Rings with Dimension

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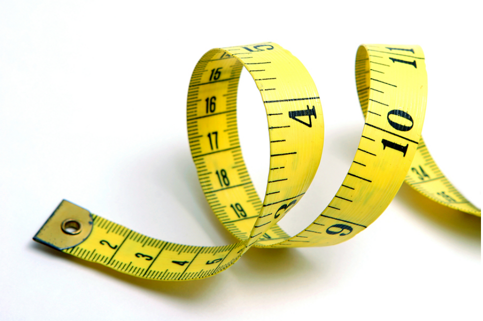


Let us imagine...

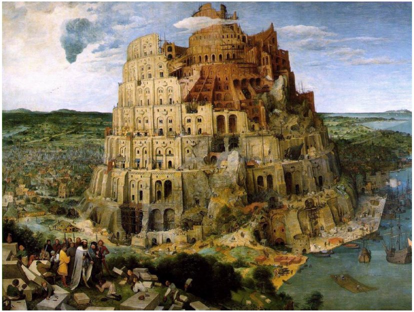
... a ring-theory perfect world:

- **Freedom** –
  all modules are free!
- **Equality** –
  no ideal with infinitely many direct summands!
- **Brotherhood** –
  nonzero elements do not annihilate each other!

Birds singing... flowers blooming...

Utopia?

What a boring world!



Long live ring diversity!

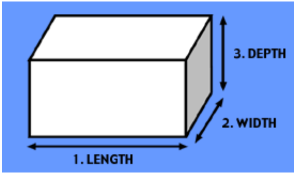
We would want the same, both general and nice enough, for

rings with dimension.

Some rings have a nice dimension...

For example, if M is a finitely generated $\mathbb{Z}$-module,

$$M = \mathbb{Z}^n \oplus \text{ torsion part}$$



so we say that

$$M \text{ has } \textbf{rank } n.$$

This scenario remind us of **vector spaces**. In fact,

$$\text{rank} M = \dim(M \otimes_{\mathbb{Z}} \mathbb{Q}).$$

Generalizes to rank of right noetherian or semiprime Goldie rings.

All rings have some dimension...

This rank-dimension does not work for **every** ring (it would be utopia if it does) but some **other** dimension can be defined on every ring.

The Goldie reduced rank

Roughly put: this rank measures how many direct summands a module allows.

Too restrictive versus too rough

- **Good** – nice properties;
- **Bad** – defined just for some rings;
- **Ugly** – tricky for "non-discrete cases".

- **Good** – defined for all the rings;
- **Bad** – all modules with ∞ many direct summands treated the same.

Finding middle ground

The year was 2010. I was looking for a class of **rings with dimension** that is not too restrictive and still general enough.

Finding middle ground during sabbatical in Málaga, Spain.

Von Neumann algebra

H – Hilbert space. $\mathcal{B}(H)$ – bounded operators.

A **von Neumann algebra** $\mathcal{A}$ is a

1) $*$-closed unital subalgebra of $\mathcal{B}(H)$,
2) closed in some sense.

Either

equal to its double commutant $\mathcal{A}''$

or, equivalently,

weakly closed in $\mathcal{B}(H)$.

Five types

type	**dimension range**
finite, discrete	$\{1, 2, \ldots, n\}$
infinite, discrete	$\{1, 2, \ldots\}$
finite, continuous	[0,1]
infinite, continuous	$\mathbb{R}$
very infinite	$\{0, \infty\}$

Tracing the dimension

A finite VNA $\mathcal{A}$ has a normal and faithful linear **trace** $\mathrm{tr}_{\mathcal{A}} : \mathcal{A} \to \mathbb{C}$.

The trace extends to matrices: $\mathrm{tr}([a_{ij}]) = \sum_{i=1}^{n} \mathrm{tr}(a_{ii})$.

Examples.

1. Usual trace on $M_n(\mathbb{C})$.
2. "Kaplansky trace" on group rings:

$$\mathrm{tr}(\sum a_g g) = a_1.$$

Extends from $\mathbb{C}G$ to $l^2(G)$, then to $\mathcal{N}G$ by

$$\mathrm{tr}(f) = \mathrm{tr}(f(1)).$$

Define dimension in two steps [Lück]

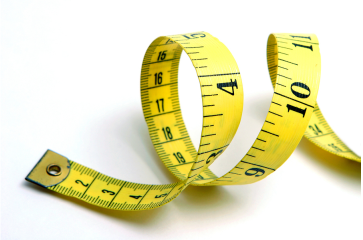
1. If P is a fin. gen. proj. module,

$$\dim_{\mathcal{A}}(P) = \operatorname{tr}(f) \quad \in [0, \infty).$$

where $f : \mathcal{A}^n \to \mathcal{A}^n$ is a projection with image P.

2. If M is **any** module,

$\dim_{\mathcal{A}}(M) =$ supremum of dimensions of fin. gen. proj. submodules $\in [0, \infty]$.



Nice properties

1. **Extension**: the two steps agree.
2. **Additivity** for short exact sequences.
3. **Cofinality**: dimension of directed union is supremum of dimensions.
4. **Continuity**: closure and dimension agree.
5. Every fin. gen. module **splits** as

finitely generated projective $\oplus$ torsion part

and the dimension faithfully measures the projective part.

Nice properties (cont.)

6. A finite VNA $\mathcal{A}$ has a regular overing Q.
   $Q = Q^r_{\mathrm{cl}}(\mathcal{A}) = Q^r_{\max}(\mathcal{A})$ and

$$\dim_{\mathcal{A}}(M) = \dim_Q(M \otimes_{\mathcal{A}} Q).$$

[V. 2005] This whole story **generalizes** to Baer *-rings satisfying certain ~~nine~~ **eight** ([V. 2006]) axioms. Let us call this class

Von-Neumann-algebra-like rings

Algebraic avenues

Berberian 1972. “Von Neumann algebras are blessed with an excess of structure – algebraic, geometric, topological – so much, that one can easily obscure, through proof by overkill, what makes a particular theorem work.”

“If all the functional analysis is stripped away … what remains should (be) completely accessible through algebraic avenues”.

Our goal

- Algebraic avenue to dimension of VNA.
- Preferably to get rid of requirements for some fancy axioms.
- To have dimension on much wider class of rings keeping all nice properties.

"but this is too general!" **"but this can be generalized!"**

Suitable class of rings

Right strongly semihereditary rings.

Few equivalent definitions.

Definition 1.	**Useful for**
R is right nonsingular and every fin. gen. nonsingular module is projective.	getting a splitting of a fin. gen. M as fin. gen. proj $PM \oplus$ torsion TM. $\dim_R(M) = \dim_R(PM)$ and $\dim_R(TM) = 0$.

Definition 2

Definition 2.	**Useful for**
R is right semihereditary and $Q = Q^r_{\max}(R)$ is a perfect left ring of quotients of R.	defining $\dim_Q$ first and getting $\dim_R(M)$ as $\dim_Q(M \otimes_R Q)$ for any M.

Definitions 3 and 4

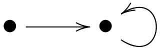
Definition 3.	**Useful for**
R is right nonsingular and R^n is CS, or extending, (CS = "complements are summands") for all n. **And so is Q.**	defining $\dim_Q(P)$ via the closure of the image of a map f $f : Q^n \to Q^n$ with image $f = P$.

Definition 4. R is right nonsingular and R^2 is CS.

Examples

Many rings for which you may want a dimension are on this list.

1. Regular and self-injective rings.
2. Commutative, semihereditary and noetherian rings.
3. Finite AW^*-algebras. More generally, finite von-Neumann-algebra-like rings.
4. There are some non-Baer-* rings on this list. For example, Leavitt path algebra over:



Examples with: semiher. $\gneq$ strongly semiher.; left $\neq$ right.

Even nicer with * around

If R is right strongly semihereditary **and** has an involution, then

- R is left strongly semihereditary as well.
- $M_n(R)$ is Baer for every n. (Note: not Baer *-necessarily).
- Q is unit-regular and directly finite.

- If * is **positive definite**

$$\sum_{i=1}^{n} x_i^* x_i = 0 \Rightarrow x_i = 0$$

for all i, for all n

then $M_n(Q)$ is *-regular Baer *-ring for every n.

These rings have dimension!

Strongly semihereditary rings with positive definite involution have dimension...

... and all the nice properties (additivity, cofinality etc) of dimension hold.

Idea of the proof: given in " Useful for" parts of the definitions.

Some corollaries

1. Von-Neumann-algebra-like-rings have dimension by algebraic arguments.

Q. Didn't we know that before?
A. Yes, but this time we used just seven axioms. Particularly ugly axioms 8 (and 9) are not needed.

2. Leavitt path algebras over finite no-exit graphs have dimension.

Q. Why is that relevant?
A. It gives us hope for non Rickart *-rings.

Another really cool corollary

In his book on Baer *-rings, Berberian asks

> If R is Baer *-ring when is $M_n(R)$ also Baer *-ring?

If R satisfies nine axioms (recall: 8 and 9 are bad), then **yes**.
[V. 2006] Axiom 9 is not needed.

Now we know,

Neither 8 nor 9 are needed.

Rougher lego world

Another Berberian's agenda: Study general relation $\sim$ on projections.

The real "main result": everything is formulated in terms of axioms on $\sim$.

The fact that a strongly semihereditary *-ring has a dimension is just a corollary of this general statement if $\sim$ is interpreted as

$p \stackrel{a}{\sim} q$ iff $xy = p$ and $yx = q$ for some x, y.

References

- Strongly semihereditary rings and rings with dimension, *Algebras and Representation Theory*, 15 (6) (2012) 1049 – 1079.

Preprints on

http://www.usciences.edu/∼lvas and on arXiv.