Dimension and Torsion Theories for a Class of Baer *-Rings

Lia Vaš

Department of Mathematics, Physics and Computer Science, University of the Sciences in Philadelphia, 600 S. 43rd St., Philadelphia, PA 19104

Abstract

Many known results on finite von Neumann algebras are generalized, by purely algebraic proofs, to a certain class $\mathcal{C}$ of finite Baer *-rings. The results in this paper can also be viewed as a study of the properties of Baer *-rings in the class $\mathcal{C}$.

First, we show that a finitely generated module over a ring from the class $\mathcal{C}$ splits as a direct sum of a finitely generated projective module and a certain torsion module. Then, we define the dimension of any module over a ring from $\mathcal{C}$ and prove that this dimension has all the nice properties of the dimension studied in [11] for finite von Neumann algebras. This dimension defines a torsion theory that we prove to be equal to the Goldie and Lambek torsion theories. Moreover, every finitely generated module splits in this torsion theory.

If R is a ring in $\mathcal{C}$, we can embed it in a canonical way into a regular ring Q also in $\mathcal{C}$. We show that $K_0(R)$ is isomorphic to $K_0(Q)$ by producing an explicit isomorphism and its inverse of monoids $\mathrm{Proj}(P) \to \mathrm{Proj}(Q)$ that extends to the isomorphism of $K_0(R)$ and $K_0(Q)$.

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1 Introduction

This paper is motivated by the remark of Sterling K. Berberian from the introduction to his book [2] on Baer *-rings: "The subject of Baer *-rings has its roots in von Neumann theory of 'rings of operators' (now called von Neumann algebras) ... Von Neumann algebras are blessed with an excess of

Email address: l.vas@usip.edu (Lia Vaš).

structure – algebraic, geometric, topological – so much, that one can easily obscure, through proof by overkill, what makes a particular theorem work." Relying just on algebra, we generalize some results from [11], [17] and [18] to a certain class of finite Baer-* rings that contains the class of all finite von Neumann algebras. The proofs in [11] rely on some of the geometric or topological structure of finite von Neumann algebras. All proofs in this paper rely strictly on algebraic properties. We follow Berberian's idea: "if all the functional analysis is stripped away ... what remains should (be) completely accessible through algebraic avenues".

We impose some restrictions onto the Baer *-ring R that are sufficient for defining the dimension function, the regular ring of R, and ensuring that all the matrix algebras over R are sufficiently nice (we need the lattice of projections of all the matrix algebras to be complete). In Section 2, we list the axioms imposed onto the Baer *-ring. These axioms are the same ones Berberian uses in [2] in order to ensure that the matrix rings over R are Baer *-rings. Baer *-rings that satisfy those axioms form a class that we shall denote by $\mathcal{C}$. Every finite AW^*-algebra (so a finite von Neumann algebra in particular) is in $\mathcal{C}$.

As in [17], torsion theories are used to study the modules over the rings of interest. In Section 3, we recall the definition of an arbitrary torsion theory and some related notions. Then we list the examples of torsion theories we shall use in the rest of the paper (Lambek, Goldie, classical, etc.).

In Section 4, we prove the main results. First, we show that a finitely generated module over a ring from class $\mathcal{C}$ splits as a direct sum of a finitely generated projective module and a certain torsion module (Theorem 11). This generalizes an analogous result for finite von Neumann algebras proven in [11].

Secondly, if R is a ring from the class $\mathcal{C}$, we prove (Theorem 17) that a dimension of *any* R-module can be defined so that it has all the nice properties of the dimension defined in [11] (i.e. we prove that Theorem 0.6 from [11] holds for the class $\mathcal{C}$). This dimension defines a torsion theory that proves to be equal to the Goldie and Lambek torsion theories and every finitely generated module splits in this torsion theory (Theorem 19). In Theorem 23, we demonstrate how the torsion theories reflect the ring-theoretic properties of R.

If R is a finite von Neumann algebra, our construction gives us precisely the central-valued dimension considered in [10] for finitely presented modules. Moreover, Theorem 17, guarantees that we can extend the definition to any R-module. Thus, in the case of finite von Neumann algebras, we can define the real-value dimension (as in [11] or [12]) and the central-valued dimension. In Section 5, we show that both dimensions define the same torsion theory (Corollary 24). Thus, our Theorem 19 generalizes Proposition 4.2 from [17].

In Section 5 we also generalize Theorem 5.2 from [17] and show that K_0 of R

is isomorphic to K_0 of the regular ring Q of R. Specifically, in Corollary 25, we show that the map $\mu : \text{Proj}(R) \to \text{Proj}(Q)$ given by $[P] \mapsto [P \otimes_R Q]$ is the isomorphism of monoids with the inverse $[S] \mapsto [S \cap R^n]$ if S is a direct summand of Q^n, and that μ induces the isomorphism $K_0(R) \cong K_0(Q)$.

2 Class $\mathcal{C}$ of Baer *-Rings

2.1 Basics.

Let R be a ring. R is a **-ring* (or *ring with involution*) if there is an operation $* : R \to R$ such that

$$(x+y)^* = x^* + y^*, \quad (xy)^* = y^*x^*, \quad (x^*)^* = x \quad \text{for all } x, y \in R.$$

If R is also an algebra over k with involution $*$, then R is an **-algebra* if $(ax)^* = a^*x^*$ for $a \in k$, $x \in R$.

An element p of a *-ring R is called a *projection* if p is idempotent ($p^2 = p$) and self-adjoint ($p^* = p$). There is a partial ordering on the set of projections of R defined by $p \leq q$ iff $p = pq$. The following conditions are equivalent $p \leq q$; $p = qp$; $pR \subseteq qR$; $Rp \subseteq Rq$.

There is an equivalence relation on the set of projections of a *-ring R defined by $p \sim q$ iff $w^*w = p$ and $ww^* = q$ for some $w \in R$. Such an element w is called a *partial isometry.*

Define another relation on the set of projections of a *-ring $R : p \preceq q$, if p is equivalent to a subprojection of q (i.e. $p \sim r \leq q$ for some projection r). The relation $\preceq$ is reflexive and transitive.

A *Rickart *-ring* is a *-ring R such that, for every $x \in R$, the right annihilator $\text{ann}_r(x) = \{y \in R \mid xy = 0\}$ is generated by a projection p

$$\text{ann}_r(x) = pR.$$

The projection p from the above definition is unique. Also, if R is a Rickart *-ring, the left annihilator of each element x of R is Rq for some (unique) projection q since $\text{ann}_l(x) = (\text{ann}_r(x^*))^*$.

Every element x of a Rickart *-ring R determines a unique projection p such that $xp = x$ and $\text{ann}_r(x) = \text{ann}_r(p) = (1-p)R$ and a unique projection q such that $qx = x$ and $\text{ann}_l(x) = \text{ann}_l(q) = R(1-q)$. p is called the *right projection* *of* x and is denoted by $\mathrm{RP}(x)$. q is the *left projection of* x and is denoted by $\mathrm{LP}(x)$.

The involution in every Rickart *-ring is *proper*: $x^*x = 0$ implies $x = 0$ (Proposition 2, p. 13, [2]). From this condition it easily follows that a Rickart *-ring is a nonsingular ring (since in a proper *-ring R $\mathrm{ann}_r(x) \cap x^*R = 0$ for all $x \in R$). Let us also recall that a *-ring is called *n-proper* if $x_1^*x_1 + x_2^*x_2 + \ldots + x_n^*x_n = 0$ imply $x_1 = x_2 = \ldots = x_n = 0$. It is easy to see that a *-ring R is n-proper if and only if the ring of $n \times n$ matrices over R is proper.

The projections in a Rickart *-ring form a lattice (Proposition 7, p. 14 in [2]).

A *Rickart C^*-algebra* is a C^*-algebra (complete normed complex algebra with involution such that $||a^*a|| = ||a||^2$) that is also a Rickart *-ring.

A *Baer *-ring* is a *-ring R such that, for every nonempty subset S of R, the right annihilator $\mathrm{ann}_r(S) = \{y \in R \mid xy = 0 \text{ for all } x \in S\}$ is generated by a projection p

$$\mathrm{ann}_r(S) = pR.$$

Since $\mathrm{ann}_l(S) = (\mathrm{ann}_r(S^*))^*$, it follows that $\mathrm{ann}_l(S) = Rq$ for some projection q.

A *-ring is Baer *-ring if and only if it is Rickart *-ring and the lattice of projections is complete (Proposition 1, p. 20, [2]).

If R is a Baer *-ring and $\{p_i | i \in I\}$ is a nonempty family of projections in R, then

$$(\inf_{i \in I} p_i)R = \bigcap_{i \in I} p_i R \tag{1}$$

This is an easy exercise (Exercise 1A in [2]).

A C^*-algebra that is a Baer *-ring is called an *AW^*-algebra.*

If H is a Hilbert space and $\mathcal{B}(H)$ the algebra of bounded operators on H, then $\mathcal{B}(H)$ is an AW^*-algebra. If A is a $*$-subalgebra of $\mathcal{B}(H)$ such that $A = A''$ where A' is the commutant of A, then A is called a *von Neumann algebra.*

A von Neumann algebra is an AW^*-algebra ([2], Proposition 9). The converse is not true (see [5]) namely there is an AW^*-algebra that cannot be represented as a von Neumann algebra on any Hilbert space.

2.2 Dimension.

We now focus our attention on a special class of Baer *-rings.

(A1) A Baer *-ring R is *finite* if $x^*x = 1$ implies $xx^* = 1$ for all $x \in R$.

The Baer *-ring R satisfies the *generalized comparability (GC)* axiom if: for every two projections p and q, there is a central projection c such that

$$cp \preceq cq \quad \text{and} \quad (1-c)q \preceq (1-c)p.$$

We are interested in finite Baer *-rings with (GC) because of the dimension function that we can define on the set of all projections. Let R be a finite Baer $*$-ring with (GC). Let Z denote the center of R. The projection lattice $P(Z)$ of Z is a complete Boolean algebra and, as such, may be identified with the Boolean algebra of closed-open subspaces of a Stonian space X. The space X can be viewed as the set of maximal ideals in $P(Z)$; $p \in P(Z)$ can be identified with the closed-open subset of X that consist of all maximal ideals that exclude p.

The algebra $C(X)$ of continuous complex-valued functions on X is a commutative AW^*-algebra. An element $p \in P(Z)$ can be viewed as an element of $C(X)$ by identifying p with the characteristic function of the closed-open subset of X to which p corresponds.

If R is an AW^*-algebra, then Z is the closed linear span of $P(Z)$ and we may identify Z with $C(X)$.

For more details on this construction, see [2].

Theorem 1 *If R is a finite Baer $*$-ring that satisfies (GC), then there exist unique a function $d : P(R) \to C(X)$ such that*

(D1) $p \sim q$ *implies* $d(p) = d(q)$,
(D2) $D(p) \geq 0$,
(D3) $d(c) = c$ *for every* $c \in P(Z)$,
(D4) $pq = 0$ *implies* $d(p+q) = d(p) + d(q)$.

The function D will be called the dimension function. *It satisfies the following properties:*

(D5) $0 \leq d(p) \leq 1$,
(D6) $d(cp) = cd(p)$ *for every* $c \in P(Z)$,
(D7) $d(p) = 0$ *iff* $p = 0$,
(D8) $p \sim q$ *iff* $d(p) = (q)$,
(D9) $p \preceq q$ *iff* $d(p) \leq d(q)$,
(D10) If p_i is an increasingly directed family of projections with supremum p, then $d(p) = \sup d(p_i)$,
(D11) If p_i is an orthogonal family of projections with supremum p, then $d(p) = \sum d(p_i)$.

Chapter 6 of [2] is devoted to the proof of this theorem.

2.3 The Regular Ring of R

Next, we would like to be able to enlarge our Baer *-ring to a regular Baer *-ring. Recall that a ring Q is *regular* if, for every $x \in Q$ there is $y \in Q$ such that $xyx = x$. Equivalently, a ring is regular if every right (left) module is flat. A regular ring can also be characterized by the condition that all finitely presented right modules are projective.

If Q is a regular Rickart *-ring and $x \in Q$, then $xQ = pQ$ for some projection $p \in Q$ (Proposition 3, p. 229 in [2]).

Every finite von Neumann algebra A can be enlarged (in a canonical way) to a regular ring Q of certain unbounded operators affiliated (densely defined and closed) with A. In chapter 8 of [2], this construction is generalized for a certain class of finite Baer *-rings. The conditions that we must impose onto a finite Baer *-ring R in order to be able to follow this construction are the following:

(A2) R satisfies *existence of projections (EP)-axiom:* for every $0 \neq x \in R$, there exist an self-adjoint $y \in \{x^*x\}''$ such that $(x^*x)y^2$ is a nonzero projection;
R satisfies the *unique positive square root (UPSR)-axiom:* for every $x \in R$ such that $x = x_1^*x_1 + x_2^*x_2 + \ldots + x_n^*x_n$ for some n and some $x_1, x_2, \ldots, x_n \in R$ (such x is called *positive*), there is a unique $y \in \{x^*x\}''$ such that $y^2 = x$ and y positive. Such y is denoted by $x^{1/2}$.
(A3) Partial isometries are addable.
(A4) R is *symmetric*: for all $x \in R$, $1 + x^*x$ is invertible.
(A5) There is a central element $i \in R$ such that $i^2 = -1$ and $i^* = -i$.
(A6) R satisfies the *unitary spectral (US)-axiom:* for each unitary $u \in R$ such that $\mathrm{RP}(1-u) = 1$, there exist an increasingly directed sequence of projections $p_n \in \{u\}''$ with supremum 1 such that $(1-u)p_n$ is invertible in p_nRp_n for every n.
(A7) R satisfies the *positive sum (PS)-axiom;* if p_n is orthogonal sequence of projections with supremum 1 and $a_n \in R$ such that $0 \leq a_n \leq f_n$, then there is $a \in R$ such that $ap_n = a_n$ for all n.

By Theorem 1, p. 80, from [2], the generalized comparability (GC) follows from (A2). Thus, all the Baer *-rings satisfying (A1) and (A2) have the dimension function.

In the presence of (A2), the notion of positivity can be simplified so that $x \in R$ is positive if and only if $x = y^*y$ for some $y \in R$.

(A2) – (A5) imply that R is n-proper (Lemma, p. 227 in [2]).

Theorem 2 *If R is a Baer *-ring satisfying (A1)– (A7), then there is a regular Baer *-ring Q satisfying (A1) – (A7) such that R is *-isomorphic to a *-subring of Q, all projections, unitaries and partial isometries of Q are in R, and Q is unique up to *-isomorphism.*

If R is also an algebra over involutive field F, then so is Q.

This result is contained in Theorem 1, p. 217, Theorem 1 and Corollary 1 p. 220, Corollary 1, p. 221, Theorem 1 and Corollary 1 p. 223, Proposition 3 p. 235, Theorem 1 p. 241, Exercise 4A p. 247 in [2].

A ring Q as in Theorem 2 is called the *regular ring of Baer *-ring R.*

Proposition 3 *If R is a Baer *-ring satisfying (A1)– (A7) with Q its regular ring, then*

(1) Q is the classical ring of quotients $Q_{\mathrm{cl}}(R)$ of R.
(2) Q is the maximal ring of quotients $Q_{\max}(R)$ of R and, thus, self-injective and equal to the injective envelope $E(R)$ of R.

PROOF. 1. First, let us show that $x \in R$ is a non-zerodivisor if and only if it is invertible in Q. It is easy to see that $x \in R$ that is invertible in Q cannot have nontrivial left and right annihilators. Conversely, if x does not have a right inverse, then the right annihilator of x in Q is nontrivial. Since Q is Rickart, there is a nontrivial projection p that generates the right annihilator. But p is in R by Theorem 2. Thus, $xp = 0$. The proof for the left handed version is similar.

By Proposition 5, p. 241 of [2], for every $x \in Q$ there is a partial isometry $w \in R$ such that $x = w(x^*x)^{1/2}$. $(x^*x)^{1/2}$ is positive and thus self-adjoint. By Proposition 2, p. 228 [2], there is a unitary u such that $1-u$ is invertible in Q and $(x^*x)^{1/2} = i(1+u)(1-u)^{-1}$. But u is in R by Theorem 2, so $1-u \in R$. Thus, every element $x \in Q$ can be represented as the right fraction $x = at^{-1}$, $a = wi(1+u) \in R$, $t = 1-u \in R$.

This proves the right Ore condition for R. Applying involution, we have the left Ore condition. Since the non-zerodivisors of R are invertible in Q, the isomorphism of Q and $Q_{\mathrm{cl}}(R)$ exists because of the universal property of $Q_{\mathrm{cl}}(R)$. It is easy to check that the isomorphism is a *-isomorphism.

2. In [13], a *-extension of a finite Baer *-ring is constructed that is (under suitable assumptions) *-isomorphic to the maximal ring of quotients ([13], Theorem 5.2) and *-isomorphic to the regular ring of that finite Baer *-ring

([13], Theorem 5.3) with *-isomorphisms that fix the original ring. Thus, to prove part 2. it is sufficient to show that R satisfies all the assumptions of Theorems 5.2 and 5.3 from [13].

The assumptions for Theorem 5.2 are that the Baer *-ring is finite (given by (A1)), every nonzero right ideal contains a nonzero projection (guaranteed by EP, thus (A2)), LP $\sim$ RP (which follows from (A2) by Corollary, p. 131 [2]) and certain condition called Utumi's condition. By Corollary 3.7 from [13], Utumi's condition is satisfied for every Baer *-ring that is finite (A1), 2-proper (we have shown that (A2)-(A5) imply n-proper for any positive n) and that (EP) and (SR) hold. (SR) is an axiom that follows from (A2) and (A3) (see Exercise 7C p. 131, [2]). Thus, all the assumptions are satisfied by R.

The assumptions for Theorem 5.3 are the same as (A1) – (A6) with the exception that (UPSR) in (A2) is replaced by (SR). But (SR) follows from (A2) and (A3) and thus the assumptions of Theorem 5.3 hold for R. □

2.4 Matrix Rings over R

Let $M_n(R)$ denotes the ring of $n \times n$ matrices over R.

If R is a Baer *-ring, the lattice of projections of R is complete. In order to ensure the completeness of lattice of projections of $M_n(R)$ it is necessary for $M_n(R)$ to be Baer. To ensure that we need two more axioms.

(A8) $M_n(R)$ satisfies the *parallelogram law (P):* for every two projections p and q,
$$p - \inf\{p, q\} \sim \sup\{p, q\} - q.$$
(A9) Every sequence of orthogonal projections in $M_n(R)$ has a supremum.

If Q is a regular ring of Baer *-ring R that satisfies (A1)– (A9), then $M_n(Q)$ is a regular Rickart *-ring that has the same projections, unitaries and partial isometries as $M_n(R)$ (Propositions 2 and 3, p. 250 in [2]). Thus, Q satisfies (A1) – (A7) by Theorem 2 and statements (A8) and (A9) are true in $M_n(Q)$ (they are statements about projections and the projections in $M_n(Q)$ and $M_n(R)$ are the same). Thus, Q satisfies (A1)– (A9).

Theorem 4 *If R is a Baer *-ring satisfying (A1)– (A9), then $M_n(R)$ is a finite Baer *-ring with (GC).*

*If Q is a regular Baer *-ring satisfying (A1)– (A9), then $M_n(Q)$ is a regular Baer *-ring.*

This result is Theorem 1 and Corollary 2, p. 262 in [2].

Corollary 5 *(1) If R is a Baer *-ring satisfying (A1) – (A7), then $M_n(R)$ is semihereditary (i.e., every finitely generated submodule of a projective module is projective or, equivalently, every finitely generated ideal is projective) for every positive n.*

*(2) If R is a Baer *-ring satisfying (A1) – (A9), then the lattice of projections of $M_n(R)$ is complete for every positive n.*

PROOF. 1. (A1) – (A7) guarantees that $M_n(R)$ is a Rickart *-ring (Theorem 1, p. 251 in [2]).

A ring is right semihereditary if and only if the algebra of $n \times n$ matrices is right Rickart for every positive n (see e.g. Proposition 7.63 in [9]). Note that this result has a corollary that if R is right semihereditary, then $M_n(R)$ is right semihereditary for every positive n (simply identify $M_m(M_n(R))$ with $M_{mn}(R)$ and use the result).

Thus, $M_n(R)$ is semihereditary for every positive n.

2. Since every Baer *-ring has a complete lattice of projections, this is a simple corollary of the fact that $M_n(R)$ is a Baer *-ring. $\square$

Definition 6 *Let $\mathcal{C}$ be the class of Baer *-rings that satisfy the axioms (A1) – (A9.)*

Every finite AW^*-algebra satisfies the axioms (A1) – (A9) (remark 1, p. 249 in [2]). Thus, the class $\mathcal{C}$ contains the class of all finite AW^*-algebras and, in particular, all finite von Neumann algebras.

3 Torsion Theories

To study the properties of the class $\mathcal{C}$, we shall use a notion that will facilitate the understanding of modules over a ring from $\mathcal{C}$.

We begin with a general setting: Let R be any ring. A *torsion theory* for R is a pair $\tau = (\mathcal{T}, \mathcal{F})$ of classes of R-modules such that

i) $\mathrm{Hom}_R(T, F) = 0$, for all $T \in \mathcal{T}$ and $F \in \mathcal{F}$.
ii) $\mathcal{T}$ and $\mathcal{F}$ are maximal classes having the property *i*).

Thus, if $(\mathcal{T}, \mathcal{F})$ is a torsion theory, the class $\mathcal{T}$ is closed under quotients, direct sums and extensions and the class $\mathcal{F}$ is closed under taking submodules, direct products and extensions.

Conversely, if $\mathcal{M}$ is a class of R-modules closed under quotients, direct sums and extensions, then it is a torsion class for a torsion theory $(\mathcal{M}, \mathcal{F})$ where $\mathcal{F} = \{ F \mid \mathrm{Hom}_R(M, F) = 0, \text{ for all } M \in \mathcal{M} \}$. Dually, if $\mathcal{M}$ is a class of R-modules closed under submodules, direct products and extensions, then it is a torsion-free class for a torsion theory $(\mathcal{T}, \mathcal{M})$ where $\mathcal{T} = \{ T \mid \mathrm{Hom}_R(T, M) = 0, \text{ for all } M \in \mathcal{M} \}$.

The modules in $\mathcal{T}$ are called *τ-torsion modules* (or torsion modules for τ) and the modules in $\mathcal{F}$ are called *τ-torsion-free modules* (or torsion-free modules for τ).

If $\tau_1 = (\mathcal{T}_1, \mathcal{F}_1)$ and $\tau_2 = (\mathcal{T}_2, \mathcal{F}_2)$ are two torsion theories, we say that τ_1 is *smaller* than τ_2 ($\tau_1 \leq \tau_2$) iff $\mathcal{T}_1 \subseteq \mathcal{T}_2$ (equivalently $\mathcal{F}_1 \supseteq \mathcal{F}_2$).

If $\mathcal{M}$ is a class of R-modules, then torsion theory *generated* by $\mathcal{M}$ is the smallest torsion theory $(\mathcal{T}, \mathcal{F})$ such that $\mathcal{M} \subseteq \mathcal{T}$. The torsion theory *cogenerated* by $\mathcal{M}$ is the largest torsion theory $(\mathcal{T}, \mathcal{F})$ such that $\mathcal{M} \subseteq \mathcal{F}$.

If $(\mathcal{T}, \mathcal{F})$ is a torsion theory for a ring R and M is a R-module, there exists submodule N such that $N \in \mathcal{T}$ and $M/N \in \mathcal{F}$ (Proposition 1.1.4 in [3]). From this it follows that every module M has the largest submodule that belongs to $\mathcal{T}$ (i.e. submodule generated by the union of all torsion submodules of M). We call it the *torsion submodule* of M and denote it with $\mathcal{T}M$. The quotient $M/\mathcal{T}M$ is called the *torsion-free quotient* and we denote it $\mathcal{F}M$.

We say that a torsion theory $\tau = (\mathcal{T}, \mathcal{F})$ is *hereditary* if the class $\mathcal{T}$ is closed under taking submodules. A torsion theory is hereditary if and only if the torsion-free class is closed under formation of injective envelopes (Proposition 1.1.6, [3]). Also, a torsion theory cogenerated by a class of injective modules is hereditary (easy to see) and, conversely, every hereditary torsion theory is cogenerated by a class of injective modules (Proposition 1.1.17, [3]).

The notion of the closure of a submodule in a module is another natural notion that can be related to a torsion theory. Let M be an R-module and K a submodule of M. The *closure* $\mathrm{cl}^M_{\mathcal{T}}(K)$ of K in M with respect to the torsion theory $(\mathcal{T}, \mathcal{F})$ is

$$\mathrm{cl}^M_{\mathcal{T}}(K) = \pi^{-1}(\mathcal{T}(M/K)) \text{ where } \pi \text{ is the natural projection } M \twoheadrightarrow M/K.$$

If it is clear in which module we are closing the submodule K, we suppress the superscript M from $\mathrm{cl}^M_{\mathcal{T}}(K)$ and write just $\mathrm{cl}_{\mathcal{T}}(K)$. If K is equal to its closure in M, we say that K is *closed* submodule of M.

For more details on closure see Proposition 3.2 in [17].

3.1 Examples.

3.1.1

The torsion theory cogenerated by the injective envelope $E(R)$ of R is called the *Lambek torsion theory*. We denote it τ_L. It is hereditary, as it is cogenerated by an injective module, and faithful. Moreover, it is the largest hereditary faithful torsion theory.

3.1.2

The class of nonsingular modules over a ring R is closed under submodules, extensions, products and injective envelopes. Thus, it is a torsion-free class of a hereditary torsion theory. This theory is called the *Goldie torsion theory*. Let us denote it with $\tau_G = (\mathbf{T}, \mathbf{P})$.

The Lambek theory is smaller than the Goldie theory (see example 3, p. 26 in [3]). If R is nonsingular, then the Lambek and Goldie theories coincide (also see [3] for details).

Here we mention a few results that we shall be using in the sequel. By Corollary 7.30 in [9], if M an R-module and K a submodule of M, then the Goldie closure of K in M is complemented in M. By Proposition 7.44 in [9], if R is a nonsingular ring and M nonsingular R-module, then the Goldie closure of K in M is the largest submodule of M in which K is essential. From this it follows that a submodule K is Goldie closed in M if and only if K is a complement in M.

The above has the following result of R.E. Johnson (introduced in [7]) as a corollary.

Corollary 7 *Let R be any ring and M a nonsingular R-module. There is an one-to-one correspondence*

$$\{\textit{complements in } M\} \longleftrightarrow \{\textit{direct summands of } E(M)\}$$

given by $K \mapsto$ the Goldie closure of K in $E(M)$ that is equal to a copy of $E(K)$. The inverse map is given by $L \mapsto L \cap M$.

The proof can be found also in [9] (Corollary 7.44').

3.1.3

If R is an Ore ring with the set of regular elements T (i.e., $Tr \cap Rt \neq 0$, for every $t \in T$ and $r \in R$), we can define a hereditary torsion theory by the condition that a right R-module M is a torsion module iff for every $m \in M$, there is a nonzero $t \in T$ such that $mt = 0$. This torsion theory is called the *classical torsion theory of an Ore ring.* It is faithful and so it is contained in the Lambek torsion theory.

3.1.4

The class of flat modules is closed under extensions. If R is semihereditary, the class of flat modules is closed under direct products (Theorem 4.47 and Example 4.46 b), [9]). If R is subflat (i.e. every submodule of a flat module is flat), it is closed under submodules. Since every semihereditary ring R is subflat (Theorem 4.67, [9]), semihereditary R has a torsion theory in which the class of all flat modules is the torsion-free class. Denote this torsion theory with τ_{flat}.

3.1.5

Let R be a subring of a ring S. Let us look at a collection of all R-modules M such that $S \otimes_R M = 0$. This collection is closed under quotients, extensions and direct sums. Moreover, if S is flat as an R-module, then this collection is closed under submodules and, hence, defines a hereditary torsion theory. In this case we denote this torsion theory by τ_S.

From the definition of τ_S it follows that

1. The torsion submodule of M in τ_S is the kernel of the natural map $M \to S \otimes_R M$.
2. All flat modules are τ_S-torsion-free.

By 2., τ_S is faithful. Thus, τ_S is contained in the Lambek torsion theory.

If R is an Ore ring, then $\tau_{Q_{\text{cl}}(R)}$ is the classical torsion theory.

If R is right semihereditary ring R, the ring of maximal right quotients that is left R-flat (Theorem 2.10 in [15]) and all torsion-free modules in $\tau_{Q^r_{\max}(R)}$ are flat (Theorem 2.1 in [16]). Thus, if R is Ore and semihereditary ring with $Q_{\text{cl}}(R) = Q_{\max}(R)$ (as is the case with any ring from the class $\mathcal{C}$) , then

$$\text{Classical torsion theory } = \tau_{Q_{\text{cl}}(R)} = \tau_{Q_{\max}(R)} = \tau_{\text{flat}}.$$

In this case, let us denote this torsion theory by $(\mathbf{t}, \mathbf{p})$.

3.1.6

If R is any ring, let $(\mathbf{b}, \mathbf{u})$ be the torsion theory cogenerated by the ring R. We call a module in $\mathbf{b}$ a *bounded module* and a module in $\mathbf{u}$ an *unbounded module*. This theory is not necessarily hereditary.

The Lambek and $(\mathbf{b}, \mathbf{u})$ torsion theory are related such that M is a Lambek torsion module if and only if every submodule of M is bounded. This is a direct corollary of the fact that $\mathrm{Hom}_R(M, E(R)) = 0$ if and only if $\mathrm{Hom}_R(N, R) = 0$, for all submodules N of M, that is an exercise in [4]. Also, it is easy to show that $(\mathbf{b}, \mathbf{u})$ is equal to the Lambek torsion theory if and only if $(\mathbf{b}, \mathbf{u})$ is hereditary.

$(\mathbf{b}, \mathbf{u})$ is the largest torsion theory in which R is torsion-free. Thus, for a ring from class $\mathcal{C}$

$$(\mathbf{t}, \mathbf{p}) \leq (\mathbf{T}, \mathbf{P}) = \text{ Lambek } \leq (\mathbf{b}, \mathbf{u}).$$

4 Torsion Theories for Rings from Class $\mathcal{C}$

For the remainder of this section, let R denote a ring from class $\mathcal{C}$ with Q the regular ring of R. If p is a matrix from $M_n(R)$, we will identify p with the R-map $R^n \to R^n$ defined by $r \mapsto pr$.

4.1 Splitting of $(\mathbf{b}, \mathbf{u})$ for Finitely Generated Modules.

First, we shall show that $M = \mathbf{b}M \oplus \mathbf{u}M$ for every finitely generated M and that $\mathbf{u}M$ is finitely generated projective. We need a few preliminary results.

Lemma 8 *Let P be a right R-module.*

(1) If P is a submodule of R^n, then the following conditions are equivalent
- *i) P is a complement in R^n.*
- *ii) There is a projection $p \in M_n(R)$ such that $P = \mathrm{im}p$.*
- *iii) P is a direct summand of R^n.*

(2) P is finitely generated projective if and only if there is a nonnegative integer n and a projection $p \in M_n(R)$ such that $P = \mathrm{im}p$.

PROOF. (1) i)$\Rightarrow$ ii) Let P be a complement in R^n. By Corollary 7, $E(P)$ is a direct summand of $E(R^n) = E(R)^n = Q^n$. The projection from Q^n onto $E(P)$ is an idempotent element $q \in M_n(Q)$ such that $\mathrm{im}q = E(P)$. Since $M_n(Q)$ is a Rickart *-ring (by Theorem 4), there is a projection $p \in M_n(Q)$ such that

$pM_n(Q) = \mathrm{ann}_r(1-q) = qM_n(Q)$. But the projections in $M_n(Q)$ and $M_n(R)$ are the same so $p \in M_n(R)$.

P is a complement, so $P = E(P) \cap R^n$ by Corollary 7. Thus, $P = p(Q^n) \cap R^n$. Since $p \in M_n(R)$, $p(R^n) \subseteq R^n$ and so $p(R^n) \subseteq p(Q^n) \cap R^n = P$. Conversely, if $p(r)$ is an element of $p(Q^n) \cap R^n = P$, then $p(r) \in R^n$ has unique decomposition as $p(r') + (1-p)(r'')$. But that decomposition still holds in Q^n. Thus, $p(r) = p(r')$ and $(1-p)(r'') = 0$. Since $r' \in R^n$, $p(r) = p(r')$ is in $p(R^n)$. This proves that $P = p(R^n)$.

ii)$\Rightarrow$ iii) Trivial.

iii)$\Rightarrow$ i) Trivial.

(2) If P is finitely generated projective, then there is a nonnegative integer n such that P is a direct summand in R^n. Then $P = \mathrm{im} p$ for some projection $p \in M_n(R)$ by (1). The converse is obvious. $\square$

The following lemma asserts that we can separate a direct summand and an element in the image of a projection out of the direct summand, with an R-valued map. This will turn out to be the key ingredient in the proof that a finitely generated module M splits as $\mathbf{b}M \oplus \mathbf{u}M$.

Lemma 9 *If P is a direct summand of R^n, $p \in M_n(R)$ a projection, and $a \in R^n$ any element such that $p(a) \notin P$, then there is a map $f \in \mathrm{Hom}_R(R^n, R)$ such that $f(P) \equiv 0$ and $f(p(a)) \neq 0$.*

PROOF. Let S be the complement of P and pr_S be the projection of R^n onto S. $p(a) = r_P + r_S$ where $r_P \in P$ and $r_S \in S$. Since $p(a) \notin P$, r_S is nontrivial. Let $(q_1, q_2, \ldots, q_n)$ be the coordinates of r_S in the standard basis. Define the map $q : R^n \to R$ by

$$g : (a_1, a_2, \ldots, a_n) \mapsto \sum_{i=1}^{n} q_i^* a_i$$

Now define map $f \in \mathrm{Hom}_R(R^n, R)$ as $f = g \circ pr_S$. Clearly $P \in \ker f$. $f(p(a)) = q(r_S) = \sum_{i=1}^{n} q_i^* q_i \neq 0$ since R is n-proper and $r_S \neq 0$. $\square$

The idea here is to study finitely generated projective modules by treating the projections and benefit from the nice properties of the matrix rings over R. This is possible since if $p, q \in M_n(R)$, then

$$pM_n(R) = qM_n(R) \quad \text{if and only if} \quad p(R^n) = q(R^n) \tag{2}$$

as basic matrix algebra shows. Now, we can understand the closures in R^n better.

Proposition 10 *If P is a submodule of R^n, then the following sets are equal and are direct summands of R^n:*

(1) $\mathrm{cl}_{\mathbf{b}}(P) = \{x \in R^n | f(x) = 0$ *for all* $f \in \mathrm{Hom}_R(R^n, R)$ *s.t.* $P \subseteq \ker f\} = \bigcap\{\ker f | f \in \mathrm{Hom}_R(R^n, R)$ *s.t.* $P \subseteq \ker f\}$,
(2) $\bigcap\{S | S$ *is a direct summand of* R^n *and* $P \subseteq S\}$,
(3) $\inf\{p | p \in M_n(R)$ *a projection with* $P \subseteq p(R^n)\}(R^n) = \bigcap\{p(R^n) | p \in M_n(R)$ *a projection with* $P \subseteq p(R^n)\}$,
(4) $\mathrm{cl}_{\mathbf{T}}(P) =$ *(largest submodule of R^n in which P is essential) = (smallest submodule of R^n that contains P and that is a complement in R^n) =* $E(P) \cap R^n$.

Moreover, if P is a right ideal in R, then the above sets are equal to $\mathrm{ann}_r(\mathrm{ann}_l(P))$.

PROOF. All sets in (1) are equal by the definition of closure in the torsion theory $(\mathbf{b}, \mathbf{u})$. Also, if P is a right ideal of R, then it is easy to see that $\mathrm{cl}_{\mathbf{b}}(P) = \mathrm{ann}_r(\mathrm{ann}_l(P))$.

The sets in (3) are equal by formula (1) in subsection 2.1 and formula (2) above.

The first three sets in (4) are equal by Proposition 7.44 from [9]. From Corollary 7 also follows that $\mathrm{cl}_{\mathbf{T}}(P) = E(\mathrm{cl}_{\mathbf{T}}(P)) \cap R^n$. But $P \subseteq_e \mathrm{cl}_{\mathbf{T}}(P)$ and so $E(P) = E(\mathrm{cl}_{\mathbf{T}}(P))$. Thus $\mathrm{cl}_{\mathbf{T}}(P) = E(\mathrm{cl}_{\mathbf{T}}(P)) \cap R^n = E(P) \cap R^n$.

(2) $\subseteq$ (1) since every f as in (1) determines one S as in (2).

(2) = (3) by Lemma 8.

(2) = (4) The intersection of complements is a complement by Proposition 7.44 in [9]. Thus, the set in (4) is the intersection of all complements in R^n containing P. But by Lemma 8, this is the same as the intersection of all direct summands of R^n containing P. Moreover, the set in (4) is a complement itself and, thus a direct summand in R^n.

The set in (1) is also a complement since it is the intersection of complements. Thus, the set in (1) is a direct summand as well.

Let us show now that (1) $\subseteq$ (2). Since the set in (1) is a direct summand, there is a projection $p \in M_n(R)$ such that $\mathrm{cl}_{\mathbf{b}}(P) = p(R^n)$. To show (1) $\subseteq$ (2) it is sufficient to show that $p(a)$ is contained in all direct summands of R^n that contain P for all $a \in R^n$ (because then $p(R^n) \subseteq$ (2)).

Suppose the contrary: there is $a \in R^n$ and a direct summand S such that $P \subseteq S$ and $p(a) \notin S$. By Lemma 9, there is a map $f \in \mathrm{Hom}_R(R^n, R)$ such that $f(S) \equiv 0$ and $f(p(a)) \neq 0$. But $p(a)$ is in $\mathrm{cl}_{\mathbf{b}}(P)$ and f is a map such that $P \subseteq S \subseteq \ker f$ so $p(a)$ has to be in the kernel of f as well. Contradiction. Thus, $p(a) \in S$. $\square$

Theorem 11 *If M is finitely generated R-module and K submodule of M, then $M/\mathrm{cl}_{\mathbf{b}}(K)$ is finitely generated projective and $\mathrm{cl}_{\mathbf{b}}(K)$ is a direct summand of M. In particular, for $K = 0$ we have that $\mathbf{u}M$ is finitely generated projective and $M = \mathbf{b}M \oplus \mathbf{u}M$.*

PROOF. If M is R^n, $\mathrm{cl}_{\mathbf{b}}(K)$ is a direct summand of M by Proposition 10. Moreover, the inclusion of $\mathrm{cl}_{\mathbf{b}}(K)$ in M splits since $\mathrm{cl}_{\mathbf{b}}(K) = p(R^n)$ for some projection $p \in M_n(R^n)$. Thus the claim follows for $M = R^n$.

Now let M be any finitely generated module. There is a nonnegative integer n and an epimorphism $f : R^n \to M$.

First, we shall show that $\mathrm{cl}_{\mathbf{b}}(f^{-1}(K)) = f^{-1}(\mathrm{cl}_{\mathbf{b}}(K))$.

Let x be in $\mathrm{cl}_{\mathbf{b}}(f^{-1}(K))$. Then $g(x) = 0$, for every $g \in \mathrm{Hom}_R(R^n, R)$ such that $f^{-1}(K) \subseteq \ker g$. We need to show that $f(x)$ is in $\mathrm{cl}_{\mathbf{b}}(K)$, i.e. that $h(f(x)) = 0$ for every $h \in \mathrm{Hom}_R(M, R)$ with $K \subseteq \ker h$. Let h be one such map. Letting $g = hf$, we obtain a map in $\mathrm{Hom}_R(R^n, R)$ such that $g(f^{-1}(K)) = hff^{-1}(K) = h(K)$ (since f is onto). But $h(K) = 0$, and so $f^{-1}(K) \subseteq \ker g$. Hence, $g(x) = 0$ i.e. $h(f(x)) = 0$.

To show the converse, let x be in $f^{-1}(\mathrm{cl}_{\mathbf{b}}(K))$. Then $h(f(x)) = 0$ for every $h \in \mathrm{Hom}_R(M, R)$ such that $K \subseteq \ker h$. We need to show that $g(x) = 0$ for every $g \in \mathrm{Hom}_R(R^n, R)$ such that $f^{-1}(K) \subseteq \ker g$. Let g be one such map. Since $f^{-1}(0) \subseteq f^{-1}(K) \subseteq \ker g$, we have $\ker f \subseteq \ker g$. This condition enables us to define a homomorphism $h : M \to R$ such that $h(f(p)) = g(p)$ for every $p \in R^n$. Then $h(K) = h(f(f^{-1}(K))) = g(f^{-1}(K)) = 0$, and so $h(f(x)) = 0$. But from this $g(x) = 0$.

It is easy to see that $f : R^n \to M$ induces an isomorphism $R^n/f^{-1}(\mathrm{cl}_{\mathbf{b}}(K)) \to M/\mathrm{cl}_{\mathbf{b}}(K)$. But $\mathrm{cl}_{\mathbf{b}}(f^{-1}(K)) = f^{-1}(\mathrm{cl}_{\mathbf{b}}(K))$, so we obtain that $M/\mathrm{cl}_{\mathbf{b}}(K)$ is finitely generated projective (since $R^n/\mathrm{cl}_{\mathbf{b}}(f^{-1}(K))$ is). So $0 \to \mathrm{cl}_{\mathbf{b}}(K) \to M \to M/\mathrm{cl}_{\mathbf{b}}(K) \to 0$ splits. $\square$

Given that the dimension function on R and on all rings $M_n(R)$ exist (Theorem 1 and Theorem 4) it would be desirable to have the dimensions on $M_m(R)$ and $M_n(R)$ agree for $m \geq n$ i.e. $d_m \upharpoonright_{M_n(R)} = d_n$ for all $m \geq n$.

The dimension on R is determined by its values on the central projections (see chapter 6 of [2]). The centers of R and $M_n(R)$ are isomorphic under the identification of $\text{diag}(a, a, ..., a) \in Z(M_n(R))$ with $a \in Z(R)$. If we identify $\text{diag}(a, a, ..., a) \in Z(M_n(R))$ with $na \in Z(R)$, we get the desired result on the dimensions.

Now let us define the function $\dim_R$ on the class of all right R modules Mod_R and values in $C(X)$ by

(1) If P is a finitely generated projective R-module, then there is a nonnegative integer n and a matrix $p \in M_n(R)$ such that $p^2 = p^* = p$ and $\text{im}p \cong P$. It is clear that an idempotent matrix q with image isomorphic to p exist. Choose p to be the projection such that $pM_n(R) = \text{ann}_r(1-q)$. Recall that we can do that because $M_n(R)$ is a Rickart *-ring. Then define

$$\dim_R(P) = d(p).$$

The values of $\dim_R$ are in $C_{[0,\infty)}(X)$, the algebra of functions from $C(X)$ with values in $[0, \infty)$. The algebra $C_{[0,\infty)}(X)$ is a boundedly complete lattice with respect to the pointwise ordering (see pages 161 and 162 in [2]). Note, however, that the infinite lattice operations might differ from the pointwise operations.

(2) If M is any R-module, define

$$\dim'_R(M) = \sup\{\dim_R(P) \mid P \text{ fin. gen. projective submodule of } M\}$$

where the supremum on the right side is an element of $C(X)$ if it exists and is a new symbol ∞ otherwise. We define $a+\infty = \infty+a = \infty = \infty+\infty$ and $a \leq \infty$ for every $a \in C(X)$.

Our first goal is to show that the following theorem (proven by Wolfgang Lück in [11]) holds for R with $[0, \infty)$ replaced by $C_{[0,\infty)}(X)$ and $[0, \infty]$ replaced by $C_{[0,\infty)}(X) \cup \{\infty\}$.

Theorem 12 *Let R be a ring such that there exist a dimension function* dim *that assigns to any finitely generated projective right R-module an element of $[0, \infty)$ and such that the following two conditions hold*

(L1) If P and Q are finitely generated projective modules, then

$$P \cong Q \Rightarrow \dim(P) = \dim(Q)$$

$$\dim(P \oplus Q) = \dim(P) + \dim(Q),$$

(L2) If K is a submodule of finitely generated projective module Q, then $\mathrm{cl}_{\mathbf{b}}(K)$ is a direct summand of Q and

$$\dim(\mathrm{cl}_{\mathbf{b}}(K)) = \sup\{\dim(P) \mid P \text{ is a fin. gen. projective submodule of } K\}.$$

Then, for every R-module M, we can define a dimension

$$\dim'_R(M) = \sup\{\dim_R(P) \mid P \text{ fin. gen. projective submodule of } M\} \in [0, \infty]$$

that satisfies the following properties:

(1) Extension: $\dim(P) = \dim'(P)$ for every finitely generated projective module P.
(2) Additivity: If $0 \to M_0 \to M_1 \to M_2 \to 0$ is a short exact sequence of R-modules, then

$$\dim'(M_1) = \dim'(M_0) + \dim'(M_2).$$

(3) Cofinality: If $M = \bigcup_{i \in I} M_i$ is a directed union, then

$$\dim'(M) = \sup\{\ \dim'(M_i) \mid i \in I\ \}.$$

(4) Continuity: If K is a submodule of a finitely generated module M, then

$$\dim'(K) = \dim'(\mathrm{cl}_{\mathbf{b}}(K)).$$

(5) If M is a finitely generated module, then

$$\dim'(M) = \dim(\mathbf{u}M) \quad \text{and} \quad \dim'(\mathbf{b}M) = 0.$$

(6) The dimension $\dim'$ is uniquely determined by (1) – (4).

For proof see Theorem 6.7, p 239 of [12] or Theorem 0.6 and Remark 2.14 in [11].

First, we show that the condition (L1) from Theorem 12 holds for R.

Proposition 13 *If P and S are finitely generated projective R-modules, then*

(1) $P \cong S$ if and only if $\dim_R(P) = \dim_R(S)$,
(2) $\dim_R(P \oplus S) = \dim_R(P) + \dim_R(S)$.

PROOF. (1) Let $P \cong S$. Let p and s be projections such that $\dim_R(P) = d(p)$ and $\dim_R(S) = d(s)$. p and s might be matrices of different size. Then there is an integer n such that

$$p_n = \begin{bmatrix} p & 0 \\ 0 & 0 \end{bmatrix} \text{ and } s_n = \begin{bmatrix} s & 0 \\ 0 & 0 \end{bmatrix}$$

are both in $M_n(R)$ and there is an invertible matrix $u \in M_n(R)$ such that $up_n = s_n u$ (see Lemma 1.2.1. in [14] for details).

Similar elements are algebraically equivalent (i.e a is algebraically equivalent to b iff $xy = a$, $yx = b$ for some x and y). Algebraic equivalence implies $\sim$ equivalence in all *-rings with (SR) (Exercise 8A, p. 9 in[2]). Since (SR) holds if (A2) and (A3) hold, we have that $p_n \sim s_n$. Thus, $d(p) = d(p_n) = d(s_n) = d(s)$.

Conversely, if $\dim_R(P) = \dim_R(S)$, then $d(p) = d(p_n) = d(s_n) = d(s)$ (we might have to enlarge p and s again). So $p_n \sim s_n$. It is easy to see (Exercise 5A, p. 8 in [2]) that then $\mathrm{im} p_n$ is isomorphic to $\mathrm{im} s_n$. But then P is isomorphic to S.

(2) Let P and S be finitely generated projective modules with p and s projections such that $\dim_R(P) = d(p)$ and $\dim_R(S) = d(s)$. Then we can use

$$p \oplus s = \begin{bmatrix} p & 0 \\ 0 & s \end{bmatrix}$$

to compute the dimension of $P \oplus S$. There is an integer n such that

$$p_n = \begin{bmatrix} p & 0 \\ 0 & 0 \end{bmatrix} \text{ and } s_n = \begin{bmatrix} 0 & 0 \\ 0 & s \end{bmatrix}$$

are both in $M_n(R)$. Then, $p_n s_n = s_n p_n = 0$ and so $\dim_R(P \oplus S) = d(p \oplus s) = d(p_n + s_n) = d(p_n) + d(s_n) = d(p) + d(s)$. $\square$

Note that this Proposition implies that

$$\dim_R(P) = 0 \quad \text{iff} \quad P = 0$$

for every finitely generated projective module P.

In order to prove that R satisfies condition (L2) from Theorem 12, we need two lemmas.

Recall that the regular ring Q of R is also in the class $\mathcal{C}$ (see subsection 2.4). Thus, we can define its dimension function $\dim_Q$. The following result relates the dimensions of R and Q and is leading us one step closer to (L2) of Theorem 12.

Lemma 14 *(1) If P is a direct summand of R^n, then*

$$\dim_R(P) = \dim_Q(E(P))$$

(2) If S is a direct summand of Q^n, then

$$\dim_Q(S) = \dim_R(S \cap R^n).$$

(3) If S is a submodule of Q^n, then

$$\dim_Q(\mathrm{cl}_{\mathbf{b}}(S)) = \sup\{d(q) \mid q \in M_n(R) \text{ a projection, } q(Q^n) \subseteq S\}$$

(4) If P is a submodule of R^n, then

$$\dim_R(\mathrm{cl}_{\mathbf{b}}(P)) = \sup\{d(q) \mid q \in M_n(R) \text{ a projection, } q(R^n) \subseteq P\}$$

PROOF. (1) If P is a direct summand of R^n, $P = p(R^n)$ for some projection $p \in M_n(R)$ by Lemma 8. By definition of $\dim_R$ it follows that $\dim_R(P) = d(p)$.

From the proof of i)$\Rightarrow$ ii) in Lemma 8, it follows that $p(R^n) = p(Q^n) \cap R^n$. Thus, $P = p(Q^n) \cap R^n$, and so $E(P) = E(p(Q^n) \cap R^n) = p(Q^n)$ by Corollary 7. Thus $\dim_Q(E(P)) = d(p)$.

(2) If S is a direct summand of Q^n, $S = p(Q^n)$ for some projection $p \in M_n(R)$. Then $\dim_Q(S) = d(p)$. Then, $S \cap Q^n = p(Q^n) \cap R^n = p(R^n)$ again by the proof of i)$\Rightarrow$ ii) in Lemma 8. Thus, $\dim_R(S \cap Q^n) = d(p)$.

(3) First we shall show that

$$\mathrm{cl}_{\mathbf{b}}(S) = \sup\{q | q \in M_n(R) \text{ a projection with } q(Q^n) \subseteq S\}(Q^n).$$

Let p denote the projection $\sup\{q|q \in M_n(R)$ a projection with $q(Q^n) \subseteq S\}$, and r denote the projection such that $\mathrm{cl}_{\mathbf{b}}(S) = r(Q^n)$. We shall show that $p = r$.

Since $q(Q^n) \subseteq S \subseteq r(Q^n)$ for all projections q with $q(Q^n) \subseteq S$, then $q \leq r$ for all such q and so $p \leq r$.

Conversely, it is sufficient to show $S \subseteq p(Q^n)$ since then $p(Q^n) \supseteq \inf\{q|q \in M_n(R)$ a projection with $S \subseteq q(Q^n)\}(Q^n) = \mathrm{cl}_{\mathbf{b}}(S) = r(Q^n)$ by Proposition

10 and thus $p \geq r$. So, let $x \in S$. Consider a matrix $X \in M_n(Q)$ such that the entries in the first column are coordinates of x in the standard basis and the entries in all the other columns equal zero. Since $M_n(Q)$ is a regular Rickart *-ring (Theorem 4), there is a projection $p_x \in M_n(Q)$ such that $XM_n(Q) = p_xM_n(Q)$. But $x \in X(Q^n) = xQ \subseteq S$ and so we have that $p_x(Q^n) \subseteq S$ for all $x \in S$. So, $p_x \leq p$ for all $x \in S$. Thus, $x \in p_x(Q^n) \subseteq p(Q^n)$ for all $x \in S$ and so $S \subseteq p(Q^n)$.

Now it is easy to see that

$$\begin{aligned}\dim_Q(\mathrm{cl}_{\mathbf{b}}(S)) &= \dim_Q(\sup\{q|q \in M_n(R) \text{ a projection with } q(Q^n) \subseteq S\}(Q^n)) \\ &= \sup\{d(q)|q \in M_n(R) \text{ a projection with } q(Q^n) \subseteq S\}\end{aligned}$$

by property (D10) of Theorem 1.

(4) Let p denote the projection $\sup\{q|q \in M_n(R)$ a projection with $q(R^n) \subseteq P\}$, and r denote the projection such that $\mathrm{cl}_{\mathbf{b}}(P) = r(R^n)$. Since $q(R^n) \subseteq P \subseteq r(R^n)$, for all projections q such that $q(R^n) \subseteq P$, then $q \leq r$. Thus, $p \leq r$ and so

$$\dim_R(\mathrm{cl}_{\mathbf{b}}(P)) = d(r) \geq d(p) = \sup\{d(q)|q \in M_n(R) \text{ projection}, q(R^n) \subseteq P\}$$

For the converse, first note that $E(\mathrm{cl}_{\mathbf{b}}(P)) = E(\mathrm{cl}_{\mathbf{T}}(P))$ by Proposition 10, $E(\mathrm{cl}_{\mathbf{T}}(P)) = E(P)$ since P is essential in $\mathrm{cl}_{\mathbf{T}}(P)$, and $E(P) = E(P) \cap Q^n = \mathrm{cl}^Q_{\mathbf{b}}(P)$ again by Proposition 10. Thus, $E(\mathrm{cl}_{\mathbf{b}}(P)) = \mathrm{cl}^Q_{\mathbf{b}}(P)$. Then,

$$\begin{aligned}\dim_R(\mathrm{cl}_{\mathbf{b}}(P)) &= \dim_Q(E(\mathrm{cl}_{\mathbf{b}}(P))) && \text{(by part (1))} \\ &= \dim_Q(\mathrm{cl}^Q_{\mathbf{b}}(P))) && \text{(by remark above)} \\ &= \sup\{d(q)|q \in M_n(R) \text{ proj.}, q(Q^n) \subseteq P\} && \text{(by part (3))} \\ &\leq \sup\{d(q)|q \in M_n(R) \text{ proj.}, q(R^n) \subseteq P\} && (q(R^n) \subseteq q(Q^n)).\end{aligned}$$

□

Lemma 15 *(1) If P is a finitely generated projective module in R^n, then*

$$\dim_R(P) = \dim_R(\mathrm{cl}_{\mathbf{b}}(P)).$$

(2) If P and S are finitely generated projective modules in R^n, then

$$P \subseteq S \textit{ imples } \quad \dim_R(P) \leq \dim_R(S).$$

PROOF. (1) Let $p \in M_n(R)$ be a projection such that $p(R^n) \cong P$ and q be a projection such that $q(R^n) = \mathrm{cl}_{\mathbf{b}}(P)$. To prove the claim, it is sufficient to show that $p \sim q$ (since then $\dim_R(P) = d(p) = d(q) = \dim_R(\mathrm{cl}_{\mathbf{b}}(P))$). For $p \sim q$, it is sufficient to show that $q(Q^n) \cong p(Q^n)$ by the same argument we used in the proof of part (1) of Proposition 13. $q(Q^n) = E(P)$ since $q(Q^n) \cap R^n = q(R^n) = \mathrm{cl}_{\mathbf{b}}(P) = E(P) \cap R^n$ (by Corollary 7). Thus,

$$q(Q^n) = E(P) \cong E(p(R^n)) = E(p(Q^n) \cap R^n) = p(Q^n).$$

(2) Let p be a projection such that $p(R^n) = \mathrm{cl}_{\mathbf{b}}(P)$ and s a projection with $s(R^n) = \mathrm{cl}_{\mathbf{b}}(S)$. $P \subseteq S$ implies $p(R^n) = \mathrm{cl}_{\mathbf{b}}(P) \subseteq \mathrm{cl}_{\mathbf{b}}(S) = s(R^n)$. Thus, $p \leq s$ and so $d(p) \leq d(s)$. Hence

$$\dim_R(P) = \dim_R(\mathrm{cl}_{\mathbf{b}}(P)) = d(p) \leq d(s) = \dim_R(\mathrm{cl}_{\mathbf{b}}(S)) = \dim_R(S)$$

by part (1). $\square$

Now we can prove the formula from Condition (L2), Theorem 12.

Proposition 16 *If K is a submodule of a finitely generated projective module S, then*

$$\dim_R(\mathrm{cl}_{\mathbf{b}}^S(K)) = \sup\{\dim_R(P) \mid P \textit{ is a fin. gen. projective submodule of } K\}$$

PROOF. Since S is a finitely generated projective, there is a nonnegative integer n such that S is a direct summand of R^n. $\mathrm{cl}_{\mathbf{b}}^S(K)$ is a direct summand of S by Theorem 11. Thus, $\mathrm{cl}_{\mathbf{b}}^{R^n}(K) \subseteq \mathrm{cl}_{\mathbf{b}}^S(K)$ by Proposition 10. $S \subseteq R^n$ implies $\mathrm{cl}_{\mathbf{b}}^S(K) \subseteq \mathrm{cl}_{\mathbf{b}}^{R^n}(K)$. Hence, $\mathrm{cl}_{\mathbf{b}}^S(K) = \mathrm{cl}_{\mathbf{b}}^{R^n}(K)$ so we can work in R^n.

$$\begin{aligned}\dim_R(\mathrm{cl}_{\mathbf{b}}^{R^n}(K)) &= \sup\{d(q) \mid q \text{ a projection in } M_n(R) \text{ such that } q(R^n) \subseteq K\} \\ &\leq \sup\{\dim_R(P) \mid P \text{ is a fin. gen. proj. submodule of } K\}\end{aligned}$$

by Lemma 14.

Conversely,

$$\begin{aligned}&\sup\{\dim_R(P) \mid P \text{ is a fin. gen. projective submodule of } K\} \leq \\ &\sup\{\dim_R(P) \mid P \text{ is a fin. gen. projective submodule of } \mathrm{cl}_{\mathbf{b}}^{R^n}(K)\} = \\ &\dim_R(\mathrm{cl}_{\mathbf{b}}^{R^n}(K)).\end{aligned}$$

The last equality holds since we have monotony for dimensions of finitely generated projective modules by the Lemma 15 (this gives us $\leq$). The converse follows since $\mathrm{cl}_{\mathbf{b}}^{R^n}(K)$ is finitely generated projective by Theorem 11. $\square$

Finally, we can prove that our dimension is just as in Lück's Theorem 12. Recall that we need to replace $[0,\infty)$ by $C_{[0,\infty)}(X)$ and $[0,\infty]$ by $C_{[0,\infty)}(X) \cup \{\infty\}$ in the formulation of the theorem. Luckily, that will not influence the proof.

Theorem 17 *Theorem 12 holds for R and* $\dim_R$.

PROOF. R satisfies condition (L1) of Theorem 12 by Proposition 13 and the condition (L2) by Theorem 11 and by Proposition 16.

The Extension property holds by Proposition 16 and Lemma 15. The proof of the rest of the theorem is identical to the original proof by Lück (proof of Theorem 6.7, p 239 of [12] or Theorem 0.6 and Remark 2.14 in [11]). $\square$

From now on we shall not distinguish between $\dim_R$ and $\dim'_R$ and will denote them both by $\dim_R$.

Further, the dimension has the following properties.

Corollary 18 *If $M = \bigoplus_{i \in I} M_i$, then* $\dim_R(M) = \sum_{i \in I} \dim_R(M_i)$.

PROOF. This is an easy corollary of Cofinality of $\dim_R$. $\square$

Theorem 17 and Corollary 18 enable us to define another torsion theory: for an R-module M define $\mathbf{T}_{\dim}M$ as the submodule generated by all submodules of M of zero dimension. It is zero-dimensional by Cofinality of $\dim_R$. So, $\mathbf{T}_{\dim}M$ is the largest submodule of M of zero dimension.

Let us denote the quotient $M/\mathbf{T}_{\dim}M$ by $\mathbf{P}_{\dim}M$.

The class $\mathbf{T}_{\dim} = \{M \in \mathrm{Mod}_R | M = \mathbf{T}_{\dim}M\}$ is closed under submodules, quotients and extensions by additivity of dimension. The closure under the formation of direct sums follows from Corollary 18. Thus, $\mathbf{T}_{\dim}$ defines a hereditary torsion theory with torsion-free class equal to $\mathbf{P}_{\dim} = \{M \in \mathrm{Mod}_R | M = \mathbf{P}_{\dim}M\}$.

Since R is semihereditary and a nontrivial finitely generated projective module has nontrivial dimension, R is in $\mathbf{P}_{\dim}$. Thus, the torsion theory $(\mathbf{T}_{\dim}, \mathbf{P}_{\dim})$ is faithful. Since the Lambek torsion theory is the largest hereditary and faithful theory,

$$(\mathbf{T}_{\dim}, \mathbf{P}_{\dim}) \leq (\mathbf{T}, \mathbf{P}) \leq (\mathbf{b}, \mathbf{u}).$$

Theorem 19

$$(\mathbf{T}_{\dim}, \mathbf{P}_{\dim}) = (\mathbf{T}, \mathbf{P}).$$

If M is finitely generated, then $\mathbf{T}_{\dim}(M) = \mathbf{b}M$. Thus, M splits as $\mathbf{T}_{\dim}M \oplus \mathbf{P}_{\dim}M$.

The theorem allows us to drop the subscript dim from $(\mathbf{T}_{\dim}, \mathbf{P}_{\dim})$. The proof is the same as the proof of Proposition 4.2 from [17]. We quote it for completeness.

PROOF. If M is finitely generated, then $\dim_R(\mathbf{b}M) = \dim_R(\mathrm{cl}_{\mathbf{b}}(0)) = 0$ by Continuity Property of dimension. Thus, $\mathbf{b}M \subseteq \mathbf{T}_{\dim}M$. Since the converse always holds, the claim follows. The splitting follows from Theorem 11.

Since $(\mathbf{T}_{\dim}, \mathbf{P}_{\dim}) \leq (\mathbf{T}, \mathbf{P}) =$ Lambek torsion theory, to prove the equality it is sufficient to prove that every Lambek torsion module M has dimension zero. Recall that M is Lambek torsion module iff all submodules of M are bounded. This means that all finitely generated submodules of M are in $\mathbf{T}_{\dim}$. The dimension of M is equal to the supremum of the dimensions of finitely generated submodules of M by Cofinality. But that supremum is 0, so M is in $\mathbf{T}_{\dim}$. $\square$

4.3 Comparing the Torsion Theories for $\mathcal{C}$

Let us summarize the situation with various torsion theories of a ring R from $\mathcal{C}$. The trivial torsion theory is the theory $(0, \mathrm{Mod}_R)$, where Mod_R is the class of all R-modules and the improper torsion theory is the theory $(\mathrm{Mod}_R, 0)$. The various torsion theories for R are ordered as follows:

$$\text{Trivial} \leq \text{Classical} = \tau_Q = (\mathbf{t}, \mathbf{p}) \leq (\mathbf{T}_{\dim}, \mathbf{P}_{\dim}) = (\mathbf{T}, \mathbf{P}) \leq (\mathbf{b}, \mathbf{u}) \leq \text{Improper}$$

where all of the above inequalities can be strict. The theory $(\mathbf{t}, \mathbf{p})$ can be nontrivial by Example 2.9 in [12]. The inequality $(\mathbf{t}, \mathbf{p}) \leq (\mathbf{T}, \mathbf{P})$ can be strict by Example 8.34 in [12]. The inequality $(\mathbf{T}, \mathbf{P}) \leq (\mathbf{b}, \mathbf{u})$ can be strict by example given in Exercise 6.5 in [12]. Note that all of the above examples are given for finite von Neumann algebras.

For any nontrivial R the theory $(\mathbf{b}, \mathbf{u})$ is not improper since R is a module in $\mathbf{u}$.

We have seen that the classes $\mathbf{T}$ and $\mathbf{b}$ coincide for finitely generated modules. Next, we shall show that the classes $\mathbf{T}$ and $\mathbf{t}$ coincide when restricted on the class of finitely presented R-modules. First, we need the following result proven in [17] for finite von Neumann algebras.

Proposition 20 *If P is finitely generated projective R-module, then*

$$P \otimes_R Q = E(P).$$

For the proof, see Theorem 5.1 from [17]. In [17], this result was proven for a finite von Neumann algebra but the proof transfers to any $R \in \mathcal{C}$. The idea is to show that $P \subseteq_e P \otimes_R Q$ (which holds because Q is the classical ring of quotients) and that $P \otimes_R Q$ is injective (which holds because P is a direct summand of some finitely generated free R-module). The Lemma 5.1 from [17] also holds for a ring in $\mathcal{C}$.

Now we can prove the following.

Proposition 21 *If M is finitely presented R-module, then $\mathbf{T}M = \mathbf{t}M$.*

PROOF. Since $\mathbf{t}M \subseteq \mathbf{T}M$, it is sufficient to prove the converse. We shall show that if M has dimension zero (i.e. $M \in \mathbf{T}$), then $M \otimes_R Q = 0$ (i.e. $M \in \mathbf{t}$). Since M is finitely presented, there are finitely generated projective modules P_0 and P_1 such that

$$0 \to P_1 \to P_0 \to M \to 0.$$

$\dim_R(M) = 0$ and so $\dim_R(P_0) = \dim_R(P_1)$. Then

$$\dim_Q(P_0 \otimes_R Q) = \dim_Q(E(P_0)) = \dim_R(P_0) =$$

$$= \dim_R(P_1) = \dim_Q(E(P_1)) = \dim_Q(P_1 \otimes_R Q)$$

by Proposition 20 and part (1) of Lemma 14. Q is R-flat, so

$$0 \to P_1 \otimes_R Q \to P_0 \otimes_R Q \to M \otimes_R Q \to 0$$

is an exact sequence. Thus, $\dim_Q(M \otimes_R Q) = \dim_Q(P_0 \otimes_R Q) - \dim_Q(P_1 \otimes_R Q) = 0$. Moreover, the modules $P_0 \otimes_R Q$ and $P_1 \otimes_R Q$ are finitely generated projective Q-modules and hence $M \otimes_R Q$ is finitely presented Q-module. But Q is a regular ring so all finitely presented modules are projective. Thus,

$M \otimes_R Q$ is a finitely generated projective module of dimension zero and so it is trivial. $\square$

Before proving the main result of this subsection, let us prove another corollary of Theorem 17, Proposition 20 and Lemma 14.

Corollary 22 *For any R-module M,*

$$\dim_Q(M \otimes_R Q) = \dim_R(M).$$

PROOF. If M is a finitely generated projective module, this follows from Lemma 14 and Proposition 20. If M is a submodule of any projective R-module, write M as a directed union its finitely generated modules M_i, $i \in I$. All the modules M_i are projective as they are finitely generated submodules of a projective module and R is semihereditary. Thus,

$$\dim_R(M) = \sup_{i\in I} \dim_R(M_i) = \sup_{i\in I} \dim_Q(M_i \otimes_R Q) = \dim_Q(M \otimes_R Q)$$

by Cofinality of $\dim_R$ and $\dim_Q$ and the fact that tensor commutes with direct limit.

Finally, if M is an arbitrary R-module, then M is a quotient of some projective module P and its submodule K. Then, $\dim_R(M) = \dim_R(P) - \dim_R(K) = \dim_Q(P \otimes_R Q) - \dim_Q(K \otimes_R Q) = \dim_Q(M \otimes_R Q)$ by Additivity of dimensions $\dim_R$ and $\dim_Q$ and since Q is R-flat. $\square$

We now show how torsion theories reflect the ring-theoretic properties of R.

Theorem 23 *(1) R is regular if and only if $(\mathbf{t}, \mathbf{p})$ is trivial.*
(2) If R is self-injective, then $(\mathbf{T}, \mathbf{P}) = (\mathbf{b}, \mathbf{u})$.
(3) The regular ring Q of R is semisimple if and only if $(\mathbf{t}, \mathbf{p}) = (\mathbf{T}, \mathbf{P})$ for R.
(4) The following are equivalent
  (a) R is semisimple,
  (b) $(\mathbf{b}, \mathbf{u})$ is trivial,
  (c) $(\mathbf{T}, \mathbf{P})$ is trivial.
(5) R is trivial if and only if $(\mathbf{b}, \mathbf{u})$ is improper.

PROOF. (1) R is regular if and only if all the R-modules are flat. But, $(\mathbf{t}, \mathbf{p})$ is trivial if and only if all R-modules are in $\mathbf{p}$ i.e. flat.

(2) If R is self-injective, then $R = E(R)$. Thus, the torsion theories cogenerated with R and $E(R)$ coincide.

(3) Suppose that Q is semisimple. We show that $\mathbf{T} \subseteq \mathbf{t}$ by showing that every R-module M with dimension zero is such that $M \otimes_R Q = 0$. If $\dim_R(M) = 0$, then $\dim_Q(M \otimes_R Q) = \dim_R(M) = 0$ by Corollary 22. Thus, $M \otimes_R Q$ is a projective (since Q is semisimple) and of dimension 0. Hence, $M \otimes_R Q = 0$.

Conversely, if $\mathbf{t} = \mathbf{T}$, we shall show that every right ideal I of Q is a direct summand (thus Q is semisimple). Since $\mathrm{cl}^Q_{\mathbf{T}}(I)$ is a direct summand by Proposition 10, it is sufficient to show that $I = \mathrm{cl}^Q_{\mathbf{T}}(I)$. Let us look at $\mathrm{cl}^R_{\mathbf{T}}(I \cap R)$.

$\mathrm{cl}^R_{\mathbf{T}}(I \cap R)/(I \cap R) = \mathbf{T}(R/(I \cap R)) = \mathbf{t}(R/(I \cap R))$ by assumption that $\mathbf{t} = \mathbf{T}$. Thus, $\mathrm{cl}^R_{\mathbf{T}}(I \cap R) \otimes_R Q = (I \cap R) \otimes_R Q$. Then,

$$\begin{aligned}
\mathrm{cl}^Q_{\mathbf{T}}(I) &= E(I) && \text{(by Proposition 10)}\\
&= E(I \cap R) && (\text{since } I \cap R \subseteq_e I)\\
&= E(\mathrm{cl}^R_{\mathbf{T}}(I \cap R)) && (\text{since } I \cap R \subseteq_e \mathrm{cl}^R_{\mathbf{T}}(I \cap R))\\
&= \mathrm{cl}^R_{\mathbf{T}}(I \cap R) \otimes_R Q && \text{(by Proposition 20)}\\
&= (I \cap R) \otimes_R Q && \text{(by what we showed above)}\\
&\subseteq I && (I \text{ is a right ideal}).
\end{aligned}$$

Since $I \subseteq \mathrm{cl}^Q_{\mathbf{T}}(I)$ always holds, $I = \mathrm{cl}^Q_{\mathbf{T}}(I)$.

(4) (a) $\Rightarrow$ (b) If R is semisimple, then all nontrivial R-modules are projective and, thus, in $\mathbf{u}$. So, $(\mathbf{b}, \mathbf{u})$ is trivial.

(b) $\Rightarrow$ (c) is trivial since $(\mathbf{T}, \mathbf{P}) \leq (\mathbf{b}, \mathbf{u})$.

(c) $\Rightarrow$ (a) If $(\mathbf{T}, \mathbf{P})$ is trivial, then $(\mathbf{t}, \mathbf{p})$ is trivial and then R is regular by (1). Thus, $R = Q$. But $(\mathbf{T}, \mathbf{P})$ is trivial implies that $(\mathbf{t}, \mathbf{p}) = (\mathbf{T}, \mathbf{P})$ and so Q is semisimple by (3). Then, $R = Q$ is semisimple.

(5) If $(\mathbf{b}, \mathbf{u})$ is improper, then $R \cong \mathrm{Hom}_R(R, R) = 0$. The converse is trivial. $\square$

Part (3) of the above theorem generalizes the result (Theorem 6.6) from [18] proven there for group von Neumann algebras. Part (2) and Theorem 19 generalize Theorem 5.1 from [18].

The order of torsion theories for R implies that for every R-module M, we have a filtration:

$$\underbrace{0 \subseteq \mathbf{t}}_{\mathbf{t}M} \underbrace{M \subseteq \mathbf{T}}_{\mathbf{Tp}M} \underbrace{M \subseteq M}_{\mathbf{P}M}$$

The quotient $\mathbf{T}M/\mathbf{t}M$ is equal to $\mathbf{Tp}M = \mathbf{T}(M/\mathbf{t}M) = \mathrm{cl}_{\mathbf{T}}(\mathbf{t}M)/\mathbf{t}M$ since $\mathrm{cl}_{\mathbf{T}}(\mathbf{t}M) = \mathbf{T}M$ as one can easily show (see Proposition 4.3 in [17]).

For M finitely generated, the finitely generated quotient $\mathbf{p}M$ splits as the direct sum of $\mathbf{Tp}M$ and $\mathbf{Pp}M = \mathbf{P}M$ and thus we have a short exact sequence $0 \to \mathbf{t}M \to M \to \mathbf{Tp}M \oplus \mathbf{P}M \to 0$. Of course, M also splits as $\mathbf{T}M \oplus \mathbf{P}M$ by Theorem 19.

If Q is a regular ring from $\mathcal{C}$, then it is its own regular ring (by Theorem 2) and is self-injective by Proposition 3. By Theorem 23, the torsion theories of Q are ordered as follows:

$$\text{Trivial} = (\mathbf{t}, \mathbf{p}) \leq (\mathbf{T}, \mathbf{P}) = (\mathbf{b}, \mathbf{u}) \leq \text{Improper}$$

The first inequality is strict if and only if Q is not semisimple and the second if and only if Q is nontrivial. Thus, for general regular ring in $\mathcal{C}$, there seem to be only one nontrivial and proper torsion theory of interest.

5 Corollaries

5.1 Finite Von Neumann Algebras.

If A is a finite von Neumann algebra, the center Z of A can be identified with $C(X)$ (because Z is the closed linear span of central projections). Thus, the dimension $\dim_A$ is central-valued.

Also, a finite von Neumann algebra A has a function $\mathrm{tr}_A : A \to Z$ uniquely determined by the following properties:

(T1) tr_A is $\mathbb{C}$-linear,
(T2) $\mathrm{tr}_A(ab) = \mathrm{tr}_A(ba)$,
(T3) $\mathrm{tr}_A(c) = c$ for every $c \in Z$,
(T4) $\mathrm{tr}_A(a)$ is positive for every positive a (i.e. $a = b^*b$ for some b).

The restriction of tr_A to the set of projections in A satisfies the properties (D1) – (D4) of Theorem 1 (for proof and details see section 8.4 of [8]). Thus, we can use the center-valued trace to define the center-valued dimension function $\dim_A$ using the approach given in this paper.

In [12], [11] and [17] the real valued dimension $\dim_A^{\mathbb{R}}$ of a module over a finite von Neumann algebra A was considered and results analogous to those we proved here for a ring in $\mathcal{C}$, were proven for A. Since the real-valued dimension $\dim_A^{\mathbb{R}}$ depends on the complex-valued trace used (that is not unique), it was surprising that the torsion theory $(\mathbf{T}_{\dim_A^{\mathbb{R}}}, \mathbf{P}_{\dim_A^{\mathbb{R}}})$ coincided with Lambek and Goldie theories (Proposition 4.2 in [17]) regardless of the complex-valued trace used to define the dimension.

A direct corollary of our Theorem 19 and Proposition 4.2 in [17] is the following

Corollary 24

$$(\mathbf{T}_{\dim_A}, \mathbf{P}_{\dim_A}) = (\mathbf{T}_{\dim_A^{\mathbb{R}}}, \mathbf{P}_{\dim_A^{\mathbb{R}}}).$$

i.e. for every A-module M, the central-valued dimension of M is zero if and only if a real-valued dimension of M is zero.

It is interesting to note that in the case of a finite von Neumann algebra A, the algebra of affiliated operators U does not automatically come equipped with a trace (and thus a dimension) function since U might not be a finite von Neumann algebra. In [12] it is shown that we can still get the real-valued U-dimension of any U-module in the following way: if S is a finitely generated projective U-module, the U-dimension of S is the A-dimension of a finitely generated projective A-module P such that $P \otimes_A U \cong S$ (this is well defined by Theorem 8.22, [12]). Then one proves that U satisfies Theorem 12. So, the U-dimension can be extended to all U-modules (Lemma 8.27, [12]). Moreover, Corollary 22 holds for A and U (Theorem 8.29, [12]).

The regular ring Q of a ring R from the class $\mathcal{C}$ is automatically equipped with a dimension since Q is also in $\mathcal{C}$ (see subsection 2.4). Thus, by the results of this paper, it readily follows that U has the dimension function $\dim_U$ satisfying Theorem 12 and Corollary 22.

Moreover, Q has all the properties of R and more (regularity, self-injectiveness etc). In contrast, the algebra of affiliated operators U of a finite von Neumann algebra A is regular but might not be a von Neumann algebra (i.e. U may loose some properties of A).

5.2 Cofinal-measurable Modules.

Using the dimension function, we can view the theory $(\mathbf{t}, \mathbf{p})$ for $R \in \mathcal{C}$ from a different angle. We say that an R-module M is *measurable* if it is a quotient of a finitely presented module of dimension zero. M is *cofinal-measurable* if each finitely generated submodule is measurable. The class of cofinal-measurable modules is equal to the class $\mathbf{t}$ by Lemma 8.36 (2) from [12]. The proof there is given for a group von Neumann algebra but it converts to any R from $\mathcal{C}$.

5.3 K_0-theorem.

Theorem 9.20 (1) from [12] states that the inclusion $i : A \to U$ of a finite von Neumann algebra A to its algebra of affiliated operators U (same as regular ring of A) induces an isomorphism of monoids $\mu : \mathrm{Proj}(A) \to \mathrm{Proj}(U)$ given by $[P] \mapsto [P \otimes_A U]$ and an isomorphism $\mu : K_0(A) \to K_0(U)$.

The proof of this theorem relies solely on Theorem 8.22 in [12]. Theorem 8.22 holds for a ring $R \in \mathcal{C}$ and its regular ring Q. Thus, the result on K_0-theories holds for $R \in \mathcal{C}$ and its regular ring Q.

In [17], the inverse of the isomorphism μ is described. Namely, the following corollary is proven for a finite von Neumann algebras. The proof transfers, word-for-word, to a ring from class $\mathcal{C}$.

Corollary 25 *For every finitely generated projective R-module M, there is an one-to-one correspondence*

$$\{\textit{direct summands of } M\} \longleftrightarrow \{\textit{direct summands of } E(M)\}$$

given by $K \mapsto K \otimes_R Q = E(K)$. The inverse map is given by $L \mapsto L \cap M$. This correspondence induces an isomorphism of monoids $\mu : \mathrm{Proj}(R) \to \mathrm{Proj}(Q)$ and an isomorphism

$$\mu : K_0(R) \to K_0(Q)$$

given by $[P] \mapsto [P \otimes_R Q]$ with the inverse $[S] \mapsto [S \cap R^n]$ if S is a direct summand of Q^n.

Since we have Corollary 7, Proposition 20, and Theorem 11 (recall that complements are $(\mathbf{T}, \mathbf{P})$-closed modules and $(\mathbf{T}, \mathbf{P}) = (\mathbf{b}, \mathbf{u})$ for finitely generated modules), it is not hard to see why Corollary 25 holds for $R \in \mathcal{C}$.

Let us mention that Handelman proved (Lemma 3.1 in [6]) that for every finite Rickart C^*-algebra A such that every matrix algebra over A is also Rickart, the inclusion of A into its regular ring Q induces an isomorphism $\mu : K_0(A) \to K_0(Q)$. By Theorem 3.4 in [1], a matrix algebra over a Rickart C^*-algebra is a Rickart C^*-algebra. Thus, $K_0(A)$ is isomorphic to $K_0(Q)$ for every finite Rickart C^*-algebra. Corollary 25 describes the inverse of this isomorphism.

5.4 Questions.

We conclude by listing some interesting questions.

(1) Is it possible to obtain the same results using the weaker axioms than (A1) – (A9)? Note that (A8) and (A9) are particularly strong.
(2) Let R be in $\mathcal{C}$ and Q be the regular ring of R. Does Q semisimple imply R semisimple? Note that this is the case for group von Neumann algebras by Exercise 9.11 from [12] (if the algebra of affiliated operators of a group von Neumann algebra $\mathcal{N}G$ is semisimple, then the group G is finite so $\mathcal{N}G$ is finite dimensional over $\mathbb{C}$ and, thus, semisimple).
(3) Let R be in $\mathcal{C}$ and $(\mathbf{T}, \mathbf{P}) = (\mathbf{b}, \mathbf{u})$ for R. Is R self-injective? The converse holds by Theorem 23.

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