Extending higher derivations to rings and modules of quotients

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Motivation

Split personality:

teaching	$\Rightarrow$	calculus
research	$\Rightarrow$	ring theory

A cure: Bland's "Differential Torsion Theories" (JPAA 204, 2006).

Rings with derivations

A **derivation** on a ring R is an additive mapping $\delta : R \to R$ such that

$$\boxed{\delta(rs) = \delta(r)s + r\delta(s)}$$

for all $r, s \in R$.

A δ-**derivation** on a right R-module M is an additive mapping $d : M \to M$ such that

$$\boxed{d(xr) = d(x)r + x\delta(r)}$$

for all $x \in M$ and $r \in R$.

Some examples: Inner derivations $d_a(x) = ax - xa$, "usual" derivation on polynomial rings, special actions of Hopf algebra on an algebra, Lie algebras...

Generalization – Higher derivations

A **higher derivation (HD)** on R is an indexed family $\Delta = \{\delta_n\}$ of additive maps δ_n such that δ_0 is the identity mapping on R and

$$\boxed{\delta_n(rs) = \sum_{i=0}^{n} \delta_i(r)\delta_{n-i}(s)}$$

for all $r, s \in R$.

For a HD Δ, a **higher Δ-derivation** (Δ-HD) on a right R-module M is an indexed family $D = \{d_n\}$ of additive maps d_n such that d_0 is the identity mapping on M and

$$\boxed{\delta_n(xr) = \sum_{i=0}^{n} d_i(x)\delta_{n-i}(r)}$$

for all $x \in M$ and $r \in R$.

Example: $\{\delta^n/n!\}$ is a HD for any ring derivation δ.

Objective – when a derivation/HD on ring (module) extends to a ring (module) of quotients.

Dr. Jekyll version: Having the product rule, when can we have the quotient rule also?

Mr. Hyde version:

rings/modules of quotients	$\Leftrightarrow$	torsion theories
extensions possible	$\Leftrightarrow$	derivations agree with torsion theories

Torsion theories – unified treatment of different rings of quotients

A **torsion theory** for a ring R is a pair $\tau = (\mathcal{T}, \mathcal{F})$ of classes of R-modules such that

i) $\text{Hom}_R(T, F) = 0$, for all $T \in \mathcal{T}$ and $F \in \mathcal{F}$.
ii) $\mathcal{T}$ and $\mathcal{F}$ are maximal classes with property i).

τ is **hereditary** if $\mathcal{T}$ is closed under taking submodules.

For every R-module M, we have:

1. **Torsion submodule** $\mathcal{T}M$, largest submodule that belongs to $\mathcal{T}$,
2. **Torsion-free quotient** $\mathcal{F}M = M/\mathcal{T}M$,
3. **Filter** $\mathfrak{F} = \{ I \text{ right ideal } \mid R/I \in \mathcal{T} \}$. Think "big" ideals.

Examples of torsion theories

1. **Classical** theory of a right Ore ring R:

   $$\boxed{\mathcal{T}M = \ker(M \to M \otimes_R Q^r_{\mathrm{cl}}).}$$

   Thus,

   $$m \in \mathcal{T}M \text{ iff } mr = 0 \text{ for some regular } r \in R.$$

   For $R = \mathbb{Z}$, torsion is "torsion" and torsion-free is "torsion-free".
2. **Lambek**: Largest hereditary torsion theory in which R is torsion-free.
3. **Goldie**: nonsingular = torsion-free.
4. **"Tensoring"** torsion theory τ_S. If $R \leq S$,

   $$\boxed{\mathcal{T}M = \ker(M \to M \otimes_R S).}$$

   τ_S is hereditary if S is left flat.

Rings and modules of quotients

If τ is hereditary, define:

- **Right ring of quotients.** $R_{\mathfrak{F}} = \varinjlim_{I \in \mathfrak{F}} \mathrm{Hom}_R(I, \mathcal{F}R)$

Think: hull, closure. For R torsion-free, $q \in R_{\mathfrak{F}}$ iff

$$qI \subseteq R \text{ for some } I \in \mathfrak{F}.$$

- **Right module of quotients.** $M_{\mathfrak{F}} = \varinjlim_{I \in \mathfrak{F}} \mathrm{Hom}_R(I, \mathcal{F}M)$

The natural (localization) map $q_M : M \to M_{\mathfrak{F}}$ composition of projection $M \to \mathcal{F}M$ and injection $\mathcal{F}M \to M_{\mathfrak{F}}$.

If M is torsion-free, q_M is given by

$$m \mapsto \text{left multiplication by } m.$$

Examples

1. For classical, the ring of quotients is Q^r_{cl}.
2. For Lambek, the ring of quotients is $Q^r_{\max}$.
3. For Goldie, the ring of quotients is $E(R/Z(R))$.
4. **Perfect quotients.** If

$$\boxed{M_{\mathfrak{F}} \cong M \otimes_R R_{\mathfrak{F}}}$$

for every M, then the torsion theory is said to be **perfect**.

This is a way to **generalize the classical torsion theory** in cases when the ring might not be right Ore.

Differential torsion theories

A torsion theory is **differential** if *any* of the following holds:

- Every derivation d on any M extends uniquely to $M_{\mathfrak{F}}$.

$$\begin{array}{ccc} M & \xrightarrow{q_M} & M_{\mathfrak{F}} \\ \downarrow d & & \downarrow d \\ M & \xrightarrow{q_M} & M_{\mathfrak{F}} \end{array}$$

- $\mathfrak{F}$ is δ-**invariant** (for every δ and $I \in \mathfrak{F}$ there is $J \in \mathfrak{F}$ with $\delta(J) \subseteq I$).
- $\mathcal{T}$ is d-**invariant** (for every d on any M, $d(\mathcal{T}M) \subseteq \mathcal{T}M$).

Higher differential torsion theories

A torsion theory is **higher differential** if any of the following holds:

- Every HD $\{d_n\}$ on any M extends uniquely to $M_{\mathfrak{F}}$.

$$\begin{array}{ccc} M & \xrightarrow{q_M} & M_{\mathfrak{F}} \\ \downarrow d_n & & \downarrow d_n \\ M & \xrightarrow{q_M} & M_{\mathfrak{F}} \end{array}$$

- $\mathfrak{F}$ is δ_n-**invariant** for every n.
- $\mathcal{T}$ is d_n-**invariant** for every n.

Overview of past results

Golan (1981). Derivation extensions on module of quotients is possible for

- all modules whose torsion modules are d-invariant.

Bland (2006). Derivation extensions on module of quotients exists and it is unique **iff** torsion theory is d-invariant.

Known: The classical torsion theory of an right Ore ring is differential.

Overview of past results (cont.)

Theorem [V.]

1. Lambek
2. Goldie
3. Any perfect

torsion theories are differential (2007), higher differential (2008).

Are all hereditary torsion theories differential?

Recently answered by Christian Lomp: Yes!

Lomp: interested in actions of Hopf algebra H on an algebra A and conditions when such action extends to a localization $A_{\mathfrak{F}}$. Goal:
If A is a semiprime algebra and H a semisimple Hopf algebra acting on A, is the smash product $A\sharp H$ semiprime again?

S. Montgomery (1993)

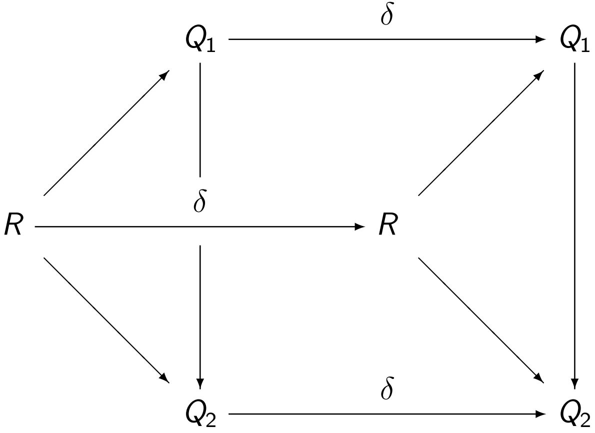
- Action extends if $\mathfrak{F}$ is invariant for such action.
- If H is pointed then every $\mathfrak{F}$ is invariant.

Lomp's proof translates to HDs too (using induction). So **every hereditary torsion theory is HD**.

New Directions

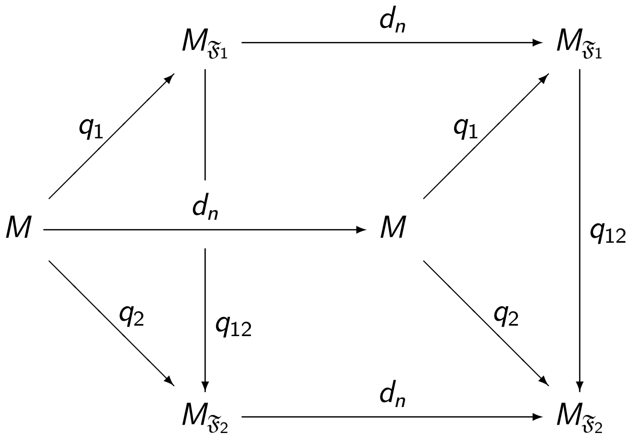
Q_1 and Q_2 two rings of quotients with $Q_1 \subseteq Q_2$.

Question: If δ extends to both Q_1 and Q_2, does the following diagram commute? I.e. do extensions **agree**?



Generalization to modules, to HDs. Answers.

If $\mathfrak{F}_1 \subseteq \mathfrak{F}_2$ and a $\{\delta_n\}$-HD $D = \{d_n\}$ extends such that:



for every n then the extensions of D on $M_{\mathfrak{F}_1}$ and $M_{\mathfrak{F}_2}$ **agree.**

HDs agree. Corollaries

Proposition [V.]. Every D on M extends both to $M_{\mathfrak{F}_1}$ and $M_{\mathfrak{F}_2}$ such that the extensions agree.

Corollary. The extensions to the modules of quotients with respect to the following agree.

1.	any hereditary and faithful t.t.	Lambek t.t
2.	any hereditary t.t	Goldie t.t

Symmetric rings of quotients

Symmetric torsion theories – mimic the commutative case.

Conference in Athens, Ohio, 2008.

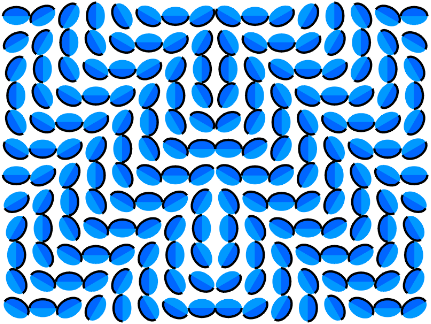
- Barbara Osofsky – "There is no reason that if something happens to the right side it would also happen to the left side."
- Keith Nicholson – "It is better to be two sided."
- T. Y. Lam – story about doctor's advise.

Solution: symmetric rings of quotients.

Symmetric rings of quotients (cont.)

- symmetric Martindale rings of quotients,
- Symmetric version of $Q^r_{\max}$ called **the maximal symmetric ring of quotient.** Studied by Lanning in 1990s and Ortega in 2000s.

Schelter's (1970s) – work on symmetric rings of quotients that parallels Gabriel's work on right rings of quotients.
Ortega (2000s) – symmetric modules of quotients.



Symmetric filters and modules of quotients

Schelter's idea.

$$\text{Symmetric for } R = \text{Right for } R \otimes_{\mathbb{Z}} R^{op}.$$

Symmetric Filter: Start with $\mathfrak{F}_l$ left and $\mathfrak{F}_r$ right filters. Define $\mathfrak{F}$ = right ideals of $R \otimes_{\mathbb{Z}} R^{op}$ containing an ideal of the form

$$J \otimes R^{op} + R \otimes I$$

$I \in \mathfrak{F}_l$ and $J \in \mathfrak{F}_r$. Corresponds to torsion theory τ for which

$$\mathcal{T}(M) = \mathcal{T}_l(M) \cap \mathcal{T}_r(M).$$

Ortega – symmetric modules of quotients:

$$M_{\mathfrak{F}} = \varinjlim_{K \in \mathfrak{F}} \mathrm{Hom}(K, \tfrac{M}{\mathcal{T}(M)})$$

Differentiability of symmetric torsion theories

Proposition [V.]

1. (2007) Symmetric versions of Golan's and Bland's results on differential torsion theories hold.

2. (2008) Symmetric versions of Golan's and Bland's results on higher differential torsion theories hold.

3. (2007) The extensions of two symmetric theories agree.

Further Generalizations

1. **Generalized Derivations.** If δ is a ring derivation and f an additive map $R \to R$, f is a **generalized δ-derivation** if

$$f(ab) = f(a)b + a\delta(b)$$

2. **Skew Derivations.** If α and β are monomorphisms $R \to R$, an (α, β)**-derivation** is an additive map $\delta : R \to R$ such that

$$\delta(ab) = \delta(a)\alpha(b) + \beta(a)\delta(b)$$

Open problem

After Lomp's result, we know that every hereditary torsion theory is differential. Let us define that any (not necessarily hereditary) torsion theory is

- differential iff $\boxed{d(\mathcal{T}M) \subseteq \mathcal{T}M}$,
- HD iff $\boxed{d_n(\mathcal{T}M) \subseteq \mathcal{T}M}$ for every n.

Is every torsion theory differential (HD)?

References. Preprints of my papers are available on

http://www.usp.edu/∼lvas
and on arXiv.