Classes of Almost Clean rings

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Mr. Clean Mr. Almost Clean

The story of clean rings begins...

Original Mr. Clean – Keith Nicholson

Nicholson introduced clean rings in 1977.

Ohio U., Zanesville, 2007.

Clean Rings

A ring R is **clean** if

every element = unit + idempotent

Additive version of unit-regular.

Examples: Unit-regular, local, semiperfect...

Non-examples: $\mathbb{Z}$, $R[x]$ for R commutative, not all regular ("Bergman example")...

How it all started for me?

At the conference at Ohio Univ., in Athens, OH, March 2005.

T. Y. Lam asked a question...

Which von Neumann algebras are clean as rings?

Von Neumann algebras – "The Idea"

A VNA R has a **unit-regular** ring of quotients Q with the **same projections**.

Go up: $a \in Q$

Have: $a = u + e$, e idempotent
Want: $a = u + p$, p projection

Start here: $a \in R$

$p \in R$ and so $u = a - p \in R$
End with: $a = u + p$ in R

Two problems: (1) want idempotent, have projection; (2) u may not be unit in R.

Fix for (2) – Almost Clean Rings

A ring R is **almost clean** if

element = regular el. + idempotent

Additive version of (abelian) Rickart.

Examples: clean, abelian Rickart,...

$\mathbb{Z}$ is almost clean and not clean.

Non-examples: Couchot's paper.

Fix for (1) – Introducing stars

Von Neumann algebras are $*$-rings (have involution).
Involution $*$: is additive, $(xy)^* = y^*x^*$, and $(x^*)^* = x$.

For $*$-rings **projections** take over the role of **idempotents.**

- Baer becomes Baer *-ring,
- Rickart becomes Rickart *-ring,
- regular becomes *-regular.
- **So clean should become...**

*-clean

A *-ring R is ***-clean** if

element = unit + projection

A *-ring R is **almost *-clean** if

element = regular el. + projection

Using this concept, I could show that:

An AW^*-algebra (in particular von Neumann algebra) of type I_f is almost $*$-clean.

How to extend ”The Idea“?

It works for **any** ring that has a
<u>clean overing with the same idempotents</u>.

Start here:
$a \in R$

$\uparrow$

Go up:
$a \in Q$

$\longrightarrow$

Have: $a = u + e$, e idempotent, u unit

$\downarrow$

$e \in R$ and so $u = a - e \in R$
End with: $a = u + e$, u regular.

Exploring "The Idea" with Evrim Akalan

- R r. quasi-continuous $\Rightarrow$

  $E(R)$ and R have same idempotents.

- R r. quasi-continuous + r. nonsingular $\Rightarrow$

  $E(R) = Q^r_{\max}(R)$ is clean, has same idempotents as R.

So,

R is almost clean.

Less obvious: **works for modules as well.**

Work with Evrim continued: C1 – C3 conditions

- R. quasi-continuous = C1 + C3.
- C1 = r. CS (or r. extending).

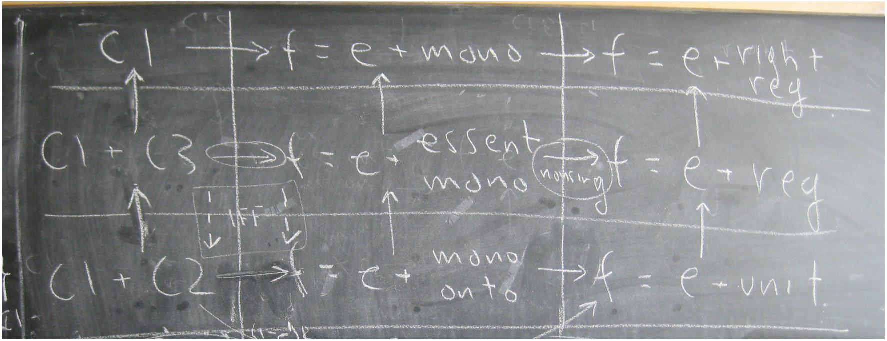
  CS is for "complements are summands".
- R. continuous = C1 + C2.

  C2 $\Rightarrow$ C3 so,

right continuous $\Rightarrow$
r. quasi-continuous $\Rightarrow$
right CS.

C1–C3 situation. Module case.

Let $f \in \mathrm{End}(M)$ be arbitrary, $e \in \mathrm{End}(M)$ be idempotent.



C1, C3 situation. Ring case.

Let $a \in R$ be arbitrary, $e \in R$ be idempotent.

$$
\begin{array}{ccc}
\begin{array}{c}\text{right CS} \\ \cancel{\text{r. quasi-continuous}}\end{array} & \begin{array}{c}\text{right nonsingular} \\ \longrightarrow\end{array} & a = e + \text{ regular} \\
\uparrow & & \uparrow \\
\text{right continuous} & \longrightarrow & a = e + \text{ unit}
\end{array}
$$

Thus,

Right CS + right nonsingular $\Rightarrow$ almost clean.

Corollary:

Finite AW^*-algebras are almost clean.

Camillo-Khurana Theorem and special clean

Camillo-Khurana:

$$\boxed{\begin{array}{c}\text{unit-regular}\\ a = eu\end{array}} \quad \longleftrightarrow \quad \boxed{\begin{array}{c}\text{special clean}\\ a = e + u, \quad aR \cap eR = 0\end{array}}$$

Almost clean – Rickart connection:

$$\boxed{\begin{array}{c}\text{Rickart}\\ a = er\end{array}} \quad \overset{\text{abelian}}{\longleftrightarrow} \quad \boxed{\begin{array}{c}\text{special almost clean}\\ a = e + r, \quad aR \cap eR = 0\end{array}}$$

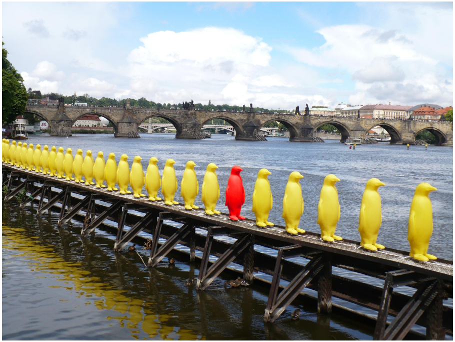
Cool Corollary

Known:

1. Left and right nonsingular, left and right CS $\Rightarrow$ Rickart.
2. Right nonsingular, right CS ring $\Rightarrow$ right Rickart.

$$\begin{array}{cccc} \text{r. nonsingular} + & \text{r. continuous} & \longrightarrow & \text{regular} \\ & \downarrow & & \downarrow \\ \textbf{r. nonsingular} + & \textbf{r. quasi-continuous} & \longrightarrow & \textbf{Rickart} \end{array}$$

Uniqueness



Rickart $a = er$	abelian r. quasi-cont. $\longleftrightarrow$	uniquely special almost clean $a = e + r$ unique, $aR \cap eR = 0$
$\uparrow$		$\uparrow$
unit-regular $a = eu$	abelian $\longleftrightarrow$	uniquely special clean $a = e + u$ unique, $aR \cap eR = 0$

Adapting the results to *-rings

$$\begin{array}{ccc} \boxed{\text{Rickart *-ring}} & \overset{\text{abelian}}{\longleftrightarrow} & \boxed{\text{special almost *-clean}} \\ \uparrow & & \uparrow \\ \boxed{\text{*-regular}} & \overset{\text{abelian}}{\longleftrightarrow} & \boxed{\text{special *-clean}} \end{array}$$

Questions

1. **Weaken abelian?** Cannot completely drop it: $\mathrm{End}(V)$, V inf. dim., is Rickart and not special almost clean.

2. **Condition (C)**: Any essential mono is an iso.

$$\begin{tikzcd}
\boxed{\text{quasi-continuous}} \arrow[rr, "\text{nonsingular}"] \arrow[d, shift left, "\text{iff (C)}"] & & \boxed{\text{almost clean}} \arrow[d, shift left, "\text{iff (C)?}"] \\
\boxed{\text{continuous}} \arrow[u, shift left] \arrow[rr] & & \boxed{\text{clean}} \arrow[u, shift left]
\end{tikzcd}$$

3. **VNAs?** For AW^*-algebras:

$$\begin{tikzcd}
\text{finite, type } I \arrow[r] \arrow[d] & \text{finite} \arrow[d] \\
\text{almost } *\text{-clean} \arrow[r] & \text{almost clean}
\end{tikzcd}$$

Some references

L. Vaš, $*$-Clean Rings; Some Clean and Almost Clean Baer $*$-rings and von Neumann Algebras, *Journal of Algebra*, 324 (12) (2010), 3388 – 3400.

E. Akalan, L. Vaš, Classes of almost clean rings, *Algebras and Representation Theory*, in print.

Preprints of these papers are available on

http://www.usciences.edu/$\sim$lvas and on arXiv.