2} = \sqrt{r^2(4r^2 + 1)} = r \sqrt{4r^2 + 1}$.

   The bounds for the integration are determined by the projection in the $xy$-plane which is the region between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$. Thus $0 \leq t \leq 2\pi$ and $1 \leq r \leq 2$. So, the surface area is $S = \int_0^{2\pi} \int_1^2 r \sqrt{4r^2 + 1} \, dr = 2\pi 4.91 = 30.85$.

   (b) The cone can be parametrized by $x = r \cos t$, $y = r \sin t$, $z = \sqrt{x^2 + y^2} = r$. $\mathbf{r}_r = (\cos t, \sin t, 1)$ and $\mathbf{r}_t = (-r \sin t, r \cos t, 0)$. $\mathbf{r}_r \times \mathbf{r}_t = (-2r \cos^2 t, -2r \sin^2 t, r)$.

   The length of this product is $\sqrt{r^2 \cos^2 t + r^2 \sin^2 t + r^2} = \sqrt{r^2 + r^2} = \sqrt{2r^2} = \sqrt{2}r$.

   The bounds for the integration are determined by the projection in the $xy$-plane which is the region between the circles $x^2 + y^2 = 4$ and $x^2 + y^2 = 9$. Thus $0 \leq t \leq 2\pi$ and $2 \leq r \leq 3$. So, the surface area is $S = \int_0^{2\pi} \int_2^3 \sqrt{2}r \, dr = 2\pi \sqrt{2} (\frac{9}{2} - \frac{4}{3}) = 5\pi \sqrt{2}$.

2. (a) Solve the equation of the plane for $z$ and parametrize as $\mathbf{r} = (x, y, 8 - 4x - 2y) \Rightarrow \mathbf{r}_x = (1, 0, -4), \mathbf{r}_y = (0, 1, -2) \Rightarrow \mathbf{r}_x \times \mathbf{r}_y = (4, 2, 1) \Rightarrow dS = \sqrt{16 + 4 + 1} \, dxdy = \sqrt{21} \, dxdy$. The bounds of integration are determined by the triangle in $xy$-plane $z = 0$ which is bounded by the coordinate axes and the line $0 = 8 - 4x - 2y \Rightarrow y = 4 - 2x$. So, the bounds are $0 \leq x \leq 2, 0 \leq y \leq 4 - 2x$. $\int \int_S xz \, dS = \int_0^2 \int_0^{4 - 2x} x(8 - 4x - 2y) \sqrt{21} \, dxdy = \frac{2\pi}{3} \int_0^2 (8xy - 4x^2y - xy^2)_{0}^{4 - 2x} \, dx = \frac{16\sqrt{21}}{3}$(simplify).

   (b) You can parametrize the plane as $\mathbf{r} = (x, y, y + 3)$ and compute $dS$ to be $|(0, -1, 1)| \, dxdy = \sqrt{1 + 1} \, dxdy = \sqrt{2} \, dxdy$. The integral becomes $\int \int y(y + 3) \sqrt{2} \, dxdy$ and it is taken over the disc in $xy$ plane of radius 1. Use the polar coordinates to get nice bounds $0 \leq t \leq 2\pi$ and $0 \leq r \leq 1$. The Jacobian is $r$ and the integral becomes $\int_0^{2\pi} \int_0^1 r \sin t(r \sin t + 3) \sqrt{2} \, drdt = \sqrt{2} \int_0^{2\pi} \sin t(\frac{1}{4} \sin t + 1) = \frac{\sqrt{2}}{4}$.

   Alternatively, you can use the parametrization $x = r \cos t$, $y = r \sin t$, $z = r \sin t + 3$ with $0 \leq t \leq 2\pi$ and $0 \leq r \leq 1$. Then $|\mathbf{r}_x \times \mathbf{r}_t| = r \sqrt{2}$ and the formula $\int_0^{2\pi} \int_0^1 r \sin t(r \sin t + 3) \sqrt{2} \, drdt$ gives you the same answer.
(c) Use spherical coordinates to parametrize the sphere as \( r = (2 \cos \theta \sin \phi, 2 \sin \theta \sin \phi, 2 \cos \phi) \). Calculate the derivatives \( r_\theta \) and \( r_\phi \) and find their cross product and then its length. Obtain that \( |r_\theta \times r_\phi| = 4 \sin \phi \) so that \( dS = 4 \sin \phi \theta d\theta d\phi \). The bounds are \( 0 \leq \theta \leq 2\pi \) and \( 0 \leq \phi \leq \frac{\pi}{2} \) (we need to stay above the \( xy \)-plane) so the integral is \( \int \int_S zdS = \int_0^{2\pi} \int_0^{\pi/2} 2 \cos \phi \ 4 \sin \phi \ d\theta d\phi = 2\pi \int_0^{\pi/2} 2 \cos \phi \ 4 \sin \phi \ d\phi = 2\pi \cdot \frac{1}{2} = 8\pi \).

3. Use the parametrization of the previous problem so \( dS = 4 \sin \phi \theta d\theta d\phi \) again. The mass can be computed as \( m = \int \int_S adS = a \int_0^{2\pi} \int_0^{\pi/2} 4 \sin \phi \ d\theta d\phi = 8a\pi \).

**Surface Integrals of Vector Fields. Flux**

If \( r(u, v) \) is a surface, vector \( r_u \times r_v \) is perpendicular to the surface (i.e. the tangent plane). Considering the normalization of this vector, \( \frac{r_u \times r_v}{|r_u \times r_v|} \), we arrive to the concept of the unit normal vector \( n \).

\[
\mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}
\]

If \( \mathbf{n} \) is a unit normal vector, then \(-\mathbf{n}\) is also a unit-length vector perpendicular to the surface, so both vectors \( \mathbf{n} \) and \(-\mathbf{n}\) can be used as unit normal vectors. Thus, we would need to be able to make a consistent choice of surface normal vector at every point. If that is possible, a surface is said to be **orientable** or **two-sided**. In this case, vector \( \mathbf{n} \) corresponds to unit normal vector of one side and \(-\mathbf{n}\) to the unit normal vector of the other side.

**Orientable and non-orientable surfaces.** Examples of orientable surfaces include planes, cylinders, and spheres.

A *Möbius strip* (or Möbius band) is an example of a surface that is not orientable. A model can be created by taking a paper strip and giving it a half-twist (180°-twists), and then joining the ends of the strip together to form a loop.

The Möbius strip has several curious properties: it is a surface with **only one side and only one boundary**. Convince yourself of these facts by creating your own Möbius strip or studying many animations on the web.

Another interesting property is that if you cut a Möbius strip along the center line, you will get one long strip with two full twists in it, not two separate strips. The resulting strip will have two sides and two boundaries. So, cutting created a second boundary. Continuing this construction you can deduce that a strip with an odd-number of half-twists will have only one surface and one boundary while a strip with an even-number of half-twists will have two surfaces and two boundaries.

There are many applications of Möbius strip in science, technology and everyday life. For example, Möbius strips have been used as conveyor belts (that last longer because the entire surface area of the belt gets the same amount of wear), fabric computer printer and typewriter ribbons. Medals often have a neck ribbon configured as a Möbius strip that allows the ribbon to fit comfortably around
the neck while the medal lies flat on the chest. Examples of Möbius strip can be encountered: in physics as compact resonators and as superconductors with high transition temperature; in chemistry as molecular knots with special characteristics (e.g. chirality); in music theory as dyads and other areas.

For more curious properties and alternative construction of Möbius strip, see Wikipedia.

If a surface is two sided, \( n \) corresponds to one side and \(-n\) to the other. For a closed surface (i.e. compact without boundary), the convention is that the positive orientation is the one that corresponds to the normal vectors pointing outward and the negative orientation corresponds to the normal vectors pointing inward.

If the surface is not closed, the positive orientation can be defined by the right hand rule. Consider any closed, simple (i.e. does not cross itself nor it has missing points), smooth curve \( C \) on the surface and consider the positive (counter-clockwise) orientation on \( C \). The surface has the positive orientation if the normal vector \( n \) is always on the left of any vector parallel with it which is transversing the curve. That is: if you imagine yourself walking along \( C \) with your head pointing in the direction of \( n \), then the region \( S \) will always be on your left. Alternatively: if your index and middle fingers follow the direction of the curve, your thumb is pointing in the same direction as the vector \( n \).

If a surface is given by implicit equation \( F(x, y, z) = 0 \), the unit normal vector \( n \) can also be found as \( n = \frac{\nabla F}{|\nabla F|} \).

**Example 7.** Find the unit normal vector of the sphere \( x^2 + y^2 + z^2 = a^2 \).

**Solutions.** Consider \( F = x^2 + y^2 + z^2 - a^2 \) so that the gradient vector is \( \nabla F = (2x, 2y, 2z) \) and \( |\nabla F| = \sqrt{4x^2 + 4y^2 + 4z^2} = 2\sqrt{x^2 + y^2 + z^2} = 2a \). Thus, \( n = \frac{(2x, 2y, 2z)}{2a} = \frac{1}{a}(x, y, z) \).

**Flux integral.** If \( r(u, v) \) is an orientable surface with a tangent plane at every point, the vector differential of the surface area element \( dS \) can be considered to be the product of \( n \) and \( dS \) up to the sign. Thus,

\[
  dS = n \ dS = \pm \frac{r_u \times r_v}{|r_u \times r_v|} \ |r_u \times r_v| \ dudv = \pm (r_u \times r_v) \ dudv.
\]

So, if \( f \) is a vector field \( f(x, y, z) = (P(x, y, z), Q(x, y, z), R(x, y, z)) \), the surface integral of \( f \) over region \( S \) on \( r(u, v) \) is given by

\[
\int \int_S f \cdot dS = \int \int_S f \cdot n \ dS = \pm \int \int_S f \cdot (r_u \times r_v) \ dudv
\]

The surface integral of a vector field is also called flux integral. The name comes from the fact that it computes the flux of fluid of density \( \rho \) and velocity field \( v \) flowing through surface region \( S \) when taking \( f \) to be the product \( \rho v \).

In fact, you can think of any flux integral of a vector function \( f \) as the measure of the total flow of \( f \) through the surface \( S \).
To evaluate a flux integral, you can follow the steps below.

1. **Parametrize the surface:** determine the parameters $u$ and $v$ and the equations $\mathbf{r}(u,v) = (x(u,v), y(u,v), z(u,v))$. Note that this is the same step as the first step for evaluating a surface integral of a scalar function.

2. **Calculate $dS$** using the formula $dS = \pm (\mathbf{r}_u \times \mathbf{r}_v) \, du \, dv$. See the practice problems to see how to determine the sign in various examples.

3. **Determine the bounds of integration.** Use the description of the surface and any constraints given in the problem to determine the bounds for $u$ and $v$. Note that this is also the same step for evaluating a surface integral of a scalar function.

4. **Determine the integrand** in terms of $u$ and $v$. Substitute the parametrization $x = x(u,v), y = y(u,v), z = z(u,v)$ in the integrand $f(x,y,z)$ and calculate the dot product $\mathbf{f} \cdot d\mathbf{S}$.

5. **Evaluate the double integral** of the integrand from step 4 in the bounds from step 3.

Besides the applications in fluid dynamics which we mentioned, the flux integral arises in other areas of physics. For example, if $\mathbf{E}$ is an electric field, the surface integral of $\mathbf{E}$ over the surface region $S$ determines the electric flux of $\mathbf{E}$ through $S$. This integral is used to formulate the Gauss’ Law stating that the net charge enclosed by a closed surface region $S$ is equal to the product of a constant $\varepsilon_0$ (the permittivity of free space) and the surface integral of $\mathbf{E}$ over $S$.

Another example of the use of this integral can be encountered in the study of heat flow. If $K$ is a constant (called conductivity) and $T$ is the temperature at point $(x, y, z)$, the heat flow is defined as $\mathbf{F} = -K \nabla T$ and the rate of heat flow across the surface $S$ is given by the surface integral of $\mathbf{F}$ over $S$.

The following table summarizes the relation of the two types of surface integrals.

| Surface integral of a scalar function $f(x,y,z)$ | $\int \int_S f(\mathbf{r}) \, dS = \int \int_S f(\mathbf{r}(u,v)) |\mathbf{r}_u \times \mathbf{r}_v| \, du \, dv$ |
| Surface integral of a vector function $\mathbf{f}(x,y,z)$ | $\int \int_S \mathbf{f}(\mathbf{r}) \cdot d\mathbf{S} = \pm \int \int_S \mathbf{f}(\mathbf{r}(u,v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v) \, du \, dv$ |

**Practice Problems.**

1. Find the flux integral of the vector field $\mathbf{f} = (y, x, z)$ over the part of the paraboloid $z = 1 - x^2 - y^2$ above the plane $z = 0$.

2. Find the flux integral of the vector field $\mathbf{f} = (y, x, z)$ over the boundary of the region enclosed by the paraboloid $z = 1 - x^2 - y^2$ and the plane $z = 0$.

3. Find the flux integral of the vector field $\mathbf{f} = (xze^y, -xze^y, z)$ over the part of the plane $x+y+z = 1$ in the first octant with the upward orientation.

4. Find the flux integral of the vector field $\mathbf{f} = (x, 2y, 3z)$ over the cube with vertices $(\pm1, \pm1, \pm1)$. 

12
Solutions. (1) In polar coordinates, the paraboloid \( z = 1 - x^2 - y^2 \) is \( z = 1 - r^2 \) so you can parametrize the paraboloid as \( \mathbf{r} = (r \cos \theta, r \sin \theta, 1 - r^2) \). Thus \( \mathbf{r}_\theta = (\cos \theta, \sin \theta, -2r) \) and \( \mathbf{r}_\theta = (-r \sin \theta, r \cos \theta, 0) \Rightarrow d\mathbf{S} = \pm(\mathbf{r}_\theta \times \mathbf{r}_\theta) d\theta = \pm(2r^2 \cos \theta, 2r^2 \sin \theta, r) d\theta \). Note that you want the normal vector to point outwards which is the case by choosing the positive sign (looking at the \( z \)-coordinate, which should be positive, can tell you that, for example). The paraboloid intersects the plane \( z = 0 \) in the circle \( 0 = 1 - x^2 - y^2 \Rightarrow x^2 + y^2 = 1 \) so the bounds are \( 0 \leq \theta \leq 2\pi \) and \( 0 \leq r \leq 1 \). Compute the dot product \( \mathbf{f} \cdot d\mathbf{S} \) to be \((r \sin \theta, r \cos \theta, 1 - r^2)((2r^2 \cos \theta, 2r^2 \sin \theta, r) d\theta = (4r^3 \sin \theta \cos \theta + r - r^3) dr d\theta \). Thus, the integral is

\[
\int_S \int f \cdot dS = \int_0^{2\pi} \int_0^1 (4r^3 \sin \theta \cos \theta + r - r^3) dr d\theta = 6 \pi (\frac{1}{2} - \frac{1}{4}) = \frac{\pi}{2}.
\]

Alternatively, you can use the parametrization \( \mathbf{r} = (x, y, 1 - x^2 - y^2) \) in which case \( d\mathbf{S} = (2x, 2y, 1) dxdy \) and the integral becomes \( \int \int_S (y, x, 1 - x^2 - y^2) \cdot (2x, 2y, 1) dxdy = \int \int_S (2xy + 2xy + 1 - x^2 - y^2) dxdy \). Then use the polar coordinates (don’t forget the Jacobian \( r \) so that \( dxdy = rdrd\theta \) and obtain \( \int_0^{2\pi} \int_0^1 (4r^2 \sin \theta \cos \theta + 1 - r^2) dr d\theta \) which is the same as above and is equal to \( \frac{\pi}{2} \).

(2) The flux integral is the sum of the integral of \( f \) over the paraboloid and the integral of \( f \) over the disc of radius 1 in the plane \( z = 0 \). The first integral is \( \frac{\pi}{2} \) by the previous problem. Let \( S_2 \) denote the disc in \( xy \)-plane \( z = 0 \) of radius 1. The plane can be parametrized by \( \mathbf{r} = (x, y, 0) \) so that \( d\mathbf{S} = \pm(0, 0, 1) dxdy \). Here, we want the normal vector to point upwards which, in this case, is downwards, so we need to chose the negative sign. Hence \( d\mathbf{S} = (0, 0, -1) dxdy \). The flux integral over \( S_2 \) is \( \int \int_{S_2} f \cdot dS = \int \int_{S_2} 0 dxdy = 0 \). So, the final answer remains \( \frac{\pi}{2} + 0 = \frac{\pi}{2} \).

(3) Parametrize the plane as \( \mathbf{r} = (x, y, 1 - x - y) \) and compute \( d\mathbf{S} \) to be \( \pm(1, 1, 1) dxdy \). The normal vector should point upwards so choose the positive sign. The integrand is \( (xze^y, -xze^y, 1 - x - y) \cdot (1, 1, 1) dxdy = (xze^y - xze^y + 1 - x - y) dxdy = (1 - x - y) dxdy \). The bounds of the integration are determined by the projection of \( S \) in \( xy \) plane which is a triangle determined by the line \( 1 - x - y = 0 \Rightarrow y = 1 - x \) (the intersection of \( z = 0 \) and the given plane) and the coordinate axes. Hence \( 0 \leq x \leq 1 \) and \( 0 \leq y \leq 1 - x \). So, the flux integral is \( \int \int_S f \cdot dS = \int_0^1 \int_0^{1-x} (1 - x - y) dxdy = \int_0^1 \int_0^{1-x} (1 - x - (1 - x) - \frac{1}{2}(1 - x)^2) = \frac{\pi}{2} \).

(4) The cube consists of 6 sides so we need to evaluate 6 surface integrals (but by the time you are done with all six, you will become a flux integral expert).

Let \( S_1 \) be the top, \( S_2 \) the bottom, \( S_3 \) the right, \( S_4 \) the left side, \( S_5 \) the front and \( S_6 \) the back.

\( S_1 \) The square \( S_1 \) is in the plane \( z = 1 \) which can be parametrized by \( \mathbf{r} = (x, y, 1) \). Compute \( d\mathbf{S} \) to be \( \pm(0, 0, 1) dxdy \) and since the normal vector should point upwards, take the positive sign. The bounds of the integration are \( -1 \leq x, y \leq 1 \) so the flux integral is \( \int \int_{S_1} (x, 2y, 3(1)) \cdot (0, 0, 1) dxdy = \int_{-1}^1 \int_{-1}^1 3dxdy = 3x \big|_{-1}^1 y \big|_{-1}^1 = 3(2)(2) = 12 \).

\( S_2 \) The square \( S_2 \) is in the plane \( z = -1 \) which can be parametrized by \( \mathbf{r} = (x, y, -1) \). Compute \( d\mathbf{S} \) to be \( \pm(0, 0, 1) dxdy \) and since the normal vector should point downwards, take the negative sign. The bounds of the integration are \( -1 \leq x, y \leq 1 \) and the flux integral is \( \int \int_{S_2} (x, 2y, 3(-1)) \cdot (0, 0, -1) dxdy = \int_{-1}^1 \int_{-1}^1 3dxdy = 3x \big|_{-1}^1 y \big|_{-1}^1 = 3(2)(2) = 12 \).

\( S_3 \) The square \( S_3 \) is in the plane \( y = 1 \) which can be parametrized by \( \mathbf{r} = (x, 1, z) \). Compute \( d\mathbf{S} \) to be \( \pm(0, 1, 0) dxdz \) and since the normal vector should point to the right, take the positive
sign. The bounds of the integration are \(-1 \leq x, z \leq 1\) so the flux integral is \(\int \int_{S_3} (x, 2(1), 3z) \cdot (01, 0) \, dx \, dz = \int_{-1}^{1} \int_{-1}^{1} 2 \, dx \, dz = 2x \big|_{-1}^{1} = 2(2)(2) = 8\).

**S_4** The square \(S_4\) is in the plane \(y = -1\) which can be parametrized by \(r = (x, -1, z)\). Compute \(dS\) to be \(\pm(0, 1, 0) \, dx \, dz\) and since the normal vector should point to the left, take the negative sign. The bounds of the integration are \(-1 \leq x, z \leq 1\) so the flux integral is \(\int \int_{S_4} (x, 2(-1), 3z) \cdot (0 - 1, 0) \, dx \, dz = \int_{-1}^{1} \int_{-1}^{1} 2 \, dx \, dz = 2x \big|_{-1}^{1} = 2(2)(2) = 8\).

**S_5** The square \(S_5\) is in the plane \(x = 1\) which can be parametrized by \(r = (1, y, z)\). Compute \(dS\) to be \(\pm(1, 0, 0) \, dy \, dz\) and since the normal vector should point to the front, take the positive sign. The bounds of the integration are \(-1 \leq y, z \leq 1\) so the flux integral is \(\int \int_{S_5} (1, 2y, 3z) \cdot (1, 0, 0) \, dy \, dz = \int_{-1}^{1} \int_{-1}^{1} 3 \, dy \, dz = 3y \big|_{-1}^{1} = (2)(2) = 4\).

**S_6** The square \(S_6\) is in the plane \(x = -1\) which can be parametrized by \(r = (-1, y, z)\). Compute \(dS\) to be \(\pm(1, 0, 0) \, dy \, dz\) and since the normal vector should point to the back, take the negative sign. The bounds of the integration are \(-1 \leq y, z \leq 1\) so the flux integral is \(\int \int_{S_6} (-1, 2y, 3z) \cdot (-1, 0, 0) \, dy \, dz = \int_{-1}^{1} \int_{-1}^{1} -3 \, dy \, dz = -3y \big|_{-1}^{1} = (2)(2) = 4\).

Thus, the total flux is \(12+12+8+8+4+4=48\).

**Stokes’ Theorem**

Stokes’ Theorem is a three-dimensional version of Green’s Theorem. Recall that Green’s theorem relates the line integral of a two-dimensional vector function \(\mathbf{f} = (P, Q)\) over a positive oriented, closed curve \(C\) and the double integral over the interior \(S\) of \(C\).

\[
\oint_C P \, dx + Q \, dy = \int \int_S \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dx \, dy.
\]

In three-dimensional analogue, we relate the line integral of a three-dimensional vector function \(\mathbf{f} = (P, Q, R)\) over a closed curve \(C\) and the surface integral of \(\text{curl} \, \mathbf{f}\) over the interior of \(C\). Recall that the curl of \(\mathbf{f}\) is defined as the vector product of \(\nabla\) and \(\vec{f}\).

\[
\text{curl} \, \vec{f} = \nabla \times \vec{f} = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
P & Q & R
\end{vmatrix} = \left( \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \hat{i} + \left( \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \hat{j} + \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \hat{k}
\]

Let \(S\) be a region on an oriented piecewise-smooth surface \(\mathbf{r}(u, v)\) that is bounded by a simple, closed, piecewise smooth curve \(C\). Recall that the orientation of \(\mathbf{r}\) induces the positive orientation of \(C\) if the normal vector \(\mathbf{n}\) of \(\mathbf{r}\) will always be on the left of any vector parallel with it that is transversing the curve (i.e. if you imagine yourself walking along \(C\) with your head pointing in the direction of \(\mathbf{n}\), then the region \(S\) will always
be on your left). In this case, if \( \mathbf{f} \) is a vector field, Stokes’ Theorem states that

\[
\oint_C \mathbf{f} \cdot d\mathbf{r} = \iint_S \text{curl} \mathbf{f} \cdot dS = \iint_S \text{curl} \mathbf{f} \cdot \mathbf{n} \, dS
\]

This relates to Green’s Theorem since if the curve \( C \) is in \( xy \)-plane, \( \mathbf{n} = (0, 0, 1) \), \( \text{curl} \mathbf{f} \cdot \mathbf{n} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \), and \( dS = \sqrt{0 + 0 + 1} \, dx \, dy = dx \, dy \) thus giving you the formula

\[
\oint_C P \, dx + Q \, dy = \iint_S \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dx \, dy.
\]

Stokes’ Theorem may be especially useful in the following cases:

1. The curl of \( \mathbf{f} \) is a simple function. In this case, evaluating the surface integral may be much easier than evaluating the line integral.

2. The curve \( C \) consists of several pieces with different parametrization. In this case, it may be much faster evaluating the surface integral than several line integrals (one for each piece of \( C \)).

3. Assume that \( S \) and \( \bar{S} \) are regions on surfaces \( \mathbf{r} \) and \( \bar{\mathbf{r}} \) that have the same boundary \( C \). Assume also that it is difficult to integrate over \( S \). The Stokes’ theorem claims that we can use \( \bar{S} \) instead since

\[
\iint_S \text{curl} \mathbf{f} \cdot dS = \oint_C \mathbf{f} \cdot d\mathbf{r} = \iint_{\bar{S}} \text{curl} \mathbf{f} \cdot d\bar{S}
\]

Practice Problems.

1. Evaluate \( \oint_C \mathbf{f} \cdot d\mathbf{r} \) for \( \mathbf{f} = (x + y^2, y + z^2, z + x^2) \) and the curve \( C \) is the intersection of the plane \( x + y + z = 1 \) and the coordinate planes. (a) Without using Stokes’ Theorem; (b) Using Stokes’ Theorem.

2. Evaluate \( \oint_C \mathbf{f} \cdot d\mathbf{r} \) for \( \mathbf{f} = (-y^2, x, z^2) \) and the curve \( C \) is the intersection of the plane \( y + z = 2 \) and the cylinder \( x^2 + y^2 = 1 \) oriented upwards. (a) Without using Stokes’ Theorem; (b) Using Stokes’ Theorem.

3. Find the work done by the force field \( \mathbf{f} = (-y, x, x^2 + y^2) \) when a particle moves under its influence along the positively oriented boundary of the part of the paraboloid \( z = 4 - x^2 - y^2 \) in the first octant. Note that the work has been found to be \( 2\pi \) in Example 3 by evaluating three line integrals. Use Stokes’ Theorem and just one flux integral to obtain the same answer.

4. Show that the total work done by the force field \( \mathbf{f} = (yz, xz, xy) \) moving the particle along the intersection of the cylinder \( x^2 + y^2 = 1 \) and the sphere \( x^2 + y^2 + z^2 = 4 \) above the \( xy \)-plane is \( \oint_C \mathbf{f} \cdot d\mathbf{r} = 0 \). When using Stokes’ Theorem, this problem becomes much shorter than without using it.

5. Find the work done by the force field \( \mathbf{f} = (x + z^2, y + x^2, z + y^2) \) when a particle moves under its influence around the positively oriented boundary of the part of the sphere \( x^2 + y^2 + z^2 = 4 \) that lies in the first octant.
Solutions. (1) (a) Solution without using Stokes’ Theorem. The curve $C$ consists of three parts $C_1$, $C_2$ and $C_3$ which are in the intersection of the plane and (1) the plane $z = 6$, (2) $xz$-plane, and (3) $yz$-plane, respectively. Positive orientation of $C$ implies that $C_1$ is traversed from $(1,0,0)$ to $(0,1,0)$, $C_2$ from $(0,1,0)$ to $(0,0,1)$ and $C_3$ from $(0,0,1)$ to $(1,0,0)$.

On $C_1: x = x, y = 1 - x$ and $z = 0 \Rightarrow dx = dx, dy = -dx$ and $dz = 0$ and the bounds are from 1 to 0. So, $\int_{C_1} \mathbf{f} \cdot d\mathbf{r} = \int_{C_1} (x+y^2)dx + (y+z^2)dy + (z+x^2)dz = \int_0^1 (x + (1-x)^2)dx + (1-x)(-1)dx = \int_1^0 (x + 1 - 2x + x^2 - 1 + x)dx = \frac{-1}{3}$.

On $C_2: x = 0, y = y, z = 1 - y \Rightarrow dx = 0, dy = dy$ and $dz = -dy$. The bounds are from 1 to 0. So, $\int_{C_2} \mathbf{f} \cdot d\mathbf{r} = \int_{C_2} (x+y^2)dx + (y+z^2)dy + (z+x^2)dz = \int_1^0 (y + (1-y)^2)dy + (1-y)(-1)dy = \int_1^0 (y + 1 - 2y + y^2 - 1 + y) = \int_1^0 y^2dy = \frac{-1}{3}$.

On $C_3: x = x, y = 0, z = 1 - x \Rightarrow dx = dx, dy = 0$ and $dz = -dx$. The bounds are from 0 to 1. So, $\int_{C_3} \mathbf{f} \cdot d\mathbf{r} = \int_{C_3} (x+y^2)dx + (y+z^2)dy + (z+x^2)dz = \int_0^1 xdx + (1-x+x^2)(-1)dx = \int_0^1 (2x - 1 - x^2)dx = 1 - \frac{1}{3} = -\frac{2}{3}$.

Thus $\int_{C} = \int_{C_1} + \int_{C_2} + \int_{C_3} = \frac{-1}{3} - \frac{1}{3} - \frac{1}{3} = -1$.

(b) Using Stokes’ Theorem, $\int_{C} = \int_{S} \text{curl} \mathbf{f} \cdot dS$, where $S$ is the part of the plane in the first octant. Calculate that $\text{curl} \mathbf{f} = (-2z, -2x, -2y)$. On the plane $z = 1 - x - y$, so $\mathbf{r} = (x, y, 1 - x - y)$ $dS = (1, 0, -1) \times (0, 1, -1) = (1, 1, 1)$. The normal vector should point upwards which is currently the case so you do not have to change the sign. The dot product $\text{curl} \mathbf{f} \cdot dS$ is $-2(1 - x - y), -2x, -2y) \cdot (1, 1, 1) = -2(-(1 - x - y) - 2x - 2y) dxdy = (-2 + 2x + 2y - 2x - 2y) dxdy = -2dxdy$ and so

$$\int_S \text{curl} \mathbf{f} \cdot dS = \int_0^1 \int_0^1 -2 dxdy = \int_0^1 -2(1 - x)dx = -2x + x^2 |_0^1 = -2 + 1 = -1.$$

(2) (a) $C$ has parametrization $x = \cos t, y = \sin t, z = 2 - y = 2 - \sin t, 0 \leq t \leq 2\pi$.

$$\int_C \mathbf{f} \cdot d\mathbf{r} = \int_C -y^2 dx + xdy + z^2dz = \int_0^{2\pi} \sin^3 tdt + \cos^2 tdt - (2 - \sin t)^2 \cos tdt.$$

You can use the calculator, Matlab or trigonometric identities and integrate by hand to obtain the answer $\pi$.

(b) Using Stokes’ Theorem, $\int_C = \int_S \text{curl} \mathbf{f} \cdot dS$, where $S$ is the part of the plane inside of the cylinder. Calculate $\text{curl} \mathbf{f} = (0, 0, 1 + 2y)$. The plane can be parametrized as $\mathbf{r} = (x, y, 2 - y)$. Calculate that $dS = (0, 1, 1) dxdy$ and the orientation of the normal vector is correct already (the vector points upwards). Thus $\int_S \text{curl} \mathbf{f} \cdot dS = \int_S (1 + 2y) dxdy$. To evaluate this integral, you can use polar coordinates so that $y = r \sin t$ and $dxdy = rdrdt$ and the bounds are $0 \leq t \leq 2\pi$ and $0 \leq r \leq 1$. Thus the integral becomes

$$\int_0^{2\pi} \int_0^1 (1 + 2r \sin t)rdrdt = \int_0^{2\pi} \left(\frac{1}{2} + \frac{2}{3} \sin t\right)dt = \pi.$$  

(3) Using Stokes’ Theorem, $\int_C = \int_S \text{curl} \mathbf{f} \cdot dS$, where $S$ is the part of the paraboloid in the first octant. Calculate the curl of $\mathbf{f}$ to be $\text{curl} \mathbf{f} = (2y, -2x, 2)$.  

The paraboloid can be parametrized by $\mathbf{r} = (r \cos \theta, r \sin \theta, 4 - r^2) \Rightarrow \mathbf{r}_r = (\cos \theta, \sin \theta, -2r)$ and $\mathbf{r}_\theta = (-r \sin \theta, r \cos \theta, 0) \Rightarrow dS = \pm (\mathbf{r}_r \times \mathbf{r}_\theta) drd\theta = \pm (2r^2 \cos \theta, 2r^2 \sin \theta, r) drd\theta$. Note that you want the normal vector to point outwards which is the case by choosing the positive sign (looking at the $z$-coordinate, which should be positive, can tell you that, for example). The bounds are $0 \leq \theta \leq \frac{\pi}{2}$ and
Compute the dot product $\nabla \times \vec{f} \cdot d\vec{S}$ to be $(2r \sin \theta, -2r \cos \theta, 2)(2r^2 \cos \theta, 2r^2 \sin \theta, r)drd\theta = (4r^3 \sin \theta \cos \theta - 4r^3 \sin \theta \cos \theta + 2r)drd\theta = 2rdrd\theta$. Thus, the work can be found as

$$
\int \int_{S} \nabla \times \vec{f} \cdot d\vec{S} = \int_{0}^{\pi/2} \int_{0}^{2} 2rdrd\theta = 2 \left[ \frac{\pi r^2}{2} \right]_{0}^{2} = \pi \approx 6.28
$$

(4) $\nabla \times \vec{f} = 0$. Thus $\int_{C} \vec{f} \cdot d\vec{r} = \int \int_{S} \nabla \times \vec{f} \cdot d\vec{S} = 0$.

(5) It is easier to evaluate the integral using Stokes’ theorem (otherwise there would be three line integrals). Calculate $\nabla \times \vec{f}$ to be $(2, 2z, 2x)$. The surface $\vec{r}$ can be parametrized by $x = 2 \cos \theta \sin \phi, y = 2 \sin \theta \sin \phi, z = 2 \cos \phi$. Thus, $d\vec{S} = (4 \sin^2 \phi \cos \theta, 4 \sin^2 \phi \sin \theta, 4 \sin \phi \cos \phi)d\phi d\theta$. One way to determine the sign of the product is to consider a convenient value of $\theta$ or $\phi$ and to determine the sign of such conveniently chosen vector. For example, when $\phi = \frac{\pi}{2}$ (the equator) and $\theta = 0$ (front meridian in $xz$ plane), the vector should point to the front too. For these two angle values, the vector is $\pm (4, 0, 0)$ and it points to the front if we take the positive sign.

Another way to determine the sign is to note that the normal vector and position vector are colinear at every point (see also Example 7) and that $d\vec{S} = (4 \sin^2 \phi \cos \theta, 4 \sin^2 \phi \sin \theta, 4 \sin \phi \cos \phi)d\phi d\theta = \pm 2 \sin \phi(2 \sin \phi \cos \theta, 2 \sin \phi \sin \theta, 2 \cos \phi)d\phi d\theta = \pm 2 \sin \phi \vec{r} d\phi d\theta$. Thus, $d\vec{S}$ has the same orientation as $\vec{r}$ if we take the positive sign.

Since the region is in the first octant, the bounds of integration are $0 \leq \theta \leq \frac{\pi}{2}$ and $0 \leq \phi \leq \frac{\pi}{2}$. The flux integral is $\int \int_{S} \nabla \times \vec{f} \cdot d\vec{S} = 16 \int_{0}^{\pi/2} \int_{0}^{\pi/2} (4 \sin^2 \phi \cos \theta \sin \theta + 4 \sin^2 \phi \sin \theta \cos \phi) d\phi d\theta = 16 \int_{0}^{\pi/2} \left( \frac{1}{2} \sin^3 \phi + 2 \sin^2 \phi \cos \phi \right) d\phi d\theta = 16$.

Divergence Theorem

Recall that the divergence of $\vec{f}$ is defined as the scalar product of $\nabla$ and $\vec{f}$:

$$
\text{div } \vec{f} = \nabla \cdot \vec{f} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}
$$

The Divergence Theorem can also be regarded as a three-dimensional version of Green’s Theorem in the form $\oint_{C} Pdy - Qdx = \int_{S} \nabla \cdot \vec{f} \cdot d\vec{S}$. Here $\vec{N}$ is the normal vector to $C$ at point $(x, y)$ and the product $\vec{N} \cdot ds$ can be calculated to be $dy \vec{i} - dx \vec{j}$. This version of Green’s theorem relates the line integral of a two-dimensional vector function $\vec{f}$ over a closed curve $C$ with the double integral of $\nabla \cdot \vec{f}$ over the interior of $C$. Adding one dimension to this formula, we relate the surface integral of a vector function $\vec{f}$ with the triple integral of $\nabla \cdot \vec{f}$.

Let $S$ be a region on a positive oriented surface that is the boundary of a simple solid region $V$. If $\vec{f}$ is a vector field, the Divergence Theorem states that

$$
\int \int_{S} \vec{f} \cdot d\vec{S} = \int \int_{V} \nabla \cdot \vec{f} \ dV
$$

where $dV$ is the volume element $dx \ dy \ dz$. 17
Practice Problems.

1. Use the Divergence Theorem to find the flux of the vector field \( \mathbf{f} = (x, 2y, 3z) \) over the cube with vertices \((\pm 1, \pm 1, \pm 1)\). (Note: the flux is found to be 48 without the use of the Divergence Theorem earlier.)

2. Find the flux of the vector field \( \mathbf{f} = (z, y, x) \) over the unit sphere.

3. Use the Divergence Theorem to find the flux of the vector field \( \mathbf{f} = (y, x, z) \) over the boundary of the region enclosed by the paraboloid \( z = 1 - x^2 - y^2 \) and the plane \( z = 0 \). (Note: the flux is found to be \( \frac{\pi}{2} \) without the use of the Divergence Theorem earlier.)

4. Find the flux of the vector field \( \mathbf{f} = (xy, yz, xz) \) over the boundary of the region enclosed by the cylinder \( x^2 + y^2 = 1 \) and the planes \( z = 0 \) and \( z = 2 \).

5. Find the flux of the vector field \( \mathbf{f} = (ye^z, 2y, xe^y) \) over the boundary of the region enclosed by the cylinder \( x^2 + y^2 = 9 \) and the planes \( z = 0 \) and \( z = 4 - y \).

6. Use the Divergence Theorem to find the flux of the vector field \( \mathbf{f} = (x, 2y, 3z) \) over the cube with vertices \((\pm 1, \pm 1, \pm 1)\) without the top.

Solutions. (1) It is much easier to evaluate the integral using the Divergence Theorem – recall that otherwise you have to do six flux integrals. Calculate \( \text{div} \mathbf{f} = 1 + 2 + 3 = 6 \) and so

\[
\int \int \mathbf{f} \cdot d\mathbf{S} = \int \int \int_V \text{div} \mathbf{f} \, dx \, dy \, dz = \int \int \int_V 6 \, dx \, dy \, dz
\]

where \( V \) is the interior of the cube. So the bounds are \(-2 \leq x, y, z \leq 1\) and the integral becomes

\[
\int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} 6 \, dx \, dy \, dz = 6|x|_1 \int_{-1}^{1} |y|_1 \int_{-1}^{1} |z|_1 = 6(2)(2)(2) = 48.
\]

(2) It is easier to evaluate the integral using the Divergence Theorem than directly because finding the flux integral directly involves long computation of \( \mathbf{dS} \). Calculate that \( \text{div} \mathbf{f} = 1 \) so that \( \int \mathbf{f} \cdot d\mathbf{S} = \int \int \int_V 1 \, dx \, dy \, dz \) where \( V \) is the interior of the unit sphere. Hence the bounds of the integration are \( 0 \leq \theta \leq 2\pi, \, 0 \leq \phi \leq \pi \) and \( 0 \leq r \leq 1 \) and the Jacobian for the spherical coordinates is \( r^2 \sin \phi \). Thus the integral becomes

\[
\int_{0}^{2\pi} \int_{0}^{\pi} \int_{0}^{1} r^2 \sin \phi \, dr \, d\phi \, d\theta = 2\pi (1 + 1) \frac{1}{3} = \frac{4\pi}{3}.
\]

(3) In this problem it is also easier to find the resulting flux using the Divergence Theorem since the alternative involves two flux integrals (see the solution of problem 2 following the section on flux integrals). Compute that \( \text{div} \mathbf{f} = 1 \) so that \( \int \mathbf{f} \cdot d\mathbf{S} = \int \int \int_V 1 \, dx \, dy \, dz \) where \( V \) is the region between the paraboloid and \( xy \)-plane. Using the cylindrical coordinates, the paraboloid is \( z = 1 - r^2 \) and the projection in \( xy \) plane is a disc of radius 1 so the bounds of the integration are \( 0 \leq \theta \leq 2\pi, \, 0 \leq r \leq 1 \) and \( 0 \leq z \leq 1 - r^2 \) and the Jacobian is \( r \). Hence the integral is

\[
\int_{0}^{2\pi} \int_{0}^{1} \int_{0}^{1-r^2} r \, dz \, dr \, d\theta = \int_{0}^{2\pi} \int_{0}^{1} (1-r^2) r \, dr \, d\theta = 2\pi (\frac{1}{2} - \frac{1}{4}) = \frac{\pi}{2}.
\]

(4) Without the Divergence Theorem, one would have to evaluate three flux integrals and with the Divergence Theorem, just one triple integral. Calculate that \( \text{div} \mathbf{f} = y + z + x \). Thus, \( \int \mathbf{f} \cdot \)
\[dS = \iint_V (x + y + z) \, dxdydz\] where \(V\) is the region between the cylinder and two planes. Using cylindrical coordinates, the bounds are \(0 \leq \theta \leq 2\pi, 0 \leq r \leq 1\) and \(0 \leq z \leq 2\). The integral is
\[\iiint_V (x + y + z) \, dxdydz = \int_0^{2\pi} \int_0^1 (r \cos \theta + r \sin \theta + 2) rdrd\theta = \int_0^{2\pi} \int_0^1 (r^2 \cos \theta + r^2 \sin \theta + 2r) drd\theta = \int_0^{2\pi} \left( \frac{1}{3} \cos \theta + \frac{1}{3} \sin \theta + 1 \right) d\theta = 0 + 0 + 2\pi = 2\pi.

(5) Without the Divergence Theorem, one would have to evaluate three flux integrals and with the Divergence Theorem, just one triple integral. Calculate that \(\text{div}\, \mathbf{f} = 2\). Thus, \(\iint_S \mathbf{f} \cdot d\mathbf{S} = \iiint_V 2 \, dxdydz\) where \(V\) is the region between the cylinder and two planes. Using cylindrical coordinates, the bounds are \(0 \leq \theta \leq 2\pi, 0 \leq r \leq 1\) and \(0 \leq z \leq 2 - y = 4 - r \sin \theta\). The integral is
\[\int_0^{2\pi} \int_0^1 \int_0^{4-r \sin \theta} 2 \, r \, dr \, d\theta \, dz = \int_0^{2\pi} \int_0^3 2r(4 - r \sin \theta) \, dr \, d\theta = \int_0^{2\pi} (36 - 18 \sin \theta) \, d\theta = 36\theta + 18 \cos \theta \bigg|_0^{2\pi} = 72\pi.

(6) Recall that \(\text{div}\, \mathbf{f} = 1 + 2 + 3 = 6\) by problem 1. In order to use the Divergence Theorem, the top has to be considered. The flux over the top has been found to be \(\iint_{\text{top}} (x, 2y, 3) \cdot (0, 0, 1) \, dxdy = \int_{-1}^1 \int_{-1}^1 3 \, dxdy = 12\) (see problem 4 in the section on flux integrals). Since
\[\iint_{\text{no top}} \mathbf{f} \cdot d\mathbf{S} + \iint_{\text{top}} \mathbf{f} \cdot d\mathbf{S} = \iiint_V 6 \, dxdydz\]
we have that
\[\iint_{\text{no top}} \mathbf{f} \cdot d\mathbf{S} = \iiint_V 6 \, dxdydz - \iint_{\text{top}} \mathbf{f} \cdot d\mathbf{S}.

The triple integral was computed in problem 1 to be 48. Thus \(\iint_{\text{no top}} \mathbf{f} \cdot d\mathbf{S} = \iiint_V 6 \, dxdydz - \int_{\text{top}} \mathbf{f} \cdot d\mathbf{S} = 48 - 12 = 36\).
Flux, surface and line integrals in cylindrical and spherical coordinates

In all the previous examples, a vector field has been given in terms of the standard basis $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ and each coordinate is a function of $x, y$ and $z$. In many applications (e.g. in Electricity and Magnetism), a vector field may be given in terms of cylindrical, spherical or some other coordinates and may be presented in terms of other coordinate systems.

**Cartesian coordinates.** Consider the position vector $\mathbf{r} = (x, y, z)$ of point $(x, y, z)$ in space. Taking the partial derivatives produces

$$\mathbf{r}_x = (1, 0, 0) = \mathbf{i}, \quad \mathbf{r}_y = (0, 1, 0) = \mathbf{j}, \quad \text{and} \quad \mathbf{r}_z = (0, 0, 1) = \mathbf{k}.$$  

Note that we usually consider these vectors exactly in this order which correspond to the fact that the cross product of the first two produces the third one. In other coordinate systems, you can use the right-hand rule to check if the product of your first two orthogonal vectors produces the third orthogonal vector or its opposite. If it is the opposite, switching any two vectors can give you the desired order. For Cartesian coordinates, we indeed have that $\mathbf{i} \times \mathbf{j} = \mathbf{k}$.

Note also that the vectors $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ have unit length so they are equal to their normalizations\(^1\). This fact is often denoted by

$$\hat{\mathbf{r}}_x = \mathbf{i}, \quad \hat{\mathbf{r}}_y = \mathbf{j}, \quad \text{and} \quad \hat{\mathbf{r}}_z = \mathbf{k}.$$  

**Cylindrical coordinates.** Let us repeat the process of normalizing the partial derivatives for the position vector $\mathbf{r} = (x, y, z)$ in cylindrical coordinates

$$\mathbf{r} = (r \cos \theta, r \sin \theta, z).$$  

Taking the partial derivatives produces

$$\mathbf{r}_r = (\cos \theta, \sin \theta, 0), \quad \mathbf{r}_\theta = (-r \sin \theta, r \cos \theta, 0), \quad \text{and} \quad \mathbf{r}_z = (0, 0, 1).$$

Note that the length of $\mathbf{r}_r$ and $\mathbf{r}_z$ is 1 and the length of $\mathbf{r}_\theta$ is $r$.

Thus, normalizing these three vectors produces

$$\hat{\mathbf{r}}_r = \mathbf{r}_r = (\cos \theta, \sin \theta, 0), \quad \hat{\mathbf{r}}_\theta = \frac{1}{r} \mathbf{r}_\theta = (-\sin \theta, \cos \theta, 0), \quad \text{and} \quad \hat{\mathbf{r}}_z = \mathbf{r}_z = (0, 0, 1).$$

---

\(^1\)Recall that the normalization of a vector $\mathbf{a}$ is

$$\hat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|}$$

where $|\mathbf{a}|$ is the length of $\mathbf{a}$. As a result, the vector $\hat{\mathbf{a}}$ has the same direction and sense as $\mathbf{a}$ but it has unit length.
Thus, for example, to compute the dot product of vector fields given by \( \mathbf{f} \) and \( \mathbf{g} \), the vectors \( \mathbf{\hat{r}}_r, \mathbf{\hat{r}}_\theta, \mathbf{\hat{r}}_z \) are particularly handy for computing dot products. In cylindrical coordinates, for example, the vectors \( \mathbf{\hat{r}}_r, \mathbf{\hat{r}}_\theta, \mathbf{\hat{r}}_z \) are orthogonal to each other and of unit length so

\[
\mathbf{\hat{r}}_r \cdot \mathbf{\hat{r}}_\theta = \mathbf{\hat{r}}_r \cdot \mathbf{\hat{r}}_z = \mathbf{\hat{r}}_\theta \cdot \mathbf{\hat{r}}_z = 0, \quad \text{and} \quad \mathbf{\hat{r}}_r \cdot \mathbf{\hat{r}}_r = \mathbf{\hat{r}}_\theta \cdot \mathbf{\hat{r}}_\theta = \mathbf{\hat{r}}_z \cdot \mathbf{\hat{r}}_z = 1.
\]

Thus, for example, to compute the dot product of vector fields given by \( \mathbf{f} = 5r\mathbf{\hat{r}}_r - 5\sin \theta \mathbf{\hat{r}}_\theta - 3\mathbf{\hat{r}}_z \) and \( \mathbf{g} = \sin \theta \mathbf{\hat{r}}_r + r\mathbf{\hat{r}}_\theta - 2\mathbf{\hat{r}}_z \), we can dot them as follows.

\[
\mathbf{f} \cdot \mathbf{g} = (5r\mathbf{\hat{r}}_r - 5\sin \theta \mathbf{\hat{r}}_\theta - 3\mathbf{\hat{r}}_z)(\sin \theta \mathbf{\hat{r}}_r + r\mathbf{\hat{r}}_\theta - 2\mathbf{\hat{r}}_z) = 5r \sin \theta \mathbf{\hat{r}}_r \cdot \mathbf{\hat{r}}_r - 5\sin \theta \mathbf{\hat{r}}_\theta \cdot \mathbf{\hat{r}}_r - 3(-2)\mathbf{\hat{r}}_z \cdot \mathbf{\hat{r}}_z = 5r \sin \theta - 5\sin \theta + 6 = 6.
\]

Check also that \( \mathbf{\hat{r}}_r \times \mathbf{\hat{r}}_\theta = \mathbf{\hat{r}}_z \) so the three vectors \( \mathbf{\hat{r}}_r, \mathbf{\hat{r}}_\theta, \mathbf{\hat{r}}_z \) are considered in the order enabling one to compute the cross product of vectors using the determinant just like in the case of Cartesian coordinates.

In some cases the three vectors \( \mathbf{\hat{r}}_r, \mathbf{\hat{r}}_\theta, \mathbf{\hat{r}}_z \) are more convenient to use than the usual vectors \( \mathbf{i}, \mathbf{j}, \mathbf{k} \) for a representation of a vector field. In cases like this, it is useful to know the conversion equations from one coordinate system to the other.

Representing the above three formula in terms of the vectors \( \mathbf{i}, \mathbf{j}, \mathbf{k} \) produces

\[
\mathbf{\hat{r}}_r = \cos \theta \mathbf{i} + \sin \theta \mathbf{j}, \quad \mathbf{\hat{r}}_\theta = -\sin \theta \mathbf{i} + \cos \theta \mathbf{j}, \quad \text{and} \quad \mathbf{\hat{r}}_z = \mathbf{k}.
\]

These three equations enable one to convert a vector field represented in terms of \( \mathbf{\hat{r}}_r, \mathbf{\hat{r}}_\theta, \mathbf{\hat{r}}_z \) into a vector field represented in term of \( \mathbf{i}, \mathbf{j}, \mathbf{k} \). For the converse problem, solving the first two equations for \( \mathbf{i} \) and \( \mathbf{j} \) produces

\[
\mathbf{i} = \cos \theta \mathbf{\hat{r}}_r - \sin \theta \mathbf{\hat{r}}_\theta, \quad \mathbf{j} = \sin \theta \mathbf{\hat{r}}_r + \cos \theta \mathbf{\hat{r}}_\theta, \quad \text{and} \quad \mathbf{k} = \mathbf{\hat{r}}_z.
\]

**Example.**

1. Convert the vector field \( \mathbf{f} = (5x, 3, x^2 + y^2) = 5x\mathbf{\hat{r}}_x + 3\mathbf{\hat{r}}_y + (x^2 + y^2)\mathbf{\hat{r}}_z \) to cylindrical coordinates and express it in the coordinate system determined by \( \mathbf{\hat{r}}_r, \mathbf{\hat{r}}_\theta, \mathbf{\hat{r}}_z \).

2. Convert the vector field \( \mathbf{f} = 5r^3\mathbf{\hat{r}}_r + 3\mathbf{\hat{r}}_\theta + r \sin \theta \mathbf{\hat{r}}_z \) to Cartesian coordinates and express it in the coordinate system determined by \( \mathbf{\hat{r}}_x, \mathbf{\hat{r}}_y, \mathbf{\hat{r}}_z \).

**Solutions.** 1. Converting the coordinates of \( \mathbf{f} \) to cylindrical coordinates produces \( \mathbf{f} = 5r \cos \theta \mathbf{\hat{r}}_x + 3\mathbf{\hat{r}}_y + r^2 \mathbf{\hat{r}}_z \). Using the conversion formulas produces

\[
\mathbf{f} = 5r \cos \theta (\cos \theta \mathbf{\hat{r}}_r - \sin \theta \mathbf{\hat{r}}_\theta) + 3(\sin \theta \mathbf{\hat{r}}_r + \cos \theta \mathbf{\hat{r}}_\theta) + r^2 \mathbf{\hat{r}}_z \Rightarrow \mathbf{f} = (5r \cos \theta \cos \theta + 15r \cos \theta \sin \theta)\mathbf{\hat{r}}_r + (-\sin \theta + 3 \cos \theta)\mathbf{\hat{r}}_\theta + r^2 \mathbf{\hat{r}}_z.
\]
2. Using the conversion formulas produces

\[ f = 5r^3 (\cos \theta \mathbf{i} + \sin \theta \mathbf{j}) + 3(-r \sin \theta \mathbf{i} + r \cos \theta \mathbf{j}) + r \sin \theta \mathbf{k}. \]

Converting the coordinates to Cartesian coordinates produces

\[ f = 5r^2 (r \cos \theta \mathbf{i} + r \sin \theta \mathbf{j}) + 3(-r \sin \theta \mathbf{i} + r \cos \theta \mathbf{j}) + r \sin \theta \mathbf{k} =
5(x^2 + y^2)(\mathbf{i} + \mathbf{j}) + 3(-y \mathbf{i} + x \mathbf{j}) + y \mathbf{k} = (5(x^2 + y^2)x - 3y)\mathbf{i} + (5(x^2 + y^2)y + 3x)\mathbf{j} + y \mathbf{k}.\]

Note that this can be written shorter as

\[ f = (5(x^2 + y^2)x - 3y, 5(x^2 + y^2)y + 3x, y). \]

**Flux and line integrals in cylindrical coordinates**

If \( C \) is a curve, \( S \) is a surface, and \( f \) is a vector field given in cylindrical coordinates, we need to express the terms \( dr, dS \) and \( dV \) in the same coordinates in order to compute the line integral \( \int_C f \cdot dr \), the surface integral \( \int_S f \cdot dS \) or to use Stokes’ or Divergence Theorems to compute any of those integrals when applicable. For those theorems, we also need to have formulas computing divergence and curl of a vector field in cylindrical coordinates.

Let us first consider the length element \( dr \) in cylindrical coordinates. Since \( dr = \hat{r}_r dr + r \hat{r}_\theta d\theta + \hat{z}_z dz \), substituting that \( \mathbf{r}_r = \hat{r}_r, \mathbf{r}_\theta = r \hat{r}_\theta \), and \( \mathbf{r}_z = \hat{z}_z \) we obtain that

\[ dr = \hat{r}_r dr + r \hat{r}_\theta d\theta + \hat{z}_z dz. \]

The product of the coefficients \( dr, r d\theta \), and \( dz \) with three vectors \( \hat{r}_r, \hat{r}_\theta \), and \( \hat{z}_z \) in the above expression computes the volume element \( dV \) in cylindrical coordinates as the figure below illustrates too. Hence,

\[ dV = r \, dr \, d\theta \, dz \]

Note that the term \( r \) in the formula \( dV = dxdydz = r \, dr \, d\theta \, dz \) matches exactly the value of the Jacobian in cylindrical coordinates.

Let us now consider the surface area element \( dS \). Let \( dS_r, dS_\theta \) and \( dS_z \) denote the three components of \( dS \) in basis \( \hat{r}_r, \hat{r}_\theta \), and \( \hat{z}_z \). Then \( dS_r \) corresponds to the surface area element of the cylinder of radius \( r \). This is area of the “parallelogram” with the sides \( r d\theta \) and \( dz \) multiplied by \( \hat{r}_r \) as the figure on the right illustrates. Similarly, \( dS_\theta \) is the surface area element of the vertical plane as
on the figure, so that \( dS_\theta \) is the area of the “parallelogram” with sides \( dr \) and \( dz \) multiplied by \( \hat{r}_\theta \). Lastly, \( dS_z \) is the surface area element of the horizontal plane and so \( dS_z \) is the product of \( dr \), \( r d\theta \) and \( \hat{r}_z \). This produces the formula for \( dS \)

\[
dS = dS_r + dS_\theta + dS_z = r \, d\theta \, dz \, \hat{r}_r + dr \, dz \, \hat{r}_\theta + r \, dr \, d\theta \, \hat{r}_z
\]

In cases when \( r, \theta \) or \( z \) are constant, the above formula can be simplified. In particular,

- If \( r \) is constant (that is when the surface is on a cylinder), then \( dr = 0 \) and
  \[
dS = r \, d\theta \, dz \, \hat{r}_r.
\]
  The first figure on the right illustrates this scenario.
- If \( z \) is constant (that is when the surface is on a horizontal plane), then \( dz = 0 \) and
  \[
dS = r \, dr \, d\theta \, \hat{r}_z.
\]
  The second figure on the right illustrates this scenario.
- If \( \theta \) is constant (that is when the surface is on a vertical plane), then \( d\theta = 0 \) and
  \[
dS = dr \, dz \, \hat{r}_\theta.
\]

**Gradient, curl and divergence in cylindrical coordinates.** Recall that the gradient operator is given by \( \nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}) \). In cylindrical coordinates, we have the relations

\[
\hat{r}_r = \frac{\partial}{\partial r} \mathbf{r}, \quad \hat{r}_\theta = \frac{1}{r} \frac{\partial}{\partial \theta} \mathbf{r}, \quad \text{and} \quad \hat{r}_z = \frac{\partial}{\partial z} \mathbf{r}
\]

which imply that the gradient operator becomes \( \nabla = \frac{\partial}{\partial r} \hat{r}_r + \frac{1}{r} \frac{\partial}{\partial \theta} \hat{r}_\theta + \frac{\partial}{\partial z} \hat{r}_z \).

If \( \mathbf{f} \) is a vector field which can be represented as \( \mathbf{f} = P \hat{r}_r + Q \hat{r}_\theta + R \hat{r}_z \) in cylindrical coordinates, it can be shown that the formulas below compute the divergence and the curl of \( \mathbf{f} \).

\[
\text{div} \mathbf{f} = \nabla \cdot \mathbf{f} = \frac{1}{r} \frac{\partial(rP)}{\partial r} + \frac{1}{r} \frac{\partial Q}{\partial \theta} + \frac{\partial R}{\partial z}
\]

\[
\text{curl} \mathbf{f} = \nabla \times \mathbf{f} = \frac{1}{r} \begin{vmatrix}
\hat{r}_r & r \hat{r}_\theta & \hat{r}_z \\
\frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\
P & Q & R
\end{vmatrix}
\]

**Example.** Calculate the divergence and curl of the vector field \( \mathbf{f} = 5r^3 \hat{r}_r + 3 \hat{r}_\theta + r \sin \theta \hat{r}_z \).

**Solutions.** For \( \mathbf{f}, \ P = 5r^3, \ Q = 3, \) and \( \ R = r \sin \theta \) so \( \text{div} \mathbf{f} = \frac{1}{r} \frac{\partial}{\partial r}(20r^3) + \frac{1}{r}(0) + 0 = 20r^2 \).

\[
\text{curl} \mathbf{f} = \frac{1}{r} \begin{vmatrix}
\hat{r}_r & \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\
5r^3 & P & Q & R \\
\end{vmatrix}
= \frac{1}{r}(r \cos \theta \hat{r}_r - r \sin \theta \hat{r}_\theta + 3 \hat{r}_z) = \cos \theta \hat{r}_r - \sin \theta \hat{r}_\theta + \frac{3}{r} \hat{r}_z.
\]
Spherical Coordinates. Recall that the position vector \( \mathbf{r} = (x, y, z) \) in spherical coordinates is

\[ \mathbf{r} = (r \cos \theta \sin \phi, r \sin \theta \sin \phi, r \cos \phi). \]

Taking the partial derivatives produces

\[ \mathbf{r}_r = (\cos \theta \sin \phi, \sin \theta \sin \phi, \cos \phi), \]
\[ \mathbf{r}_\phi = (r \cos \theta \cos \phi, r \sin \theta \cos \phi, -r \sin \phi), \]
\[ \mathbf{r}_\theta = (-r \sin \theta \sin \phi, r \cos \theta \sin \phi, 0). \]

Calculate that the length of \( \mathbf{r}_r \) is 1, the length of \( \mathbf{r}_\phi \) is \( r \), and the length of \( \mathbf{r}_\theta \) is \( r \sin \phi \). Normalizing the three partial derivative produces

\[ \hat{\mathbf{r}}_r = \frac{\mathbf{r}_r}{r}, \]
\[ \hat{\mathbf{r}}_\phi = \frac{r \mathbf{r}_\phi}{r^2}, \]
\[ \hat{\mathbf{r}}_\theta = \frac{r \mathbf{r}_\theta}{r \sin \phi}. \]

This implies that the conversion equations are as below.

\[ \hat{\mathbf{r}}_r = \cos \theta \sin \phi \hat{i} + \sin \theta \sin \phi \hat{j} + \cos \phi \hat{k}, \]
\[ \hat{\mathbf{r}}_\phi = \cos \theta \cos \phi \hat{i} + \sin \theta \cos \phi \hat{j} - \sin \phi \hat{k}, \]
\[ \hat{\mathbf{r}}_\theta = -\sin \theta \hat{i} + \cos \theta \hat{j}. \]

Since the vectors \( \hat{\mathbf{r}}_r, \hat{\mathbf{r}}_\phi, \hat{\mathbf{r}}_\theta \) are orthonormal (mutually orthogonal and of unit length), formulas \( \hat{\mathbf{r}}_r \cdot \hat{\mathbf{r}}_r = 1, \hat{\mathbf{r}}_\phi \cdot \hat{\mathbf{r}}_\phi = 1, \hat{\mathbf{r}}_\theta \cdot \hat{\mathbf{r}}_\theta = 1 \) hold and can be used to compute the dot product of two vector fields in spherical coordinates.

Check that \( \hat{\mathbf{r}}_r \times \hat{\mathbf{r}}_\phi = \hat{\mathbf{r}}_\theta \). Thus, the three vectors in the order \( \hat{\mathbf{r}}_r, \hat{\mathbf{r}}_\phi, \hat{\mathbf{r}}_\theta \) satisfy the right-hand rule.

Flux and line integrals spherical coordinates

If \( \mathbf{f} \) is a vector field given in spherical coordinates, we need to express the terms \( d\mathbf{r}, d\mathbf{S} \) and \( dV \) in the same coordinates in order to compute a line integral, a surface integral or to use Stokes’ or Divergence Theorems for any of those integrals. In spherical coordinates, \( \mathbf{r}_r = \hat{\mathbf{r}}_r, \mathbf{r}_\phi = r \hat{\mathbf{r}}_\phi \) and \( \mathbf{r}_\theta = r \sin \phi \hat{\mathbf{r}}_\theta \) and so \( d\mathbf{r} = \mathbf{r}_r \, dr + \mathbf{r}_\phi \, d\phi + \mathbf{r}_\theta \, d\theta \Rightarrow \)

\[ d\mathbf{r} = \hat{\mathbf{r}}_r \, dr + r \hat{\mathbf{r}}_\phi \, d\phi + r \sin \phi \hat{\mathbf{r}}_\theta \, d\theta. \]

As the figure on the right illustrates, the product of the coefficients \( dr, r d\phi \), and \( r \sin \phi d\theta \) with the vectors \( \hat{\mathbf{r}}_r, \hat{\mathbf{r}}_\phi, \) and \( \hat{\mathbf{r}}_\theta \) in the above expression
computes the volume element \(dV\) in spherical coordinates. Thus,

\[
dV = r^2 \sin \phi \, dr \, d\theta \, d\phi.
\]

The term \(r^2 \sin \phi\) in the formula \(dV = dx\,dy\,dz = r^2 \sin \phi \, dr \, d\theta \, d\phi\) is the Jacobian in spherical coordinates.

Let us now consider the surface area element \(dS\). Let \(dS_r, dS_\phi\) and \(dS_\theta\) denote the three components of \(dS\) in basis \(\hat{r}, \hat{\phi}, \) and \(\hat{\theta}\). Thus, \(dS_r\) corresponds to the surface area element of the sphere of radius \(r\). This is area of the “parallelogram” with the sides \(rd\phi\) and \(r \sin \phi d\theta\) multiplied by \(\hat{r}\), as the figure on the right illustrates. Similarly, \(dS_\phi\) is the surface area element of the plane as on the figure, so that \(dS_\phi\) is the area of the “parallelogram” with sides \(dr\) and \(r \sin \phi d\theta\) multiplied by \(\hat{\phi}\). Lastly, \(dS_\theta\) is the product of \(dr\), \(r d\phi\) and \(\hat{\theta}\).

This produces the formula for \(dS\)

\[
dS = r^2 \sin \phi \, d\phi \, d\theta \, \hat{r} + r \sin \phi \, dr \, d\theta \, \hat{\phi} + r \, dr \, d\phi \, \hat{\theta}
\]

In cases when \(r, \phi\) or \(\theta\) are constant, the above formula can be simplified. In particular,

- If \(r\) is constant (that is when the surface is on a sphere), then \(dr = 0\) and

\[
dS = r^2 \sin \phi \, d\phi \, d\theta \, \hat{r}_r
\]

as the figure on the right illustrates.

- If \(\phi\) is constant (that is when the surface is on a cone), then \(d\phi = 0\) and

\[
dS = r \sin \phi \, dr \, d\theta \, \hat{r}_\phi
\]

- If \(\theta\) is constant (that is when the surface is on a vertical plane), then \(d\theta = 0\) and

\[
dS = r \, dr \, d\phi \, \hat{r}_\theta
\]

**Gradient, curl and divergence in spherical coordinates.** In spherical coordinates, we have the relations

\[
\hat{r}_r = \frac{\partial}{\partial r} \hat{r}, \quad \hat{r}_\phi = \frac{1}{r} \frac{\partial}{\partial \phi} \hat{r}, \quad \text{and} \quad \hat{r}_\theta = \frac{1}{r \sin \phi} \frac{\partial}{\partial \theta} \hat{r}
\]

which imply that the gradient operator becomes

\[
\nabla = \frac{\partial}{\partial r} \hat{r}_r + \frac{1}{r} \frac{\partial}{\partial \phi} \hat{r}_\phi + \frac{1}{r \sin \phi} \frac{\partial}{\partial \theta} \hat{r}_\theta.
\]
If $\mathbf{f}$ is a vector field which can be represented as $\mathbf{f} = P\hat{r}_r + Q\hat{r}_\phi + R\hat{r}_\theta$ in spherical coordinates, it can be shown that the formulas below compute the divergence and the curl of $\mathbf{f}$.

\[
\text{div}\mathbf{f} = \nabla \cdot \mathbf{f} = \frac{1}{r^2} \frac{\partial (r^2 P)}{\partial r} + \frac{1}{r \sin \phi} \frac{\partial (r^2 \sin \phi Q)}{\partial \phi} + \frac{1}{r \sin \phi} \frac{\partial R}{\partial \theta} \\
\text{curl} \mathbf{f} = \nabla \times \mathbf{f} = \frac{1}{r^2 \sin \phi} \begin{vmatrix} \hat{r}_r & \hat{r}_\phi & \hat{r}_\theta \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial \theta} \\ P & rQ & r \sin \phi R \end{vmatrix}
\]

Example. Calculate the divergence of the vector field

\[
\mathbf{f} = r^2 \sin \phi \hat{r}_r + 4r^2 \cos \phi \hat{r}_\phi + \frac{r \sin \phi}{\cos \phi} \hat{r}_\theta.
\]

Solutions. For $\mathbf{f}$, $P = r^2 \sin \phi$, $Q = 4r^2 \cos \phi$, and $R = \frac{r \sin \phi}{\cos \phi}$ so

\[
\text{div}\mathbf{f} = \frac{1}{r^2} \frac{\partial (r^4 \sin \phi)}{\partial r} + \frac{1}{r \sin \phi} \frac{\partial (4r^2 \sin \phi \cos \phi)}{\partial \phi} + \frac{1}{r \sin \phi} \frac{\partial (r \sin \phi)}{\partial \theta} = \\
4r \sin \phi + \frac{1}{r \sin \phi} (4r^2 \cos^2 \phi - 4r^2 \sin^2 \phi) + 0 = 4r \sin \phi + \frac{4r^2 \cos^2 \phi}{r \sin \phi} - 4r \sin \phi = \frac{4r \cos^2 \phi}{\sin \phi}.
\]

Computing flux of a vector field in cylindrical or spherical coordinates

Example 1. Compute the flux of the vector field $\mathbf{f} = r \sin \phi \hat{r}_r + r \cos \phi \hat{r}_\theta$ over the unit sphere.

Solution. Since $r = 1$, $dr = 0$ and so $dS = r^2 \sin \phi \, d\phi \, d\theta \, \hat{r}_r$. As $\hat{r}_r \cdot \hat{r}_r = 1$ and $\hat{r}_\theta \cdot \hat{r}_r = 0$, the dot product $\mathbf{f} \cdot d\mathbf{S}$ is

\[
(r \sin \phi \hat{r}_r + r \cos \phi \hat{r}_\theta) \cdot r^2 \sin \phi \, d\phi \, d\theta \, \hat{r}_r = r \sin \phi r^2 \sin \phi \, d\phi \, d\theta + 0 = r^3 \sin^3 \phi \, d\phi \, d\theta.
\]

The radius of the sphere is 1 so $\mathbf{f} \cdot d\mathbf{S} = \sin^2 \phi \, d\phi \, d\theta$. Hence

\[
F = \int \int_S \mathbf{f} \cdot d\mathbf{S} = \int_0^{2\pi} \int_0^\pi \sin^2 \phi \, d\phi \, d\theta = 2\pi \int_0^\pi \sin^2 \phi \, d\phi = 2\pi \frac{\pi}{2} = \pi^2 \approx 9.87.
\]

Example 2. Use the Divergence theorem to find the flux of the vector field $\mathbf{f} = r \sin \phi \hat{r}_r + r \cos \phi \hat{r}_\theta$ over the unit sphere.

Solution. The divergence of $\mathbf{f}$ is $\text{div}\mathbf{f} = \frac{1}{r^2} \frac{\partial (r^3 \sin \phi)}{\partial r} + \frac{1}{r \sin \phi} \frac{\partial (r \cos \phi)}{\partial \theta} = \frac{1}{r^2} 3r^2 \sin \phi + 0 = 3 \sin \phi$. Hence

\[
F = \int \int \int_V \text{div}\mathbf{f} \, dV = \int \int \int_V 3 \sin \phi r^2 \sin \phi \, dr \, d\theta \, d\phi = \\
\int_0^{2\pi} \int_0^\pi \int_0^1 3r^2 \sin^2 \phi \, dr \, d\theta \, d\phi = 2\pi \int_0^\pi \frac{\pi}{2} \sin^2 \phi \, d\phi = 2\pi \frac{\pi}{2} = \pi^2 \approx 9.87.
\]

Example 3. Find the flux done by the vector field $\mathbf{f} = r \sin \theta \hat{r}_r + 2r \cos \theta \hat{r}_\theta + 3r \hat{z}$ over the boundary of the region between the paraboloids $z = r^2$ and $z = 4 - r^2$. 

26
Solution. Since computing the flux directly involves two flux integrals and using the Divergence Theorem just one, we choose to use the Divergence Theorem. Calculate the divergence as \( \text{div} \mathbf{f} = \frac{1}{r} \frac{\partial (r^2 \sin \theta)}{\partial r} + \frac{1}{r} \frac{\partial (r \cos \theta)}{\partial \theta} + \frac{\partial}{\partial z} = \frac{1}{r} 2r \sin \theta - \frac{1}{r} 2r \sin \theta + 0 = 2 \sin \theta - 2 \sin \theta = 0 \). So, the flux is \( F = \int \int \int (0) dV = 0 \). Hence, the flux is zero.

Example 4. Compute the flux of the vector field \( \mathbf{f} = r \sin \theta \hat{r} + 2r \cos \theta \hat{\theta} + 3z \hat{z} \) over the boundary of the region inside the cylinder \( x^2 + y^2 = 9 \) between the planes \( z = 0 \) and \( z = 4 \).

(a) without using the Divergence Theorem and (b) using the Divergence Theorem.

Solution. (a) The boundary consists of three surfaces: the cylinder \( S_1 \), the top plane \( S_2 \) and the bottom plane \( S_3 \).

On the cylinder, \( dS_1 = r \, d\theta \, dz \hat{r} \), since \( r = 3 \) so \( dr = 0 \). The dot product \( \mathbf{f} \cdot dS_1 \) is \( (r \sin \theta \hat{r} + 2r \cos \theta \hat{\theta} + 3z \hat{z}) \cdot r \, d\theta \, dz \hat{r} = r^2 \sin \theta \, d\theta \, dz \cdot \hat{r} \). The radius is 3 so \( \mathbf{f} \cdot dS_1 = 9 \sin \theta \, d\theta \, dz \). Hence

\[
F_1 = \int \int_{S_1} \mathbf{f} \cdot dS_1 = \int_0^{2\pi} \int_0^4 9 \sin \theta \, d\theta \, dz = \int_0^{2\pi} \sin \theta \, d\theta \int_0^4 \, dz = 0.
\]

On the top plane, \( dS_2 = r \, dr \, d\theta \hat{r} \) since \( z \) is constant so \( dz = 0 \). The dot product \( \mathbf{f} \cdot dS_2 \) is \( (r \sin \theta \hat{r} + 2r \cos \theta \hat{\theta} + 3z \hat{z}) \cdot r \, dr \, d\theta \hat{r} = 3r^2 z \, dr \, d\theta \hat{r} \cdot \hat{r} = 3r^2 z \, dr \, d\theta \). The plane is \( z = 4 \) so \( \mathbf{f} \cdot dS_2 = 12r^2 \, dr \, d\theta \). Hence

\[
F_2 = \int \int_{S_2} \mathbf{f} \cdot dS_2 = \int_0^{2\pi} \int_0^3 12r^2 \, dr \, d\theta = 2\pi 4(3)^3 = 216\pi.
\]

On the bottom plane, the surface area element \( dS_3 = -r \, dr \, d\theta \hat{r} \) because the length of this vector is the same as for \( dS_1 \) and the sense is opposite since it points outwards. The dot product \( \mathbf{f} \cdot dS_3 \) is \( (r \sin \theta \hat{r} + 2r \cos \theta \hat{\theta} + 3z \hat{z}) \cdot r \, dr \, d\theta \hat{r} = 3r^2 r \, dr \, d\theta \hat{r} \cdot \hat{r} = 3r^2 \, dr \, d\theta \). The plane is \( z = 0 \) so \( \mathbf{f} \cdot dS_3 = 0 \) and so \( F_3 = 0 \).

Thus, we have that \( F = F_1 + F_2 + F_3 = 0 + 216\pi + 0 = 216\pi \).

(b) The divergence of \( \mathbf{f} \) is \( \text{div} \mathbf{f} = \frac{1}{r} \frac{\partial (r^2 \sin \theta)}{\partial r} + \frac{1}{r} \frac{\partial (r \cos \theta)}{\partial \theta} + \frac{\partial}{\partial z} = \frac{1}{r} 2r \sin \theta - \frac{1}{r} 2r \sin \theta + 3r = 2 \sin \theta - 2 \sin \theta + 3r = 3r \). Hence

\[
F = \int \int \int_V \text{div} \mathbf{f} \, dV = \int \int \int_V 3r \, r \, dr \, d\theta \, dz = \int_0^{2\pi} d\theta \int_0^3 3r^2 \, dr \int_0^4 \, dz = 2\pi 3^3 4 = 216\pi.
\]

Example 5. Ice cream problem. Compute the flux of the vector field \( \mathbf{f} = r^2 \hat{r} + 2r^2 \sin \phi \hat{\phi} + r^4 \sin \phi \hat{\phi} \) over the boundary of the region inside the cone \( z = \sqrt{3(x^2 + y^2)} \) and the sphere of radius \( a \). To obtain the equation of the cone in spherical coordinates, note that the the value of \( \phi \) is constant on the cone and compute this value of \( \phi \). The problem is significantly shorter if you use the Divergence Theorem.
Solution. In $rz$-plane the cone is determined by revolution of the line with $\tan \alpha = \sqrt{3} \Rightarrow \alpha = \frac{\pi}{3}$ and so $\phi = \frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6}$. Hence the angle between any line radiating from the vertex of the cone and the $z$-axis is constant and equal to $\frac{\pi}{6}$ and so the equation of the cone is given by $\phi = \frac{\pi}{6}$.

The divergence of $f$ is $\text{div} f = \frac{1}{r^2} \frac{\partial (r^4)}{\partial r} + \frac{1}{r \sin \phi} \frac{\partial (2r^2 \sin^2 \phi)}{\partial \phi} + \frac{1}{r \sin \phi} \frac{\partial (r^4 \sin \phi)}{\partial \theta} = 4r + 4r \cos \phi = 4r(1 + \cos \phi)$. Hence $F = \int \int \int_V \text{div} f \, dV = 4a^4 \int_0^{\pi/6} (1 + \cos \phi) \sin \phi \, d\phi = 2a^4 \pi \left( -\cos \phi - \frac{\cos^2 \phi}{2} \right) \bigg|_0^{\pi/6} = 2a^4 \pi \left( \frac{9}{8} - \frac{\sqrt{3}}{2} \right) \approx 1.63a^4$. 


Find the flux done by the vector field $f = r \hat{r}_r + 2rz \hat{r}_\theta + r \sin \theta \hat{r}_z$ over the boundary of the region between the paraboloids $z = r^2$ and $z = a^2 - 3r^2$. Note that the problem is significantly shorter if you use the Divergence Theorem.

Solution. Calculate the divergence of the field to be $\text{div} f = \frac{1}{r} \frac{\partial (r^2)}{\partial r} + \frac{1}{r \sin \phi} \frac{\partial (2r^2 \sin^2 \phi)}{\partial \phi} + \frac{1}{r \sin \phi} \frac{\partial (r^4 \sin \phi)}{\partial \theta} = 2 + 0 + 0 = 2$. The paraboloids intersect in a circle whose radius can be obtained by solving $r^2 = a^2 - 3r^2 \Rightarrow r^2 = \frac{a^2}{4} \Rightarrow r = \frac{a}{2}$. The paraboloid $z = a^2 - 3r^2$ is upper and the paraboloid $z = r^2$ is lower surface. Hence

$F = \int \int \int_V \text{div} f \, dV = \int_0^{\pi/2} \int_0^{a/2} \int_{r/2}^{a^2 - 3r^2} 2r \, dr \, d\theta \, dz = 2\pi \int_0^{a/2} \left( (a^2 - 3r^2)^2 - r^4 \right) dr = 2\pi \int_0^{a/2} \left( a^4 - 6a^2r^2 + 9r^4 - r^4 \right) dr = 2\pi \left( \frac{a^5}{2} - \frac{a^5}{4} + \frac{a^5}{20} \right) = \frac{3a^5 \pi}{5}$.

Computing line integrals in cylindrical and spherical coordinates

By Stokes’ Theorem, the work done by a vector field $f$ acting along a a simple, closed, piecewise smooth curve $C$ on an oriented piecewise-smooth surface is given by $\oint_C f \cdot \, dr = \iint_S \text{curl} f \cdot \, dS$ where $S$ is the interior of $C$. 

28
Example 7. Find the work done by the vector field \( \mathbf{f} = \cos \theta \hat{r}_r - r \sin \theta \hat{r}_\theta - 5r \hat{r}_z \) acting along the positive oriented contour consisting of four curve segments represented on the figure on the right.

(a) directly, \hspace{1cm} (b) using Stokes’ Theorem.

Solutions. (a) Note that the curve consists of four parts \( C_1, C_2, C_3 \) and \( C_4 \).

On \( C_1, r = 2, z = 0 \) and \( 0 \leq \theta \leq \frac{\pi}{2} \). Hence \( dr = 0, dz = 0 \) and so \( d\mathbf{r} = r d\theta \hat{r}_\theta \). Hence, \( \mathbf{f} \cdot d\mathbf{r} = -r \sin \theta \hat{r}_\theta \cdot r d\theta \hat{r}_\theta = -r^2 \sin \theta d\theta \). So, \( \int_{C_1} = \int_0^{\pi/2} -4 \sin \theta d\theta = -4 \).

On \( C_2, x = 0, z = 0 \) and \( y : 2 \rightarrow 0 \). In cylindrical coordinates, \( x = 0 \Rightarrow \cos \theta = 0 \Rightarrow \theta = \frac{\pi}{2} \), so \( y = r \sin \theta = r \) with \( r : 2 \rightarrow 0 \). Since \( \theta \) and \( z \) are constant, \( d\theta = 0, dz = 0 \) and so \( d\mathbf{r} = dr \hat{r}_r \). Hence \( \mathbf{f} \cdot d\mathbf{r} = \cos \theta dr (\hat{r}_r \cdot \hat{r}_r) = \cos \theta dr = 0 \) since \( \theta = \frac{\pi}{2} \). So, \( \int_{C_2} = 0 \).

On \( C_3, y = 0, z = 0 \) and \( x : 2 \rightarrow 0 \). In cylindrical coordinates, \( y = 0 \Rightarrow \theta = 0 \) and so \( 2 = x = r \cos \theta = r \). Hence \( d\theta = 0, dr = 0 \) and so \( d\mathbf{r} = dz \hat{r}_z \), \( z : 2 \rightarrow 0 \). Thus, \( \mathbf{f} \cdot d\mathbf{r} = -5r \hat{r}_z \cdot dz \hat{r}_z = -5drdz = -10dz \) since \( r = 2 \). So, \( \int_{C_3} = \int_2^0 -10dz = -10(-2) = 20 \).

Hence, the total work done is \(-4 + 0 - 8 + 20 = 8 \).

(b) In order to use Stokes’ Theorem, calculate the curl to be \( \text{curl} = \frac{1}{r} \left| \begin{array}{ccc} \hat{r}_r & \hat{r}_\theta & \hat{r}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \end{array} \right| = \frac{1}{r} (5r \hat{r}_\theta - 2r \sin \theta \hat{r}_z + \sin \theta \hat{r}_z) = 5 \hat{r}_\theta - 2 \sin \theta \hat{r}_z + \frac{1}{r} \sin \theta \hat{r}_z \).

Let \( S_1 \) denote the quarter of the disc of radius 2 in the first quadrant of the \( xy \)-plane and \( S_2 \) denote the triangle in \( xz \)-plane with vertices (0,0), (2,2) and (2,0).

On \( S_1, z = 0 \) and so \( dz = 0 \). Thus, \( d\mathbf{S} = r dr d\theta \hat{r}_z \) and \( \text{curl} \cdot d\mathbf{S} = (-2 \sin \theta \hat{r}_z + \frac{1}{r} \sin \theta \hat{r}_z) \cdot r dr d\theta \hat{r}_z = -2r \sin \theta dr d\theta + \sin \theta dr d\theta = (-2r \sin \theta + \sin \theta) dr d\theta \) and so

\[
\int \int_S \text{curl} \cdot d\mathbf{S} = \int_0^{\pi/2} \int_0^2 (-2r \sin \theta + \sin \theta) dr d\theta = \int_0^{\pi/2} (-4 \sin \theta + 2 \sin \theta) d\theta = -4 + 2 = -2.
\]

On \( S_2, y = 0, \sin \theta = 0 \) so \( \theta = 0 \) is constant and \( d\theta = 0 \). Thus, \( d\mathbf{S} = dr dz \hat{r}_\theta \) and \( \text{curl} \cdot d\mathbf{S} = 5 \hat{r}_\theta \cdot dr dz \hat{r}_\theta = 5drdz \). The bounds for \( r \) and \( z \) are \( 0 \leq r \leq 2 \) and \( x = r \cos \theta = r \leq z \leq 2 \).

Hence,

\[
\int \int_S \text{curl} \cdot d\mathbf{S} = \int_0^2 \int_r^2 5drdz = \int_0^2 5(2-r)dr = 5(4-2) = 10.
\]

Thus the total work done is \( \int \int_{S_1} + \int \int_{S_2} = -2 + 10 = 8 \).

Example 8. Find the work done by the vector field \( \mathbf{f} = r \sin \theta \hat{r}_r + r \hat{r}_\theta + 4 \hat{r}_z \) acting along the positive oriented intersection of the sphere of radius \( R \) centered at the origin and the horizontal plane \( z = \frac{1}{2} R \).

(a) directly, \hspace{1cm} (b) using Stokes’ Theorem.
Solutions. (a) Considering the right triangle on the figure on the right, the radius of the circle in the horizontal plane is \( \sqrt{R^2 - \left(\frac{R}{2}\right)^2} = \sqrt{\frac{3R^2}{4}} = \frac{\sqrt{3}R}{2} \). Thus, the circle has parametrization \( r = \frac{\sqrt{3}R}{2}, z = \frac{R}{2} \) and \( 0 \leq \theta \leq 2\pi \) in cylindrical coordinates. Hence \( \text{d}r = 0, \text{d}z = 0 \) and so \( \text{d}r = \frac{\sqrt{3}R}{2} \text{d}\theta \text{r}_\theta \) and

\[
W = \oint_C \textbf{f} \cdot \text{d}r = \int_C \frac{\sqrt{3}R \sqrt{3}R}{2} \text{d}\theta = \frac{3R^2}{4} \int_0^{2\pi} \text{d}\theta = \frac{3R^2\pi}{2}.
\]

(b) We can consider \( S \) to be part of the horizontal plane inside of the circle. Since \( z \) is constant, \( \text{d}z = 0 \) and \( \text{d}S = r \text{d}r \text{d}\theta \text{r}_z \). Calculate the curl to be \( \text{curl} = \frac{1}{r} \begin{vmatrix} \hat{r}_r & \hat{r}_\theta & \hat{r}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ r \sin \theta & r & 0 \end{vmatrix} \) = \( \frac{1}{r}(2r - r \cos \theta)\hat{r}_z \) = \( (2 - \cos \theta)\hat{r}_z \). Hence \( \text{curl} \cdot \text{d}S = (2 - \cos \theta)r \text{d}r \text{d}\theta \).

\[
W = \int_S \text{curl} \cdot \text{d}S = \int_0^{2\pi} \int_0^{\frac{\sqrt{3}R}{2}} (2 - \cos \theta)r \text{d}r \text{d}\theta = \int_0^{\frac{\sqrt{3}R}{2}} (4\pi - 0)r \text{d}r = \frac{2\pi}{4} 3R^2 = \frac{3R^2\pi}{2}.
\]

Example 9. Find the work done by the vector field \( \textbf{f} = r^2\hat{r}_r + r^2 \sin \phi \hat{r}_\phi + 2\hat{r}_\theta \) acting along the curve from the previous problem

(a) directly, (b) using Stokes’ Theorem.

Solutions. (a) Considering the right triangle on the figure in solutions for Example 8, the value of \( \phi \) along the circle is constant and such that \( \cos \phi = \frac{R}{R} = \frac{1}{2} \Rightarrow \phi = \frac{\pi}{3} \). Hence, the circle has parametrization \( r = R, \phi = \frac{\pi}{3} \) and \( 0 \leq \theta \leq 2\pi \) in spherical coordinates. Hence \( \text{d}r = 0, \text{d}\phi = 0 \) and so \( \text{d}r = r \sin \phi \text{d}\theta \hat{r}_\theta = R \sin \frac{\pi}{3} \text{d}\theta \hat{r}_\theta = \frac{\sqrt{3}R}{2} \text{d}\theta \hat{r}_\theta \). So \( \text{f} \cdot \text{d}r = 2\frac{\sqrt{3}R}{2} \text{d}\theta \hat{r}_\theta \cdot \hat{r}_\theta = \sqrt{3} Rd\theta \)

\[
W = \int_C \textbf{f} \cdot \text{d}r = \sqrt{3} R \int_0^{2\pi} \text{d}\theta = 2\sqrt{3} R\pi.
\]

(b) Note that we can consider \( S \) to be any surface which has boundary \( C \). Since we are in spherical coordinates, it is not convenient to consider \( S \) to be the part of the horizontal plane inside of the circle, but the part of the sphere of radius \( R \) inside of the circle. This surface has very simple parametrization in spherical coordinates \( r = R \) so \( \text{d}r = 0 \) and \( \text{d}S = r^2 \sin \phi \text{d}\theta \text{d}\phi \hat{r}_r \). Calculate the curl to be

\[
\text{curl} = \frac{1}{r^2 \sin \phi} \begin{vmatrix} \hat{r}_r & r \hat{r}_\phi & r \sin \phi \hat{r}_\theta \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial \theta} \\ r^2 & r^2 \sin \phi & 2r \sin \phi \end{vmatrix} = \frac{1}{r^2 \sin \phi} (2r \cos \phi \hat{r}_r - 2r \sin \phi \hat{r}_\phi + 3r^3 \sin^2 \phi \hat{r}_\theta).
\]

Dotting by \( \text{d}S \) produces \( 2r \cos \phi \text{d}\theta \text{d}\phi \hat{r}_r \cdot \hat{r}_r = 2r \cos \phi \text{d}\theta \text{d}\phi \). Since \( r = R \), \( \text{curl} \cdot \text{d}S = 2R \cos \phi \text{d}\theta \text{d}\phi \).

\[
W = \int_C \int_S \text{curl} \cdot \text{d}S = \int_0^{2\pi} \int_0^{\frac{\pi}{3}} 2R \cos \phi \text{d}\theta \text{d}\phi = 4R\pi \int_0^{\frac{\pi}{3}} \cos \phi \text{d}\phi = 4R\pi \frac{\sqrt{3}}{2} = 2\sqrt{3} R\pi.
\]
General curvilinear coordinates

Consider a general substitution
\[ \mathbf{r} = (x(u_1, u_2, u_3), y(u_1, u_2, u_3), z(u_1, u_2, u_3)) \]
of \( \mathbf{r} = (x, y, z) \) with partial derivatives
\[ \mathbf{r}_{u_1} = (x_{u_1}, y_{u_1}, z_{u_1}), \quad \mathbf{r}_{u_2} = (x_{u_2}, y_{u_2}, z_{u_2}), \quad \text{and} \quad \mathbf{r}_{u_3} = (x_{u_3}, y_{u_3}, z_{u_3}). \]

If these three vectors are mutually orthogonal, they determine an **orthogonal curvilinear coordinate system**. Let us order the vectors so that \( \mathbf{r}_{u_1} \times \mathbf{r}_{u_2} \) has the same sense as \( \mathbf{r}_{u_3} \).

Let us use \( h_1, h_2, h_3 \) to denote the lengths of the three vectors above. Normalizing the three partial derivative produces
\[ \hat{\mathbf{r}}_{u_1} = \frac{1}{h_1} \mathbf{r}_{u_1}, \quad \hat{\mathbf{r}}_{u_2} = \frac{1}{h_2} \mathbf{r}_{u_2}, \quad \text{and} \quad \hat{\mathbf{r}}_{u_3} = \frac{1}{h_3} \mathbf{r}_{u_3}. \]

The above relations and the relation \( d\mathbf{r} = \mathbf{r}_{u_1} du_1 + \mathbf{r}_{u_2} du_2 + \mathbf{r}_{u_3} du_3 \) produce the formula for \( d\mathbf{r} \) below.
\[ d\mathbf{r} = h_1 \hat{\mathbf{r}}_{u_1} du_1 + h_2 \hat{\mathbf{r}}_{u_2} du_2 + h_3 \hat{\mathbf{r}}_{u_3} du_3. \]

The product of the coefficients \( h_1 du_1, h_2 du_2, \) and \( h_3 du_3 \) with three vectors \( \hat{\mathbf{r}}_{u_1}, \hat{\mathbf{r}}_{u_2}, \) and \( \hat{\mathbf{r}}_{u_3} \) computes the volume element \( dV \) in general curvilinear coordinates.
\[ dV = h_1 h_2 h_3 du_1 du_2 du_3 \]

Since
\[ \hat{\mathbf{r}}_{u_1} = \frac{1}{h_1} \frac{\partial}{\partial u_1} \mathbf{r}, \quad \hat{\mathbf{r}}_{u_2} = \frac{1}{h_2} \frac{\partial}{\partial u_2} \mathbf{r}, \quad \text{and} \quad \hat{\mathbf{r}}_{u_3} = \frac{1}{h_3} \frac{\partial}{\partial u_3} \mathbf{r}, \]
the gradient operator becomes
\[ \nabla = \frac{1}{h_1} \frac{\partial}{\partial u_1} \hat{\mathbf{r}}_{u_1} + \frac{1}{h_2} \frac{\partial}{\partial u_2} \hat{\mathbf{r}}_{u_2} + \frac{1}{h_3} \frac{\partial}{\partial u_3} \hat{\mathbf{r}}_{u_3}. \]

If \( \mathbf{f} \) is a vector field which can be represented as \( \mathbf{f} = P \hat{\mathbf{r}}_{u_1} + Q \hat{\mathbf{r}}_{u_2} + R \hat{\mathbf{r}}_{u_3} \), it can be shown\(^2\) that the formulas below compute the divergence and the curl of \( \mathbf{f} \).

\[ \text{div}\mathbf{f} = \nabla \cdot \mathbf{f} = \frac{1}{h_1 h_2 h_3} \frac{\partial}{\partial u_1} (h_2 h_3 P) + \frac{1}{h_1 h_2 h_3} \frac{\partial}{\partial u_2} (h_1 h_3 Q) + \frac{1}{h_1 h_2 h_3} \frac{\partial}{\partial u_3} (h_1 h_2 R) \]

\[ \text{curl}\mathbf{f} = \nabla \times \mathbf{f} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \frac{\partial}{\partial u_1} & h_2 \frac{\partial}{\partial u_2} & h_3 \frac{\partial}{\partial u_3} \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ \end{vmatrix} \]

These general formulas can be used for demonstrating the validity of the corresponding formulas in cylindrical and spherical coordinates. For example, for cylindrical coordinates with \( u_1 = r, u_2 = \)

\(^2\)One of the project topics focuses on the proof of this.
θ, u₃ = z, we have that h₁ = 1, h₂ = r, h₃ = 1 so that h₁h₂h₃ = r and the above general formula for divergence produces

\[ \text{div} \mathbf{f} = \nabla \cdot \mathbf{f} = \frac{1}{r} \frac{\partial(rP)}{\partial r} + \frac{1}{r} \frac{\partial Q}{\partial \theta} + \frac{1}{r} \frac{\partial(rR)}{\partial z} = \frac{1}{r} \frac{\partial(rP)}{\partial r} + \frac{1}{r} \frac{\partial Q}{\partial \theta} + \frac{1}{r} \frac{\partial R}{\partial z} = \frac{1}{r} \frac{\partial(rP)}{\partial r} + \frac{1}{r} \frac{\partial Q}{\partial \theta} + \frac{1}{r} \frac{\partial R}{\partial z} \]

which agrees with the formula from the section on cylindrical coordinates.

Similarly, for spherical coordinates with u₁ = r, u₂ = φ, u₃ = θ, we have that h₁ = 1, h₂ = r, h₃ = r sin φ so that h₁h₂h₃ = r² sin φ and the general formula for divergence produces

\[ \text{div} \mathbf{f} = \nabla \cdot \mathbf{f} = \frac{1}{r^2 \sin \phi} \frac{\partial(r^2 \sin \phi P)}{\partial r} + \frac{1}{r^2 \sin \phi} \frac{\partial(r \sin \phi Q)}{\partial \phi} + \frac{1}{r^2 \sin \phi} \frac{\partial(r R)}{\partial \theta} = \frac{1}{r^2} \frac{\partial(r^2 P)}{\partial r} + \frac{1}{r \sin \phi} \frac{\partial(\sin \phi Q)}{\partial \phi} + \frac{1}{r \sin \phi} \frac{\partial R}{\partial \theta} \]

which agrees with the formula from the section on spherical coordinates.

One can check that the general formula for curl of a vector field produces earlier formulas for the curl of a vector field in cylindrical and spherical coordinates also.