Some Clean and Almost Clean Von-Neumann-algebra-like rings

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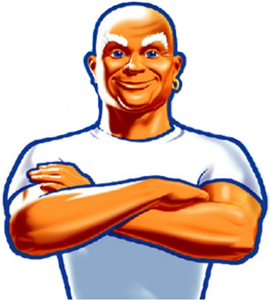


Mr. Clean meets the stars

How it all started?

At the conference at the Ohio U. in Athens, March 2005.

T. Y. Lam asked a question...

Lam's Question

Which von Neumann algebras are clean as rings?

- Background on VNAs and VNA-like rings.
- Background on clean rings.
- Introducing stars: *-cleanness.
- A class of VNAs is (almost) clean – idea of the proof.

The story of von Neumann Algebra begins...

John von Neumann's dream – to capture abstractly the concept of an algebra of observables in quantum mechanics.

- He constructed a non-commutative generalization of Hilbert space/ probability theory.

- Captured all the types of non-commutative measures that occur: (1) in classical theory, (2) in quantum systems (infinite in size or in degrees of freedom).
- Dimension function: Corresponds to normalized measure.

VNA - damsel in distress

H – Hilbert space

$\mathcal{B}(H)$ – bounded operators.

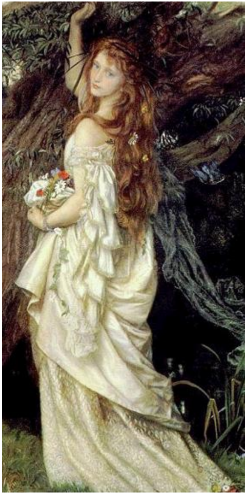
A **von Neumann algebra** $\mathcal{A}$ is a

1) $*$-closed unital subalgebra of $\mathcal{B}(H)$,

2a) equal to its double commutant $\mathcal{A}''$ (where $\mathcal{A}' = \{x \in \mathcal{B}(H) \mid ax = xa$ for all $a \in \mathcal{A}\}$)

equivalently

2b) weakly closed in $\mathcal{B}(H)$.



Five Types

finite, discrete	I_f	"sum" of I_n with μ on $\{1, 2, \ldots, n\}$
infinite, discrete	I_∞	μ on $\{1, 2, \ldots\}$
finite, continuous	II_1	μ on [0,1]
infinite, continuous	II_∞	μ on $\mathbb{R}$
very infinite	III	μ on $\{0, \infty\}$

Examples

I_n	$\mathcal{B}(H)$, $\dim(H) = n$ "finite matrices"
I_∞	$\mathcal{B}(H)$, $\dim(H) = \infty$ "infinite matrices"
II_1	group VNA for G "very infinite and nonabelian" G-invariant operators on Hilbert space $l^2(G)$ i.e. $f(xg) = f(x)g$
II_∞	"infinite matrices" over type II_1

Types I_f and II_1 are **finite von Neumann algebras.**

Von Neumann Algebra – in distress

"Von Neumann algebras are blessed with an excess of structure – algebraic, geometric, topological – so much, that one can easily obscure, through proof by overkill, what makes a particular theorem work."

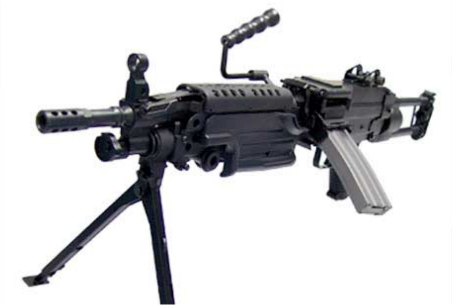
"If all the functional analysis is stripped away ... what remains should (be) completely accessible through algebraic avenues".

Berberian, S. K. Baer $*$-rings; Springer-Verlag, Berlin-Heidelberg-New York, 1972.

The overkill

The overkill that Berberian is referring to:

a mosquito



a machine gun

Law and Order – Enter the Rings

Von Neumann: studied lattice of projections. Led him to von Neumann regular rings.

Kaplansky's dream: to axiomatize (at least part of) the theory of VNAs. Followed similar path as von Neumann (looked at projections, idempotents, annihilators) – ended up defining Baer $*$-rings and AW^*-algebras.

The Knight in shining armor – Baer $*$-Ring

Baer ring – every right annihilator is generated by an idempotent.

Baer $*$-ring – every right annihilator is generated by a projection.

AW*-algebra – Baer $*$-ring that is also a C^*-algebra.

AW* generalizes VNA's; Baer $*$ generalizes AW*.

Finite "Von-Neumann-algebra-like" – Six Axioms

A1 A Baer $*$-ring R is **finite** if $x^*x = 1$ implies $xx^* = 1$ for all $x \in R$.

A2 R satisfies **existence of projections** and **unique positive square root** axioms.

A3 Partial isometries are addable.

A4 R is **symmetric**: for all $x \in R$, $1 + x^*x$ is invertible.

A5 There is a central element $i \in R$ such that $i^2 = -1$ and $i^* = -i$.

A6 R satisfies the **unitary spectral** axiom (if unitary u is such that $\text{ann}_r(1-u)$ is sufficiently small, then $1-u$ is locally invertible in a sequence of subrings that converge to R).

What do A1 – A6 bring?

Berberian: R can be embedded in a

unit-regular ring Q

satisfying A1–A6, having

the same projections

as R.

Moreover, R is **Ore** and $Q_{\mathrm{cl}}(R) = Q = Q_{\mathrm{max}}(R)$.

Original Mr. Clean – Keith Nicholson

Nicholson introduced clean rings in 1977.

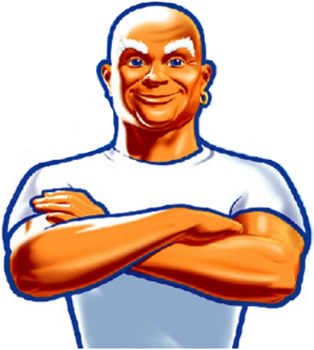
Ohio U., Zanesville, 2007.

Clean Rings

A ring R is **clean** if

every element = unit + idempotent

Additive version of unit-regular.



Examples: Unit-regular, local, semiperfect...

Non-examples: $\mathbb{Z}$, $R[x]$ for R commutative, not all regular ("Bergman example")...

Von-Neumann-algebra-like rings – "The Idea"

Recall that a VNA-like R has a **unit-regular** ring of quotients Q with **same projections**.

Start here: $a \in R$

↑

Go up: $a \in Q$

⟶

Have: $a = u + e$, e idempotent
Want: $a = u + p$, p projection

↓

$p \in R$ and so $u = a - p \in R$
End with: $a = u + p$ in R

Almost Clean Rings

A ring R is **almost clean** if

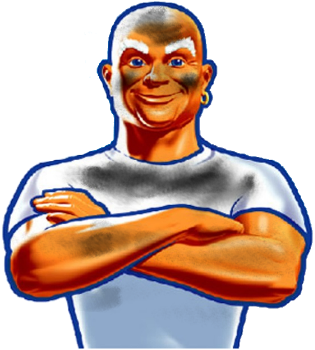
element = regular el. + idempotent

Additive version of (abelian) Rickart.

Examples: clean, abelian Rickart,...

$\mathbb{Z}$ is almost clean and not clean.

Non-examples: Couchot's paper.



Introducing stars

Von Neumann algebras (and von-Neumann-algebra-like rings) are $*$-rings (have involution).

Involution $*$: is additive, $(xy)^* = y^*x^*$, and $(x^*)^* = x$.

For $*$-rings **projections** take over the role of **idempotents.**

- Baer becomes Baer *-ring,
- Rickart becomes Rickart *-ring,
- regular becomes *-regular.
- **So clean should become...**

*-clean

A *-ring R is ***-clean** if

element = unit + projection

A *-ring R is **almost *-clean** if

element = regular el. + projection

Some corollaries:

1. (Almost) *-clean implies (almost) clean.
2. If R is *-clean, $M_n(R)$ is *-clean.
3. *-regular and abelian implies *-clean.

Von-Neumann-algebra-like rings are almost clean

Type I_n Baer *-rings that satisfying A2:

- R *-isomorphic to $M_n(Z(R))$,
- $Z(Q)$ is abelian and *-regular so it is *-clean.
- Thus, $M_n(Z(Q)) \cong Q$ is *-clean.
- R is almost *-clean.

Type I_f Baer *-rings that satisfying A2–A6:

- There are central orthogonal projections p_n such that $p_n R$ is of type I_n.
- Q is the direct product of $p_n Q$.
- Rings $p_n Q$ are *-clean so Q is *-clean.
- R is **almost *-clean.**

Corollary: If R is regular, then $Q = R$ and R is *-clean.

Back to Lam's question

Corollary.

An AW^*-algebra (in particular von Neumann algebra) of type I_f is almost $*$-clean.

If it is regular, then it is $*$-clean.

Other types?

Example. Let $G = \prod_n G_n$, where G_n are finite.
Then $\mathcal{N}G$ is *-clean.

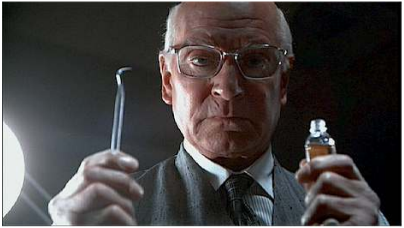
- If just finitely many G_n are not abelian, G is type I_f.
- If not, then $\mathcal{N}G$ **is not type** I_f.

Questions

1. **Other types?** Are type II_1 von Neumann algebras (almost) clean? Good start: consider $\mathcal{N}G$ for $G = \mathbb{Z} * \mathbb{Z}$.

2. **Clean and not *-clean?** Is there a *-ring that is clean but not *-clean? Known: No such example for abelian Rickart *-rings.

3. **Strongly clean?** Can "clean" (or "*-clean") be replaced by "strongly clean"?



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[Va1] L. Vaš, Dimension and Torsion Theories for a Class of Baer $*$-Rings, Journal of Algebra 289 (2005) no. 2, 614–639.

[Va2] L. Vaš, $*$-Clean Rings; Some Clean and Almost Clean Baer $*$-rings and von Neumann Algebras, submitted for publication.

Preprints of my papers are available on

http://www.usp.edu/∼lvas

and on arXiv.