COMPARABILITY IN THE GRAPH MONOID

ROOZBEH HAZRAT AND LIA VAŠ

Abstract. Let Γ be the infinite cyclic group on a generator x. To avoid confusion when working with $\mathbb{Z}$-modules which also have an additional $\mathbb{Z}$-action, we consider the $\mathbb{Z}$-action to be a Γ-action instead.

Starting from a directed graph E, one can define a cancellative commutative monoid M_E^Γ with a Γ-action which agrees with the monoid structure. This monoid is the positive cone of the Grothendieck Γ-group G_E^Γ of the Leavitt path algebra $L_K(E)$ of E over any field when this algebra is considered with its natural Γ-graded structure. The Graded Classification Conjecture states that G_E^Γ is a complete invariant of $L_K(E)$.

The monoid M_E^Γ has a natural order which agrees with the Γ-action. The order and the action enable one to label each nonzero element as being exactly one of the following: comparable (periodic or aperiodic) or incomparable. We comprehensively pair up these element features with the graph-theoretic properties of the generators of the element. These characterizations provide further progress towards a positive answer to the Graded Classification Conjecture conjecture and imply that some results of [9] hold without requiring the graph to be row-finite.

0. Introduction

There are several different ways to associate an algebra over a field K to a directed graph E. For example, one can form the path algebra $P_K(E)$ which is a vector space over K based on paths multiplied using concatenation. If one wants to add a natural involutive structure to this algebra (as, for example, when completing the path algebra over complex numbers to obtain the graph C^*-algebra $C^*(E)$), then every vertex naturally becomes a self-adjoint idempotent, a projection, and every edge e becomes a partial isometry making the projections ee^* and e^*e equivalent. If $\mathbf{s}$ and $\mathbf{r}$ are the source and range maps of E respectively, and $\mathbf{s}(e) = v$, then $ve = e$ so that $vee^* = ee^*$ and, hence, $v \geq ee^*$ (recall that the projections are ordered by $p \leq q$ if $pq = p$). On the other hand, if $w = \mathbf{r}(e)$, then $ew = e$ and so $w \geq e^*e$. The requirement that $w = e^*e$ is called the (CK1) axiom. One also aims to have that the projections v and w are equivalent if and only if e is the *only* path from v to w. This is achieved by an additional requirement, the (CK2) axiom, stating that $v = \sum_{e\in\mathbf{s}^{-1}(v)} ee^*$ if v emits at least one and only finitely many edges. The axioms (CK1) and (CK2) imposed on the involutive closure of the path algebra produce the Leavitt path algebra $L_K(E)$. If $\mathcal{V}(L_K(E))$ is the monoid of the isomorphism classes of finitely generated projective modules (or conjugation classes of idempotent matrices), the (CK1) and (CK2) axioms imply that

$$[v] = \sum_{e\in\mathbf{s}^{-1}(v)} [\mathbf{r}(e)]$$

holds in $\mathcal{V}(L_K(E))$ for every vertex v which emits at least one and only finitely many edges. If E is such that every vertex emits only finitely many edges, in which case we say that E is row-finite, one of the first papers on Leavitt path algebras [5] shows that elements $[v]$ generate $\mathcal{V}(L_K(E))$ and that the above relations are the *only* relations which hold on $\mathcal{V}(L_K(E))$. Thus, to capture $\mathcal{V}(L_K(E))$ entirely, it is sufficient to consider a free commutative monoid M_E generated by $[v]$ where v is a vertex of E subject to the above relations. In [3], the authors generalized this construction to arbitrary graphs. To handle vertices which emit infinitely many edges (infinite emitters), one adds two natural relations to the one listed above (the details are reviewed in Section 1.5) to obtain M_E.

2010 *Mathematics Subject Classification.* 06F05, 05C25, 19A49, 06F20, 16E20, 20M32.
Key words and phrases. graph, group action, graph monoid, ordered abelian group, Grothendieck group.
The first author would like to acknowledge Australian Research Council grant DP160101481.

The monoid M_E is not necessarily cancellative which is easy to see: if v is a vertex emitting two edges to itself, then the relation $[v]+[v]=[v]$ holds in the monoid but the generator $[v]$ is nonzero. So, when one forms the Grothendieck group G_E of the monoid M_E a lot of information can get lost. In particular, if E is a graph consisting only of the vertex and edges from the previous example, then $G_E = 0$.

In addition to the above mentioned downside, very different graphs give rise to isomorphic monoids and, consequently, isomorphic Grothendieck groups. For example, $\bullet$ and $\bullet \circlearrowright$. In addition, consider the graphs E_1 and E_2 below, for example.

$$\bullet^{v_1} \longrightarrow \bullet^{w_1} \qquad\qquad \bullet^{v_2} \longrightarrow \bullet \longrightarrow \bullet^{w_2}$$

The relation $[v_1] = [w_1]$ holds in the first and the relation $[v_2] = [w_2]$ holds in the second graph monoid regardless of the fact that the length of the only path from v_1 to w_1 is 1 in E_1 while the length of the only path from v_2 to w_2 is 2 in E_2. So, this type of information is also lost in the Grothendieck group.

These downsides can be avoided by taking the natural grading of a Leavitt path algebra into consideration. Namely, the elements pq^* where p and q are paths, generate the entire algebra and if p and q are such that the difference of the length of p and the length of q is an integer n, the generator pq^* is considered to be in the n-th component of $L_K(E)$. This produces a $\mathbb{Z}$-graded structure of $L_K(E)$ where $\mathbb{Z}$ is the set of integers. For a ring R graded by a group Γ, the monoid $\mathcal{V}^\Gamma(R)$ of the graded isomorphism classes of finitely generated *graded* projective modules (or conjugation classes of certain homogeneous idempotent matrices) is a natural analogue of $\mathcal{V}(R)$. The monoid $\mathcal{V}^\Gamma(R)$ has a canonical Γ-action and we refer to a monoid with this type of structure as a Γ-monoid.

To avoid confusion when working with structures which are $\mathbb{Z}$-modules but also have an additional $\mathbb{Z}$-action, we let $\Gamma = \{x^n \mid n \in \mathbb{Z}\}$ and consider the $\mathbb{Z}$-action to be a Γ-action instead. The Γ-action on $\mathcal{V}^\Gamma(L_K(E))$ is such that the relation $[v] = \sum_{e\in \mathbf{s}^{-1}(v)}[\mathbf{r}(e)]$ becomes

$$[v] = \sum_{e\in \mathbf{s}^{-1}(v)} x[\mathbf{r}(e)]$$

if $\mathbf{s}^{-1}(v)$ is nonempty and finite. The power 1 of x in this relation indicates the length of the path e from v to $\mathbf{r}(e)$. With analogous modifications of the other defining relations, we let M_E^Γ be the quotient of a free Γ-monoid F_E^Γ with basis elements labeled by the vertices and the elements related to the infinite emitters subject to the defining relations. Alternatively, if $\to_1$ is a binary relation of F_E^Γ given by these defining relations, $\to$ is the reflexive and transitive closure of $\to_1$, and $\sim$ is the congruence closure of $\to$, then M_E^Γ is the quotient Γ-monoid $F_E^\Gamma/\sim$. The Γ-monoid M_E^Γ is naturally isomorphic to $\mathcal{V}^\Gamma(L_K(E))$.

The monoid M_E^Γ has several important advantages over M_E. First, it is always cancellative by [4, Corollary 5.8] (we give an alternative proof in Proposition 3.1) and so it is exactly the positive cone of its Grothendieck group G_E^Γ. This group inherits the Γ-action from M_E^Γ so we refer to it as the Grothendieck Γ-group. Second, the information on the lengths of paths from a vertex to vertex is not lost. For example, if E_1 and E_2 are the above two graphs, the relations $[v_1] = [w_1]$ and $[v_2] = [w_2]$ of M_{E_1} and M_{E_2} become

$$[v_1] = x[w_1] \quad \text{and} \quad [v_2] = x^2[w_2]$$

in $M_{E_1}^\Gamma$ and $M_{E_2}^\Gamma$ respectively. Here, the powers of x indicate that the length of the (only) path from v_1 to w_1 is 1 in E_1 and that the length of the (only) path from v_2 to w_2 is 2 in E_2. In addition, very different graphs $\bullet$ and $\bullet \circlearrowright$ have different Grothendieck Γ-groups: G_E^Γ of the first graph is isomorphic to $\mathbb{Z}[\Gamma]$ with the natural action of Γ while G_E^Γ of the second graph is isomorphic to $\mathbb{Z}$ with the trivial action of Γ.

Because of these favorable properties of M_E^Γ and, hence, G_E^Γ, it was conjectured in [7] that G_E^Γ is a complete invariant of E in the following sense.

> For any two row-finite graphs E and F and any field K, $L_K(E)$ and $L_K(F)$ are isomorphic as Γ-graded algebras if and only if G_E^Γ and G_F^Γ are isomorphic as ordered Γ-groups with order-units.

We let the *Graded Classification Conjecture* be the above statement without the restriction that E and F are row-finite. Since the monoid M_E^Γ is always cancellative, this conjecture can also be phrased in terms of the graph Γ-monoids instead of in terms of their Grothendieck Γ-groups.

The monoid M_E^Γ has a natural pre-order $\leq$ which agrees with the Γ-action. Since M_E^Γ is cancellative, this pre-order is, in fact, an order. In [9], the authors show that the relation $a < x^n a$ is impossible for any $a \in M_E^\Gamma$ and any positive integer n if E is row-finite. In Proposition 3.3, we show that this holds for all graphs E. Hence, there are two remaining cases.

(1) $a \geq x^n a$ for some positive integer n. In this case, we say that a is *comparable*.
(2) a and $x^n a$ incomparable for any positive integer n. In this case, we say that a is *incomparable*.

If a is comparable, there are two possibilities.

(1i) $a = x^n a$ for some positive integer n. In this case, we say that a is *periodic*.
(1ii) $a > x^n a$ for some positive integer n. In this case, we say that a is *aperiodic*.

In this paper, we provide complete characterizations of all four types of elements (comparable, incomparable, periodic and aperiodic) in terms of the graph-theoretic properties of the generators of an element. We obtain this by three groups of results. First, in Section 2, we obtain a graph-theoretic characterization of the relation $\rightarrow$ (Proposition 2.2). Second, in Sections 3.4 and 3.5, we introduce and study certain well-behaved building blocks of comparable elements, the stationary elements. Third, in Section 3.6, we produce a graph-theoretic characterization of a stationary element in Proposition 3.15. This enables us to prove Theorem 3.17, the main result of Section 3, which characterizes a comparable element in terms of the graph-theoretic properties of its generators.

In Section 4, we characterize periodic and aperiodic elements in Theorems 4.1 and 4.4. We have already found a use of Theorem 4.1: it was used in [10, Theorem 3.1] to characterize Leavitt path algebras which are crossed products in terms of the properties of the underlying graphs. We also characterize graphs such that *every* element of M_E^Γ is comparable (Theorem 3.19), periodic (Theorem 4.2), graphs such that every nonzero element of M_E^Γ is aperiodic (Theorem 4.5), incomparable (Corollary 4.7), graphs such that no nonzero element of M_E^Γ is periodic (Corollary 4.3), and graphs such that no element of M_E^Γ is aperiodic (Corollary 4.6). These characterizations comprehensively pair up the monoid and the graph properties and are summarized in the table below. In the table, $\mathrm{c}(a)$, $\mathrm{p}(a)$, $\mathrm{ap}(a)$, $\mathrm{ic}(a)$ shorten the statements that $a \in M_E^\Gamma$ is comparable, periodic, aperiodic, and incomparable respectively. The formula "$(\exists a \neq 0)\ \mathrm{c}(a)$", for example, shortens "There is a nonzero comparable element in M_E^Γ".

Property of the graph Γ-monoid	**Property of the graph**
$(\exists a \neq 0)\ \mathrm{c}(a) \ = \ (\exists a \neq 0)$ not $\mathrm{ic}(a)$	There is a cycle.
$(\forall a \neq 0)\ \mathrm{ic}(a) \ = \ (\forall a \neq 0)$ not $\mathrm{c}(a)$	There is no cycle.
$(\exists a \neq 0)\ \mathrm{p}(a)$	There is a cycle with no exits.
$(\exists a)\ \ \mathrm{ap}(a)$	There is a cycle with an exit.
$(\forall a)\ \ \mathrm{c}(a) \ = \ (\forall a)$ not $\mathrm{ic}(a)$	Condition from Theorem 3.19.
$(\forall a)\ \ \mathrm{p}(a)$	Condition from Theorem 4.2.
$(\forall a \neq 0)\ \mathrm{ap}(a)$	$(\forall a)\ \mathrm{c}(a)$ and every cycle has an exit.
$(\forall a \neq 0)$ not $\mathrm{p}(a)$	Every cycle has exits.
$(\forall a)$ not $\mathrm{ap}(a)$	No cycle has exits.
$(\exists a)\ \ \mathrm{ic}(a) \ = (\exists a)$ not $\mathrm{c}(a)$	Negation of $(\forall a)\ \mathrm{c}(a)$
$(\exists a)$ not $\mathrm{p}(a)$	Negation of $(\forall a)\ \mathrm{p}(a)$.
$(\exists a \neq 0)$ not $\mathrm{ap}(a)$	Negation of $(\forall a \neq 0)\ \mathrm{ap}(a)$.

In Section 4.1, we relax the assumptions of statements in [9]. In particular, we show that the main results of [9] hold without the requirement that the graph is row-finite (Corollaries 4.8, 4.9, 4.10 and the first part of Corollary 4.11). The second part of Corollary 4.11 summarizes further progress towards a positive answer to the Graded Classification Conjecture obtained as a corollary of our earlier results.

Although Leavitt path algebras have often been mentioned in the introduction, the rest of the paper is about a graph and its graph monoid, not about its Leavitt path algebra. The definition of a Leavitt path algebra is reviewed in Section 1.4 only for context. No result of this paper mentions these algebras except one part of Theorem 4.2 presented only for context. No prior knowledge of Leavitt path algebras is needed for any part of this paper.

1. Prerequisites, notation and preliminaries

In this section only, we use Γ to denote an arbitrary group with multiplicative notation. In the other sections of the paper, Γ stands for the infinite cyclic group generated by an element x.

1.1. **Pre-ordered Γ-monoids and Γ-groups.** If M is an additive monoid with a left action of Γ which agrees with the monoid operation, we say that M is a Γ-*monoid*. If G an abelian group with a left action of Γ which agrees with the group operation, we say that G is a Γ-*group*. Such action of Γ uniquely determines a left $\mathbb{Z}[\Gamma]$-module structure on G, so G is also a left $\mathbb{Z}[\Gamma]$-module.

Let $\geq$ be a reflexive and transitive relation (a pre-order) on a Γ-monoid M (Γ-group G) such that $g_1 \geq g_2$ implies $g_1 + h \geq g_2 + h$ and $\gamma g_1 \geq \gamma g_2$ for all g_1, g_2, h in M (in G) and $\gamma \in \Gamma$. We say that such monoid M is a *pre-ordered* Γ-*monoid* and that such a group G is a *pre-ordered* Γ-*group*.

If G is a pre-ordered Γ-group, the set $G^+ = \{x \in G \mid x \geq 0\}$, called the positive cone of G, is a Γ-monoid. Any additively closed subset M of G which contains 0 and is closed under the action of Γ, defines a pre-order Γ-group structure on G such that $G^+ = M$. Such set G^+ is *strict* if $G^+ \cap (-G^+) = \{0\}$ and this condition is equivalent with the pre-order being a partial order. In this case, we say that G is an *ordered* Γ-*group*. For example, $\mathbb{Z}[\Gamma]$ is an ordered Γ-group with the positive cone $\mathbb{Z}^+[\Gamma]$ consisting of elements $a = \sum_{i=1}^n k_i\gamma_i \in \mathbb{Z}[\Gamma]$ such that $k_i \geq 0$ for all $i = 1, \dots, n$.

An element u of a pre-ordered Γ-monoid M is an *order-unit* if for any $x \in M$, there is a nonzero $a \in \mathbb{Z}^+[\Gamma]$ such that $x \leq au$. An element u of a pre-ordered Γ-group G is an *order-unit* if $u \in G^+$ and for any $x \in G$, there is a nonzero $a \in \mathbb{Z}^+[\Gamma]$ such that $x \leq au$.

If G and H are pre-ordered Γ-groups, a $\mathbb{Z}[\Gamma]$-module homomorphism $f \colon G \to H$ is *order-preserving* or *positive* if $f(G^+) \subseteq H^+$. If G and H are pre-ordered Γ-groups with order-units u and v respectively, an order-preserving $\mathbb{Z}[\Gamma]$-module homomorphism $f \colon G \to H$ is *order-unit-preserving* if $f(u) = v$.

A Γ-*order-ideal* of a pre-ordered Γ-monoid M is a Γ-submonoid I of M such that $a \leq b$ and $b \in I$ implies $a \in I$. If G is a pre-ordered Γ-group, a Γ-subgroup J of G is a Γ-*order-ideal* of G if $J \cap G^+$ is a Γ-order-ideal of G^+ and $J = \{x - y \mid x, y \in J \cap G^+\}$ (equivalently, J is a directed and convex Γ-subgroup of G using definitions of a directed set and a convex set from [6]). The lattices of Γ-order-ideals of G^+ and Γ-order-ideals of G are isomorphic by the map $I \mapsto \{x - y \mid x, y \in I\}$ with the inverse $J \mapsto J \cap G^+$.

1.2. **Graded rings.** We briefly review the concept of graded rings for context only. Other than a part of the statement of Corollary 4.10, no result of this paper refers to graded rings or requires any knowledge of their properties.

A ring R is Γ-*graded* if $R = \bigoplus_{\gamma \in \Gamma} R_\gamma$ where R_γ is an additive subgroup of R and $R_\gamma R_\delta \subseteq R_{\gamma\delta}$ for all $\gamma, \delta \in \Gamma$. The standard definitions of graded right R-modules, graded module homomorphisms and isomorphisms, and graded projective right modules can be found in [12] and [8]. If M is a graded right R-module and $\gamma \in \Gamma$, the γ-*shifted* graded right R-module $(\gamma)M$ is defined as the module M with the Γ-grading given by $(\gamma)M_\delta = M_{\gamma\delta}$ for all $\delta \in \Gamma$.

If R is a Γ-graded ring, let $\mathcal{V}^\Gamma(R)$ denote the monoid of graded isomorphism classes $[P]$ of finitely generated graded projective right R-modules P with the direct sum as the addition operation and the left Γ-action given by $(\gamma, [P]) \mapsto [(\gamma^{-1})P]$.[1] In particular, the definitions and results of [8, §3.2] carry to the

[1] If M is a graded left R-module and $\gamma \in \Gamma$, the γ-shifted graded left R-module $M(\gamma)$ is the module M with the Γ-grading given by $M(\gamma)_\delta = M_{\delta\gamma}$ for all $\delta \in \Gamma$. The monoid $\mathcal{V}^\Gamma(R)$ can be represented using the classes of left modules in which case the corresponding formula is $(\gamma, [P]) \mapsto [P(\gamma)]$. Two representations are equivalent (see [12, Section 2.4] or [8, Section 1.2.3]).

case when Γ is not necessarily abelian as it is explained in [15, Section 1.3]. The *Grothendieck Γ-group* $K_0^\Gamma(R)$ is defined as the group completion of the Γ-monoid $\mathcal{V}^\Gamma(R)$ which naturally inherits the action of Γ from $\mathcal{V}^\Gamma(R)$. The monoid $\mathcal{V}^\Gamma(R)$ is a pre-ordered Γ-monoid and the group $K_0^\Gamma(R)$ is a pre-ordered Γ-group for any Γ-graded ring R. If Γ is the trivial group, $K_0^\Gamma(R)$ is the usual K_0-group.

1.3. **Graphs.** If E is a directed graph, let E^0 denote the set of vertices, E^1 the set of edges and $\mathbf{s}$ and $\mathbf{r}$ the source and the range maps of E. The graph E is *finite* if both E^0 and E^1 are finite and E is *row-finite* if $\mathbf{s}^{-1}(v)$ is finite for every $v \in E^0$. A vertex $v \in E^0$ is a *sink* if $\mathbf{s}^{-1}(v) = \emptyset$ and a *source* if $\mathbf{r}^{-1}(v) = \emptyset$. A vertex of E is *regular* if $\mathbf{s}^{-1}(v)$ is finite and nonempty.

We use the standard definitions of a *path*, a *closed simple path* and a *cycle* (see [1, Definitions 1.2.2. and 2.0.2]). A path q is a *prefix* of a path p if $p = qr$ for some path r. If $q = \mathbf{s}(p)$, then q is a *trivial* prefix. If $r \neq \mathbf{r}(p)$, then q is a *proper* prefix. If E has no cycles, E is *acyclic.* A cycle c *has an exit* if a vertex on c emits an edge outside of c. The graph E satisfies *Condition (NE)* (and E is a *no-exit graph* in this case) if v emits just one edge for every vertex v of every cycle. The graph E satisfies *Condition (L)* if every cycle has an exit (equivalently if every closed simple path has an exit) and E satisfies *Condition (K)* if for each vertex v which lies on a closed simple path, there are at least two different closed simple paths based at v. An *infinite path* is a sequence of edges $e_1e_2\ldots$ such that $\mathbf{r}(e_i) = \mathbf{s}(e_{i+1})$ for $i = 1, 2, \ldots$. Such infinite path *ends in a cycle* if there is a positive integer n and a cycle c such that $e_ne_{n+1}\ldots$ is equal to $cc\ldots$.

If E is a finite and acyclic graph, it is well-established that it has a source. Since we were not aware of a reference for this fact and we use it in the proof of Lemma 3.10, we provide a quick proof for it.

Lemma 1.1. *If E is a finite and acyclic graph, it has a source.*

Proof. If the graph E does not have any edges, then each of its vertices is both a source and a sink. If E has edges, pick any of them, say e_0. If $\mathbf{r}^{-1}(\mathbf{s}(e_0))$ is empty, then $\mathbf{s}(e_0)$ is a source. If $\mathbf{r}^{-1}(\mathbf{s}(e_0))$ is nonempty, take $e_1 \in \mathbf{r}^{-1}(\mathbf{s}(e_0))$. Then $e_0 \neq e_1$ since otherwise $\mathbf{r}(e_0) = \mathbf{s}(e_0)$ and e_0 would be a cycle. If $\mathbf{r}^{-1}(\mathbf{s}(e_1))$ is empty, then $\mathbf{s}(e_1)$ is a source. If $\mathbf{r}^{-1}(\mathbf{s}(e_1))$ is nonempty, continue the process. At any step of the process, we obtain a different edge than any of the edges considered previously otherwise E has a cycle. Since E is finite, this process eventually ends. If it ends at the n-th step, then $\mathbf{s}(e_n)$ is a source. $\square$

1.4. **Leavitt path algebras.** We review the concept of a Leavitt path algebra for context only. No result of this paper except one part of Theorem 4.2 refers to Leavitt path algebras or requires any knowledge of these algebras. If K is any field, the *Leavitt path algebra* $L_K(E)$ of E over K is a free K-algebra generated by the set $E^0 \cup E^1 \cup \{e^* \mid e \in E^1\}$ such that, for all vertices v, w and edges e, f,

(V) $vw = 0$ if $v \neq w$ and $vv = v$,
(E1) $\mathbf{s}(e)e = e\mathbf{r}(e) = e$,
(E2) $\mathbf{r}(e)e^* = e^*\mathbf{s}(e) = e^*$,
(CK1) $e^*f = 0$ if $e \neq f$ and $e^*e = \mathbf{r}(e)$,
(CK2) $v = \sum_{e\in\mathbf{s}^{-1}(v)} ee^*$ for each regular vertex v.

By the first four axioms, $L_K(E)$ is a K-linear span of the elements of the form pq^* for paths p and q. If $L_K(E)_n$ is the K-linear span of pq^* for paths p and q with $|p| - |q| = n$ where $|p|$ denotes the length of a path p, then it is the n-component of $L_K(E)$ producing a natural grading of $L_K(E)$ by the group of integers $\mathbb{Z}$. One can also grade $L_K(E)$ by any group Γ as follows. Any function $w\colon E^1 \to \Gamma$, called the *weight* function, extends by $w(e^*) = w(e)^{-1}$ for $e \in E^1$ and $w(v) = \varepsilon$ for $v \in E^0$, and, ultimately, by $w(pq^*) = w(p)w(q)^{-1}$ for any generator pq^* of $L_K(E)$ (see [8, Section 1.6]). Thus, $L_K(E)$ becomes Γ-graded with $L_K(E)_\gamma$ being the K-linear span of the elements pq^* with weight γ.

1.5. **The monoid and the Grothendieck group of a graph.** If E is a graph, the graph monoid M_E was defined for row-finite graphs in [5] and for arbitrary graphs in [3]. We briefly review this definition.

Any edge $e \in E^1$ is a partial isometry of ee^* and $\mathbf{r}(e) = e^*e$ so that $[ee^*]$ and $[\mathbf{r}(e)]$ are the same element in $\mathcal{V}(L_K(E))$. Hence, the relation below holds in $\mathcal{V}(L_K(E))$ by the (CK2)-axiom if v is regular.

$$[v] = \sum_{e\in\mathbf{s}^{-1}(v)} [\mathbf{r}(e)]$$

For any infinite emitter v and any finite and nonempty $Z \subseteq \mathbf{s}^{-1}(v)$, one considers the element q_Z representing $v - \sum_{e \in Z} ee^*$. We refer to the elements of the form q_Z as the *improper vertices.* When we need to emphasize that q_Z is related to the infinite emitter v (in the sense that $Z \subseteq \mathbf{s}^{-1}(v)$) we write q_Z^v for q_Z. Also, whenever the notation q_Z appears, it is to be understood that there is an infinite emitter v and that Z is a finite and nonempty subset of $\mathbf{s}^{-1}(v)$. For any finite sets Z and W such that $\emptyset \subsetneq Z \subsetneq W \subsetneq \mathbf{s}^{-1}(v)$, it is direct to check that the relations

$$[v] = [q_Z] + \sum_{e \in Z} [\mathbf{r}(e)] \quad \text{and} \quad [q_{Z_1}] = [q_{Z_2}] + \sum_{e \in W - Z} [\mathbf{r}(e)]$$

also hold in $\mathcal{V}(L_K(E))$. So, one aims to define M_E so that the above three relations are the *only* relations which hold in M_E. This is achieved in the following way.

Let F_E be a free commutative monoid generated by the elements indexed by the proper and improper vertices of E. To be consistent with [3], [4] and [9], we abuse the notation and refer to the generator indexed by a proper vertex $v \in E^0$ as v and, similarly, to the generator indexed by q_Z by q_Z. The monoid M_E is the quotient of F_E with respect to the the congruence closure $\sim$ of the relation $\to_1$ defined on $F_E - \{0\}$ by

$$a + v \to_1 a + \sum_{e \in s^{-1}(v)} r(e),$$

whenever v is a regular vertex and $a \in F_E$ and by

$$a + v \to_1 a + q_Z + \sum_{e \in Z} \mathbf{r}(e) \quad \text{and} \quad a + q_Z \to_1 a + q_W + \sum_{e \in W - Z} \mathbf{r}(e)$$

whenever v is an infinite emitter and Z and W are finite and such that $\emptyset \subsetneq Z \subsetneq W \subsetneq \mathbf{s}^{-1}(v)$.

One often considers an intermediate step of this construction and lets $\to$ be the transitive and reflexive closure of $\to_1$ on F_E so that $\to$ is a pre-order. In this case, $\sim$ is the congruence on F_E generated by the relation $\to$ (i.e. the symmetric closure of the pre-order $\to$).

We use the notation $[v]$ for the congruence class of v as an element of M_E. By [1, Corollary 3.2.11] (or [3, Theorem 4.3]), the map $[v] \mapsto [vL_K(E)]$ extends to a pre-ordered monoid isomorphism of M_E and $\mathcal{V}(L_K(E))$ (here $\mathcal{V}(L_K(E))$ is given using the finitely generated projective right modules). So, the Grothendieck group completion G_E of M_E is isomorphic to $K_0(L_K(E))$.

1.6. **The Γ-monoid and the Grothendieck Γ-group of a graph.** Let Γ be a group and $w\colon E^1 \to \Gamma$ be a function which we refer to as a weight determining a Γ-grading of $L_K(E)$. The following relations hold in the Γ-monoid $\mathcal{V}^\Gamma(L_K(E))$. For every regular vertex v,

$$\gamma[v] = \sum_{e \in \mathbf{s}^{-1}(v)} \gamma w(e)[\mathbf{r}(e)],$$

and for every infinite emitter v and finite Z and W such that $\emptyset \subsetneq Z \subsetneq W \subsetneq \mathbf{s}^{-1}(v)$,

$$\gamma[v] = \gamma[q_Z] + \sum_{e \in Z} \gamma w(e)[\mathbf{r}(e)] \quad \text{and} \quad \gamma[q_Z] = \gamma[q_W] + \sum_{e \in W - Z} \gamma w(e)[\mathbf{r}(e)].$$

To adapt the original construction of M_E to this setting, the authors of [4] replaced generators v and q_Z of F_E by $v(\gamma)$ and $q_Z(\gamma)$ for any $\gamma \in \Gamma$ and considered a free commutative monoid F_E^Γ with the action of Γ given by $\delta v(\gamma) = v(\delta\gamma)$ and $\delta q_Z(\gamma) = q_Z(\delta\gamma)$ for all $\gamma, \delta \in \Gamma$. Then M_E^Γ is the quotient of F_E^Γ subject to the congruence closure $\sim$ of relation $\to_1$ defined just as in the previous section but with the three relations modified accordingly so that

$$a + \gamma v \to_1 a + \sum_{e \in s^{-1}(v)} \gamma w(e) r(e),$$

whenever v is a regular vertex and $a \in F_E$ and by

$$a + \gamma v \to_1 a + \gamma q_Z + \sum_{e \in Z} \gamma w(e)\mathbf{r}(e) \quad \text{and} \quad a + \gamma q_Z \to_1 a + \gamma q_W + \sum_{e \in W - Z} \gamma w(e)\mathbf{r}(e)$$

whenever v is an infinite emitter and Z and W are finite and such that $\emptyset \subsetneq Z \subsetneq W \subsetneq \mathbf{s}^{-1}(v)$.

One downside of this approach is that M_E^Γ is still considered to be a commutative monoid, not a commutative Γ-monoid. For example, if E is a single vertex, M_E^Γ is a direct sum of $|\Gamma|$-many copies of $\mathbb{Z}^+$ (with a natural action of Γ) instead of being a single copy of $\mathbb{Z}^+[\Gamma]$. Also, the abundance of generators can make some proofs less direct. Because of this, we adopt a simpler and more intuitive approach here: we let M_E^Γ be defined by the same set of generators as when the weight function is trivial, but we let F_E^Γ be a free commutative Γ-monoid, not a free commutative monoid. In this case, if E is a single vertex, then M_E is a single copy of $\mathbb{Z}^+$ and M_E^Γ is a single copy of $\mathbb{Z}^+[\Gamma]$. The equivalence of ours and the construction from [4] can be seen considering the graph covering $\overline{E}$ of E.

So, we let F_E^Γ be a free commutative Γ-monoid generated by proper and improper vertices. A nonzero element a of F_E^Γ has a *representation*, unique up to a permutation, as $\sum_{j=1}^n \alpha_i g_i$, where g_i are *different* generators of F_E^Γ and $\alpha_i \in \mathbb{Z}^+[\Gamma]$. The *support* $\operatorname{supp}(a)$ of a is the set $\{g_i \mid i = 1, \ldots, n\}$.

Let $k_\gamma \in \mathbb{Z}^+$ be the coefficient of $\gamma \in \Gamma$ in $\alpha_i \in \mathbb{Z}^+[\Gamma]$ in the above representation. By writing each $k_\gamma > 0$ as the sum $1 + 1 + \ldots + 1$, one obtains the format $a = \sum_{j=1}^m \gamma_j g_j$ for some positive integer m and $\gamma_j \in \Gamma, j = 1, \ldots, m$. We allow the generators g_j and g_k to be possibly equal for $j \neq k$ in this form, also unique up to a permutation. We refer to it as a *normal representation* of a and we say that each summand $\gamma_j g_j$ of this representation is a *monomial* of a. We can still write $\operatorname{supp}(a) = \{g_j \mid j = 1, \ldots, m\}$ because any possible repetition of an element does not impact $\operatorname{supp}(a)$ as a set.

For example, if Γ is the infinite cyclic group generated by x, v is a vertex of E, and $a = xv + 3v$, then $(x + 3)v$ is a representation of a and $xv + v + v + v$ is a normal representation of a.

To shorten some statements, we say that a vertex v, considered as a generator of F_E^Γ, is *regular* if v is regular as a vertex of E. We also say that a generator $v \in F_E^\Gamma$ is a sink or an infinite emitter, if v is a sink or an infinite emitter as a vertex of E. An element $a \in F_E^\Gamma$ is *regular* if every element of $\operatorname{supp}(a)$ is regular.

The Γ-monoid M_E^Γ is obtained as a quotient of F_E^Γ subject to the congruence closure $\sim$ of the relation $\to_1$ on $F_E^\Gamma - \{0\}$ defined by (A1), (A2) and (A3) below for any $\gamma \in \Gamma$ and $a \in F_E^\Gamma$.

(A1) If v is a regular vertex, then
$$a + \gamma v \to_1 a + \sum_{e \in s^{-1}(v)} \gamma w(e)\mathbf{r}(e).$$

(A2) If v is an infinite emitter and Z a finite and nonempty subset of $\mathbf{s}^{-1}(v)$, then
$$a + \gamma v \to_1 a + \gamma q_Z + \sum_{e \in Z} \gamma w(e)\mathbf{r}(e).$$

(A3) If v is an infinite emitter and $Z \subsetneq W$ are finite and nonempty subsets of $\mathbf{s}^{-1}(v)$, then
$$a + \gamma q_Z \to_1 a + \gamma q_W + \sum_{e \in W - Z} \gamma w(e)\mathbf{r}(e).$$

So, if $\to$ is the reflexive and transitive closure of $\to_1$ on F_E^Γ, then $\sim$ is the congruence on F_E^Γ generated by the relation $\to$. This means that the relation $a \sim b$ holds for some $a, b \in F_E^\Gamma - \{0\}$ if and only if there is a nonnegative integer n and $a = a_0, \ldots, a_n = b \in F_E^\Gamma - \{0\}$ such that $a_i \to_1 a_{i+1}$ or $a_{i+1} \to_1 a_i$ for all $i = 0, \ldots, n-1$. We refer to such n as the *length of the sequence* $a_0, \ldots, a_n$ and we write $a \sim^n b$ to emphasize the length. In particular, if $a \to b$, the sequence can be chosen so that $a_i \to_1 a_{i+1}$ for all $i = 0, \ldots, n-1$. In this case, we write
$$a \to^n b.$$
Note that $a \to^1 b$ is just $a \to_1 b$ and that $a \to^0 b$ is just $a = b$.

To shorten the notation in multiple proofs, if g is a generator of F_E^Γ, and one of the three axioms is applied to g, we use $\mathbf{r}(g)$ to denote the resulting term on the right side of relation $\to_1$: $\sum_{e \in \mathbf{s}^{-1}(v)} w(e)\mathbf{r}(e)$ if $g = v$ is a regular vertex, $q_Z + \sum_{e \in Z} w(e)\mathbf{r}(e)$ for some finite and nonempty subset Z of $\mathbf{s}^{-1}(v)$ if $g = v$ is an infinite emitter, or $q_W + \sum_{e\in W-Z} w(e)\mathbf{r}(e)$ for some finite Z and W such that $\emptyset \subsetneq Z \subsetneq W \subsetneq \mathbf{s}^{-1}(v)$ if $g = q_Z^v$ for an infinite emitter v. The element $\mathbf{r}(g)$ is uniquely determined just for (A1). However, for a *fixed* use of (A2) or (A3) which is not changed within a proof, the notation $\mathbf{r}(g)$ is a well-defined shortening. Such uniform treatment enables us to condense some proofs by avoiding considerations of three separate cases depending on which axiom is used.

Another benefit of our approach is that the proofs of many known statements in the case when Γ is trivial directly transfer to the case when Γ is not trivial. For example, if $[g]$ denotes the congruence class of a generator g of F_E^Γ, the map $[g] \mapsto [gL_K(E)]$ extends to a pre-ordered Γ-monoid isomorphism of M_E^Γ and $\mathcal{V}^\Gamma(L_K(E))$ and the proof of the case when Γ is trivial (see, for example, [3, Theorem 4.3]) directly adapts to the case when Γ is arbitrary. In [4, Proposition 5.7], this monoid isomorphism is shown to exist by considering the graph covering.

Lemma 1.2 greatly simplifies many proofs which involve handling relation $\sim$. Parts of this lemma can be shown by directly generalizing the proofs of [3, Lemmas 5.6 and 5.8]. We add some new elements in part (1) of Lemma 1.2 to control the length of sequences for certain relations. We also note that part (2), the Confluence Lemma, is shown for general Γ in [4, Lemma 5.9] but using the graph covering. The Confluence Lemma is key for showing that the monoid M_E^Γ has the refinement property (see [3, Proposition 5.9]).

Lemma 1.2. *Let E be a graph, Γ a group, $w\colon E^1 \to \Gamma$ a weight function and $a, b \in F_E^\Gamma - \{0\}$.*

(1) *(The Refinement Lemma) If $a = a' + a''$ for some $a', a'' \in F_E^\Gamma$ and if $a \to^n b$, then b has summands $b', b'' \in F_E^\Gamma$ and n has summands i, j such that $b = b' + b''$, $i + j = n$, $a' \to^i b'$, and $a'' \to^j b''$.*
(2) *(The Confluence Lemma) The relation $a \sim b$ holds if and only if $a \to c$ and $b \to c$ for some $c \in F_E - \{0\}$.*

Proof. We show (1) by induction on n. If $n = 0$ then $a = b$ and we can take $b' = a' = a = b$, $b'' = a'' = 0$, and $i = j = 0$. Assuming the induction hypothesis, let $a = a_0 \to_1 a_1 \to_1 \ldots \to_1 a_n = b$ and let γg be a monomial of a so that a_1 is obtained by replacing γg by $\gamma\mathbf{r}(g)$. Since $a = a' + a''$, γg is a summand of either a' or a''. Say it is a' (the case when it is a'' is analogous) and let $a' = c + \gamma g$ for some $c \in F_E^\Gamma$. For $a_1' = c + \gamma\mathbf{r}(g)$ and $a_1'' = a''$, $a' \to^1 a_1'$ and $a'' \to^0 a_1''$. The induction hypothesis implies the existence of $b', b'' \in F_E^\Gamma$ and i, j such that such that $b = b' + b''$, $i + j = n - 1$, $a_1' \to^i b'$ and $a_1'' \to^j b''$. Thus, $i + 1 + j = n$, $a' \to^1 a_1' \to^i b'$, and $a'' \to^0 a_1'' \to^j b''$ and so $a' \to^{i+1} b'$ and $a'' \to^j b''$.

The direction $\Leftarrow$ of (2) is direct since if $a \to c$ and $b \to c$, then $a \sim c$ and $b \sim c$ so that $a \sim b$. First, we show the direction $\Rightarrow$ of (2) for finite graphs using induction on n for $a \sim^n b$. If $n = 0$, $a = b$ and we can take $c = a = b$. Assuming the induction hypothesis, let $a \sim^n b$, $a_0 = a, a_n = b$ and let $a_i \to_1 a_{i+1}$ or $a_{i+1} \to_1 a_i$ for some $a_i \in F_E^\Gamma$ for $i = 0, \ldots, n-1$. Since $a_1 \sim^{n-1} b$, there is d such that $a_1 \to d$ and $b \to d$. Then either $a \to_1 a_1$ or $a_1 \to_1 a$. In the first case, we can take $c = d$. In the second case, there is a monomial γg of a_1 so that $a_1 = a' + \gamma g$ for some a' and $a = a' + \gamma\mathbf{r}(g)$. By part (1), $d = d' + d''$ for some d' and d'' such that $a' \to d'$ and $\gamma g \to^l d''$ for some $l \geq 0$. If $l = 0$, then $d'' = \gamma g$ so $d = d' + \gamma g$. Let $c = d' + \gamma\mathbf{r}(g)$. Then we have that $a = a' + \gamma\mathbf{r}(g) \to d' + \gamma\mathbf{r}(g) = c$ and $b \to d = d' + \gamma g \to_1 d' + \gamma\mathbf{r}(g) = c$.

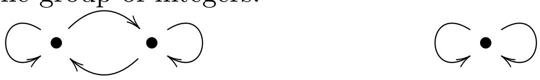
If l is positive, we use the assumption that E is finite to conclude that there are no infinite emitters so that g is necessarily a regular vertex and $a_1 \to_1 a$ is an application of (A1.) Hence, the relation $\gamma g \to d''$ necessarily decomposes as $\gamma g \to_1 \gamma\mathbf{r}(g) \to d''$ and we have that $a_1 = a' + \gamma g \to_1 a = a' + \gamma\mathbf{r}(g) \to d' + d'' = d$. So, in this case we can also take $c = d$.

To complete the proof in the case when E is an arbitrary graph, we use the argument of the proof of [4, Lemma 5.9] relying on [3, Construction 5.3]. If $R(E)$ denotes the set of regular vertices of E, the pair $(E, R(E))$, considered as an element of an appropriate category from [3, Section 3], can be represented as a direct limit of pairs (E', S) where E' is a finite subgraph of E and S is a subset of $R(E')$ (see [3, Proposition 3.3] for details). The pair (E', S) gives rise to the relative graph E'_S of E' with respect to S such that the bijection on the generators of the corresponding free Γ-monoids produces a natural Γ-monoid isomorphism (see [11, Theorem 3.7] and the graded version in [14, Lemma 2.2]). Hence, if $a, b \in F_E^\Gamma$ correspond to elements a' and b' of $F_{E'_S}^\Gamma$ for some finite subgraph E' and some subset S of $R(E')$, then the relation $a \sim b$

holds in F_E^Γ if and only if $a' \sim b'$ holds in $F_{E'_S}^\Gamma$. Assuming that $a \sim b$ holds, we have that $a' \sim b'$ holds. By the proven claim for finite graphs, there is $c' \in F_{E'_S}^\Gamma$ such that the relations $a' \to c'$ and $b' \to c'$ hold in $F_{E'_S}^\Gamma$. If $c \in F_E^\Gamma$ corresponds to c', these relations imply that $a \to c$ and $b \to c$ hold in F_E^Γ. □

One can also show the Confluence Lemma directly, by considering an arbitrary graph E and discussing possibilities that the relation $a_1 \to_1 a$ in the above proof is obtained by (A2) or (A3).

We conclude this section by a remark: the Graded Classification Conjecture is false if the pre-ordered Γ-groups (equivalently Γ-monoids) of the graphs are replaced by the free Γ-monoids. Indeed, let E and F be the graphs below and Γ be the group of integers.



The graph E is an out-split of the graph F, so the Leavitt path algebras of E and F are graded isomorphic (see [2, Theorem 2.8]). Hence, M_E^Γ and M_F^Γ are isomorphic as Γ-monoids and G_E^Γ and G_F^Γ are isomorphic as pre-ordered Γ-groups with order-units. However, F_E^Γ and F_F^Γ are not isomorphic as Γ-monoids since one has two while the other has one generator. This example illustrates that the Γ-monoid F_E^Γ of a graph E is informative only when considered together with the relation $\sim$.

2. Connectivity

In this section and the rest of the paper, $\Gamma = \{x^n \mid n \in \mathbb{Z}\}$ is the infinite cyclic group with generator x and E is an arbitrary graph. To simplify the terminology in some of the proofs, we say that n is the *degree* of the monomial $x^n g$ where g is a generator of F_E^Γ. First, we characterize the relation $\to$ in terms of the graph-theoretic properties (Proposition 2.2).

If v and w are vertices of E and p a path from v to w, one can apply (A1) or (A2) to the vertices on p to obtain that $v \to x^{|p|}w + a$ for some $a \in F_E^\Gamma$. Indeed, if p is trivial, then $v = w$ and one can take $a = 0$. If $p = e_1e_2 \dots e_n$, one can apply (A1) if v is regular and (A2) if it is not, an then apply (A1) to $\mathbf{r}(e_1)$ if it is regular and (A2) if it is not. Continuing this process, one obtains a sequence for

$$v \to x^{|p|}w + a$$

for some $a \in F_E^\Gamma$, where the "change" a reflects the existence of bifurcations from p. For example, in the graph below with $p = f$, we have that $v \to xw + xu$ so $a = xu$.

$$\bullet^u \xleftarrow{\;e\;} \bullet^v \xrightarrow{\;f\;} \bullet^w$$

We generalize this process to improper vertices also. The terminology introduced below allows uniform treatment of generators of F_E^Γ of both types and enables us to express the comparability properties in terms of the properties of the graph E.

Definition 2.1. Let g and h be generators of F_E^Γ. We say that g *connects to* h *by a path* p (written $g \rightsquigarrow^p h$) if one of the following conditions hold.

(i) $g = v$ and $h = w$ are proper vertices and p is a path from v to w. In this case, $v \to x^{|p|}w + a$ holds for some $a \in F_E^\Gamma$ as we pointed out above.

(ii) $g = v$ is a proper vertex, $h = q_Z^w$ for an infinite emitter w and some Z, and p is a path from v to w. In this case, $v \to x^{|p|}w + a' \to x^{|p|}q_Z + a$ for some $a' \in F_E^\Gamma$ and $a = a' + \sum_{e\in Z} x^{|p|+1}\mathbf{r}(e)$. Note that if $v = w$ and p is trivial then $\to$ can be chosen to be a single application of (A2). If $v = w$ and p has positive length, then v is necessary on a cycle.

(iii) $g = q_Z^v$ for an infinite emitter v and some Z, $h = w$ is a proper vertex, and $p = eq$ is a path from v to w such that $e \notin Z$. In this case,

$$q_Z \to q_{Z\cup\{e\}} + x\mathbf{r}(e) \to q_{Z\cup\{e\}} + x^{|p|}w + a' = x^{|p|}w + a$$

for some $a' \in F_E^\Gamma$ and for $a = a' + q_{Z\cup\{e\}}$. If $v = w$, then v is on a cycle.

(iv) $g = q_Z^v$ for some v and Z, $h = q_W^w$ for some w and W, p is a path from v to w, and one of the following two scenarios hold.
- If p is trivial, then $v = w$ and $Z \subseteq W$. If $Z = W$, then $q_Z \to x^0 q_Z$ and if $Z \subsetneq W$ and $a = \sum_{e \in W-Z} x\mathbf{r}(e)$, then
$$q_Z \to x^0 q_W + \sum_{e \in W-Z} x\mathbf{r}(e) = x^0 q_W + a.$$
- If p has positive length, then $p = eq$ for some $e \notin Z$. In this case,
$$q_Z \to q_{Z\cup\{e\}} + x\mathbf{r}(e) \to q_{Z\cup\{e\}} + x^{|p|}w + a' \to q_{Z\cup\{e\}} + x^{|p|}q_W + \sum_{f\in W} x^{|p|+1}\mathbf{r}(f) + a' = x^{|p|}q_W + a$$
for some $a' \in F_E^\Gamma$ and $a = a' + q_{Z\cup\{e\}} + \sum_{f\in W} x^{|p|+1}\mathbf{r}(f)$. If $v = w$, then v is on a cycle.

Definition 2.1 enables us to deal with every generator of F_E^Γ in a uniform way. In particular, in any of the above four cases, we have that
$$g \to x^{|p|}h + a$$
for some element $a \in F_E^\Gamma$ and a path p. In this case, we say that h *is obtained from* g *following the path* p. The element a reflects the existence of bifurcations from p. In Corollary 2.4, we show the converse: $g \to x^n h + a$ implies that $g \rightsquigarrow^p h$ for a path p of length n.

We say that g *connects to* h, written $g \rightsquigarrow h$, if there is a path p such that $g \rightsquigarrow^p h$. If v and w are vertices, $v \rightsquigarrow w$ is usually written $v \geq w$ (see [1, Definition 2.0.4]). However, we reserve the relation $\geq$ for the order on the monoid M_E^Γ. It is direct to check that $\rightsquigarrow$ is reflexive and transitive.

Note that a proper vertex v is on a cycle if and only if v connects to v by a path of positive length. Definition 2.1 enables us to talk about improper vertices being on cycles: we say that any generator g of F_E^Γ *is on a cycle* if g connects to g by a path of positive length. We say that g *is on an exit from a cycle* c if g is not on c and there is a generator h of F_E^Γ which is on c such that h connects to g. By Definition 2.1, q_Z^v is on a cycle if and only if there is $e \in \mathbf{s}^{-1}(v) - Z$ and a path p with $\mathbf{r}(p) = v, \mathbf{s}(p) = \mathbf{r}(e)$ such that ep is a cycle.

If $a \to b$ and $a = \sum_{i=1}^k x^{m_i} g_i$ and $b = \sum_{j=1}^l x^{t_j} h_j$ are normal representations of a and b respectively, repeated use of the Refinement Lemma 1.2(1) ensures the existence of a partition $\{I_1, \ldots, I_k\}$ of $\{1, \ldots, l\}$ and summands b_i of b such that $b = \sum_{i=1}^k b_i$, $b_i = \sum_{j\in I_i} x^{t_j} h_j$, and $x^{m_i} g_i \to b_i$. This implies that $t_j \geq m_i$ for all $j \in I_i$. Proposition 2.2 implies the existence of paths p_{ij} with $|p_{ij}| = t_j - m_i$ and $g_i \rightsquigarrow^{p_{ij}} h_j$ in this case. We introduce this idea of partitioning b according to a if $a \to b$ in Proposition 2.2 and use it again in Section 3.6. Proposition 2.2 describes the relation $a \to b$ in terms of the properties of the generators in the supports of a and b and the length of the paths connecting them.

Proposition 2.2. *Let $a, b \in F_E^\Gamma - \{0\}$ and $a = \sum_{i=1}^k x^{m_i} g_i$ and $b = \sum_{j=1}^l x^{t_j} h_j$ be normal representations of a and b respectively. The following conditions are equivalent.*

(1) *The relation $a \to b$ holds.*
(2) *There is a partition $\{I_1, \ldots, I_k\}$ of $\{1, \ldots, l\}$ and finitely many paths $p_{ij}, j \in I_i, i = 1, \ldots, k$, such that $g_i \rightsquigarrow^{p_{ij}} h_j$, $|p_{ij}| = t_j - m_i$ for all $j \in I_i, i = 1, \ldots, k$, and*
$$b = \sum_{j=1}^l x^{t_j} h_j = \sum_{i=1}^k \sum_{j \in I_i} x^{m_i + |p_{ij}|} h_j.$$
If p is a prefix of p_{ij} and $v = \mathbf{r}(p)$, let
$$P_p = \{e \in \mathbf{s}^{-1}(v) \mid e \text{ is on } p_{ij'} \text{ for some } j' \in I_i\}.$$
Then the following hold.
 (i) *If v is regular and P_p nonempty, then $P_p = \mathbf{s}^{-1}(v)$.*

(ii) *If v is an infinite emitter and P_p nonempty, then there is $j' \in I_i$ such that $h_{j'} = q_Z^v$ for some Z such that $P_p \subseteq Z$.*
(iii) *The relation $t_j = |p| + m_i$ holds if and only if $p = p_{ij}$ and $h_j = q_Z^v$ for some Z implies $P_p \subseteq Z$.*

Before presenting the proof, let us motivate it by some examples.

Example 2.3. (1) In the graph below, $u \to xv$ and $w \to xv$ so $u + w \to xv + xv$. For this last relation, $k = 2, l = 2$ and one can take $I_1 = \{1\}, I_2 = \{2\}$, $p_{11} = e$, and $p_{22} = f$ so condition (2) holds.

$$\bullet^u \xrightarrow{e} \bullet^v \xleftarrow{f} \bullet^w$$

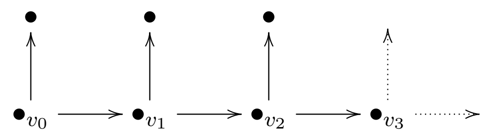
By condition (2) also, $u \to x^2v + a$ fails for any a since there is no path of length 2 from u to v.

(2) In the graph below, $v \to xu + xw$. For this relation, $k = 1, l = 2$ and one can take $I_1 = \{1, 2\}$, $p_{11} = e$, and $p_{12} = f$ so condition (2) holds.

$$\bullet^u \xleftarrow{e} \bullet^v \xrightarrow{f} \bullet^w$$

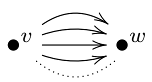
Although v connect to w by a path of length one, $v \to \alpha w$ fails for any $\alpha \in \mathbb{Z}^+[\Gamma]$ since the path from v to w has a bifurcation towards u so u must appear in any "result" obtained following a path from v to w by condition (2)(i).

(3) The relation $v_0 \to a$ fails for any a with $\operatorname{supp}(a)$ consisting of sinks only in the graph below.



Indeed, all paths from v_0 to finitely many sinks have a bifurcation on a path which does not end in any sink. Hence, if $v_0 \to a$ then a necessarily has v_i in its support for some $i \geq 0$.

(4) If $v \to^n a$ for $n > 0$ in the graph below,



condition (2) implies the existence of an improper vertex in $\operatorname{supp}(a)$. Hence, $v \to \alpha w$ fails for any $\alpha \in \mathbb{Z}^+[\Gamma]$.

Proof. Let us show direction $\Rightarrow$ by induction on n for $a \to^n b$. If $n = 0$, then $a = b$ so $k = l$ and one can permute the monomials in the normal representation of b if necessary to get that $t_i = m_i$ for all $i = 1, \ldots, k$. In this case, one can take $I_i = \{i\}$ and p_{ii} to be the trivial path which connects g_i to g_i for all $i = 1, \ldots, k$. In this case any prefix p of p_{ij} is trivial and relation $t_i = |p_{ij}| + m_i = |p| + m_i$ holds. Since $P_p = \emptyset$, conditions (i) to (iii) hold.

Considering the case $n = 1$ shortens the arguments in the inductive step. If $n = 1$, reorder the terms of the normal representation of a if necessary to assume that b is obtained by applying an axiom to $x^{m_k} g_k$ and let $x^{m_k}\mathbf{r}(g_k)$ denote the result of this application. Thus, g_k is not a sink. Reorder the terms of the normal representation of b to have that $b = \sum_{i=1}^{k-1} x^{m_i} g_i + x^{m_k}\mathbf{r}(g_k)$ and let $x^{m_k}\mathbf{r}(g_k) = \sum_{j \in J} x^{t_j} h_j$ for some finite subset J of $\{1, \ldots, l\}$. Let $I_i = \{i\}$ for $i = 1, \ldots, k-1$ and p_{ii} be the trivial path which connects g_i and g_i if $k > 1$. Let $I_k = J$ and p_{kj} be the path (of length zero or one) which connects g_k and h_j. Since g_k is not a sink, there are just three possible cases, listed below, for g_k.

1. g_k is a regular vertex v. In this case, $|J| = |\mathbf{s}^{-1}(v)|$ and we can label the elements of $\mathbf{s}^{-1}(v)$ such that $h_j = \mathbf{r}(e_j)$ for $j \in J$. Then $x^{t_j} h_j = x^{m_k+1}\mathbf{r}(e_j)$ and so $t_j = m_k + 1$. Let $p_{kj} = e_j$. If p is a prefix of e_j, then either $p = v$ in which case $t_j > |p| + m_i$ and $P_p = \mathbf{s}^{-1}(v)$, or $p = e_j$ in which case $t_j = |p| + m_i$. In this case, if $\mathbf{r}(e_j)$ is regular and $\mathbf{r}(e_j) \neq v$, then $P_p = \emptyset$ and if $\mathbf{r}(e_j) = v$, then $P_p = \mathbf{s}^{-1}(v)$.

2. g_k is an infinite emitter v. In this case, $\mathbf{r}(g_k) = q_Z + \sum_{e\in Z} x\mathbf{r}(e)$ for some Z and $|J| = |Z| + 1$. We can label the elements of Z such that $h_j = \mathbf{r}(e_j)$ for $j \in J - j_0$ and $h_{j_0} = q_Z$. Thus, $t_{j_0} = m_k$ and $t_j = m_k + 1$ for $j \in J - \{j_0\}$. Let $p_{kj_0} = v$ and $p_{kj} = e_j$ so that $|p_{kj}| = t_j - m_k$ for all $j \in J$. If p is a prefix of $p_{kj_0} = v$, then $p = v$, $t_j = |p| + m_i$, $h_{j_0} = q_Z^v$ and $P_p = Z$. If p is a prefix of e_j, then either $p = v$ or $p = e_j$. In the first case, $t_j > |p| + m_i$, and condition (ii) holds with $j = j_0$. In the second case, $t_j = |p| + m_i$, if $\mathbf{r}(e_j)$ is regular, then $\mathbf{r}(e_j) \neq v$ and so $P_p = \emptyset$. For $j \neq j_0$, h_j is a proper vertex and $h_{j_0} = q_Z$ with $P_p = Z$. Thus, condition (iii) holds.
3. g_k is an improper vertex q_Z^v. In this case, $\mathbf{r}(g_k) = q_W + \sum_{e\in W-Z} x\mathbf{r}(e)$ for some $W \supsetneq Z$ and $|J| = |W - Z| + 1$. We can label the elements of $W - Z$ such that $h_j = \mathbf{r}(e_j)$ for $j \in J - j_0$ and $h_{j_0} = q_W$. Thus, $t_{j_0} = m_k$ and $t_j = m_k + 1$ for $j \in J - \{j_0\}$. Let $p_{kj_0} = v$ and $p_{kj} = e_j$ so that $|p_{kj}| = t_j - m_k$ for all $j \in J$. If p is a prefix of $p_{kj_0} = v$, then $p = v$, $t_j = |p| + m_i$, $h_{j_0} = q_W$, and $P_p = W - Z \subseteq W$. If p is a prefix of e_j, then either $p = v$ or $p = e_j$. In the first case, $t_j > |p| + m_i$, and condition (ii) holds with $j = j_0$. In the second case, $t_j = |p| + m_i$, if $\mathbf{r}(e_j)$ is regular, then $\mathbf{r}(e_j) \neq v$ and so $P_p = \emptyset$. For $j \neq j_0$, h_j is a proper vertex and $h_{j_0} = q_W$ with $P_p = W - Z \subseteq W$. Thus, condition (iii) holds.

By construction,

$$b = \sum_{i=1}^{k-1} x^{m_i} g_i + \sum_{j\in J} x^{t_j} h_j = \sum_{i=1}^{k-1} x^{m_i + |p_{ii}|} g_i + \sum_{j\in I_k} x^{m_k + |p_{kj}|} h_j = \sum_{i=1}^{k} \sum_{j\in I_i} x^{m_i + |p_{ij}|} h_j.$$

Assuming the induction hypothesis, let us consider a sequence $a_0 = a \to_1 a_1 \to_1 \ldots \to_1 a_n = b$. Let $a_{n-1} = \sum_{j'=1}^{l'} x^{t'_{j'}} h'_{j'}$. By induction hypothesis, there is a partition $\{I'_1, \ldots, I'_k\}$ of $\{1, \ldots, l'\}$ and finitely many paths $p_{ij'}, j' \in I'_i, i = 1, \ldots, k$, such that $g_i \rightsquigarrow^{p_{ij'}} h'_{j'}$, $|p_{ij'}| = t'_{j'} - m_i$, and the required conditions hold for any prefix of $p_{ij'}$ for all $j' \in I'_i$ and $i = 1, \ldots, k$. The element b is obtained from a_{n-1} by application of one of the axioms to exactly one monomial $x^{t'_{j'}} h'_{j'}$. Reordering the terms of a_{n-1} if necessary, we can assume that it is the last one $x^{t'_{l'}} h'_{l'}$. Reorder the terms of b if necessary to have that $b = \sum_{j'=1}^{l'} x^{t'_{j'}} h'_{j'} + x^{t'_{l'}} \mathbf{r}(h'_{l'})$ and let $x^{t'_{l'}} \mathbf{r}(h'_{l'}) = \sum_{j\in J} x^{t_j} h_j$ for some finite subset J of $\{1, \ldots, l\}$. By construction, we have that $l = l' + |J|$ and that l' is in I'_{i_0} for exactly one i_0. So we let

$$I_i = I'_i, \text{ if } i \neq i_0, \text{ and } I_{i_0} = J.$$

If $i \neq i_0$, for each $j \in I_i$, $x^{t_j} h_j = x^{t'_{j'}} h'_{j'}$ for exactly one $j' \in I'_i$. So, for such j and j', we let $p_{ij} = p_{ij'}$ so that $|p_{ij}| = |p_{ij'}| = t'_{j'} - m_i = t_j - m_i$.

For i_0, we let $p_{i_0 j}$ be the concatenation of $p_{i_0 l'}$ and the path $p_{l'j}$ constructed as in the case $n = 1$ for $h'_{l'}$ and h_j for $j \in J = I_{i_0}$. Since $g_{i_0} \rightsquigarrow^{p_{i_0 l'}} h'_{l'}$ and $h'_{l'} \rightsquigarrow^{p_{l'j}} h_j$ for all $j \in J = I_{i_0}$, we have that $g_{i_0} \rightsquigarrow^{p_{i_0 j}} h_j$ for all $j \in I_{i_0}$. We have that $|p_{i_0 l'}| = t'_{l'} - m_{i_0}$ and $|p_{l'j}| = t_j - t'_{l'}$ and so

$$|p_{i_0 j}| = |p_{i_0 l'}| + |p_{l'j}| = t'_{l'} - m_{i_0} + t_j - t'_{l'} = t_j - m_{i_0} \quad \text{and}$$

$$b = \sum_{j=1}^{l} x^{t_j} h_j = \sum_{j'=1}^{l'} x^{t'_{j'}} h'_{j'} + \sum_{j\in J} x^{t_j} h_j = \sum_{i=1, i\neq i_0}^{k} \sum_{j\in I_i} x^{m_i + |p_{ij}|} h_j + \sum_{j\in I_{i_0}} x^{m_{i_0} + |p_{i_0 j}|} g_{i_0} = \sum_{i=1}^{k} \sum_{j\in I_i} x^{m_i + |p_{ij}|} h_j.$$

If p is a prefix of $p_{i_0 j}$, then it is either a prefix of $p_{i_0 l'}$ or $p = p_{i_0 l'} q$ for some prefix q of $p_{l'j}$ and one of the following three cases holds: first, p is a proper prefix of $p_{i_0 l'}$, second, q is a proper prefix of $p_{l'j}$ or, third, $q = p_{l'j}$ thus $p = p_{i_0 j}$. In the first case, $t'_{l'} > |p| + m_{i_0}$ and so $t_j \geq t'_{l'} > |p| + m_{i_0}$. In the second case, $t_j > |q| + t'_{l'} = |q| + |p_{i_0 l'}| + m_{i_0} = |p| + m_{i_0}$. In the last case, $t_j = |p| + m_{i_0}$ and if $h_j = q_Z^v$ for some Z then $P_p \subseteq Z$ since this condition holds for $a_{n-1} \to_1 b$ by the first induction step. In all three cases, if $\mathbf{r}(p)$ is regular and $P_p \neq \emptyset$, we can use induction hypothesis to conclude that $P_p = \mathbf{s}^{-1}(\mathbf{r}(p))$ and, if $\mathbf{r}(p)$ is an infinite emitter v and $P_p \neq \emptyset$, we can use induction hypothesis to conclude that there is j' such that $h_{j'} = q_Z^v$ for some Z such that $P_p \subseteq Z$. Thus, in any case, conditions (i) to (iii) hold.

Let us use induction on k to show direction $\Leftarrow$. If $k=1$ and $a=x^m g$, let $p_j, j=1,\ldots,l$ denote the paths which exist by condition (2). We show the claim using induction on $n=\sum_{j=1}^l |p_j|$. If this length is zero, then we claim that $b=a$. Indeed, since $|p_j|=0$, the relation $g \rightsquigarrow^{p_j} h_j$ implies that either $g=h_j$, or $g=v$ for some infinite emitter v and $h_j=q^v_Z$, or that $g_i=q^v_W$ and $h_j=q^v_Z$ for some v and $W\subsetneq Z$. However, in the second and third case we would have that $P_v\subseteq Z$ by condition (2) so there would have to be some paths $p_{j'}$ of length at least one which cannot happen since $n=0$. Hence, $a=b$ and, thus, $a\to b$.

Assuming the induction hypothesis, let $n=\sum_{j=1}^l |p_j|>0$. Since $n>0$, $a\neq b$ and there is $j=1,\ldots,l$ such that $|p_j|>0$. If $p_j=e_0p$ for an edge e_0 and a path p, let $v=\mathbf{s}(e_0)$. Since $e_0\in P_v$, $P_v\neq\emptyset$. We have exactly three possibilities for g, listed below.

1. $g=v$ is regular. Since P_v is nonempty, $P_v=\mathbf{s}^{-1}(v)$ by (i). Let $a_1=\mathbf{r}(v)=\sum_{e\in P_v} x^{m+1}\mathbf{r}(e)$. Note that $a\to_1 a_1$ by (A1). We claim that condition (2) holds for a_1 and b.

   For $e\in P_v=\mathbf{s}^{-1}(v)$, there is some $j=1,\ldots,l$ such that $p_j=eq_j$ for some path q_j and so the set
$$I_e=\{j\in\{1,\ldots,l\}\mid e \text{ is the first edge of } p_j\}$$
   is nonempty. Since the first edge of p_j is in $P_v=\mathbf{s}^{-1}(v)$ for any j, we have that $\bigcup_{e\in P_v} I_e=\{1,\ldots,l\}$. If $j\in I_e\cap I_{e'}$, then $e=e'$ since the first edge of a path is unique. As $t_j=|p_j|+m$, we have that $t_j=|q_j|+1+m$ and
$$b=\sum_{j=1}^l x^{t_j}h_j=\sum_{j=1}^l x^{|p_j|+m}h_j=\sum_{j=1}^l x^{|q_j|+1+m}h_j.$$
   If q is a prefix of q_j, then eq is a prefix of p_j and conditions (i) to (iii) hold for q because they hold for eq. Thus, we have that $a_1\to b$ by induction hypothesis. Since $a\to_1 a_1$, we have that $a\to b$.
2. $g=v$ is an infinite emitter. In this case, let $a_1=x^m q_{P_v}+\sum_{e\in P_v} x^{m+1}\mathbf{r}(e)$. So that $a\to_1 a_1$ by (A2). Since $P_v\neq\emptyset$, there is j such that $h_j=q^v_Z$ for some Z with $P_v\subseteq Z$ by (ii). By (iii), such j can be found so that $t_j=|p_j|+m$. Reorder the terms of b if necessary so that we can assume that $j=1$. We check that condition (2) holds for a_1 and b.

   For $e\in P_v$, there is $j=2,\ldots,l$ such that $p_j=eq_j$ for some path q_j and so the sets $I_e, e\in P_v$, defined as in the previous case, are nonempty and mutually disjoint. Let $I_1=\{1\}$ and $q_1=p_1$. Since the first edge of p_j is in P_v for every $j=2,\ldots,l$, $\bigcup_{e\in P_v} I_e=\{2,\ldots,l\}$, so $I_1\cup\bigcup_{e\in P_v} I_e=\{1,2,\ldots,l\}$. Hence, $\{I_1\}\cup\{I_e|e\in P_v\}$ is a partition of $\{1,\ldots,l\}$. For $j=2,\ldots,l$, $t_j=|p_j|+m=|q_j|+1+m$, $t_1=|p_1|+m=|q_1|+m$, and
$$b=\sum_{j=1}^l x^{|p_j|+m}h_j=x^{|q_1|+m}h_1+\sum_{j=2}^l x^{|q_j|+1+m}h_j.$$
   If q is a prefix of q_j for $j>1$, then e_iq is a prefix of p_j and conditions (i) to (iii) hold for q because they hold for e_iq. If q is a prefix of $q_1=p_1$, then the requirements also hold. Thus, $a_1\to b$ by induction hypothesis. Since $a\to_1 a_1$, we have that $a\to b$.
3. $g=q^v_Z$ for some Z. In this case, P_v is a proper superset of Z. Let $a_1=x^m q_{P_v}+\sum_{e\in P_v-Z} x^{m+1}\mathbf{r}(e)$ so that $a\to_1 a_1$ by (A3). By (ii), there is j such that $h_j=q^v_W$ for some W such that $P_v\subseteq W$ and, by (iii), such j can be found so that $t_j=|p_j|+m$. Reorder the terms of b if necessary so that we can assume that $j=1$. We check that condition (2) holds for a_1 and b.

   For $e\in P_v-Z$, there is some $j=2,\ldots,l$ such that $p_j=eq_j$ for some path q_j and so the sets $I_e, e\in P_v-Z$, defined as in the previous cases, are nonempty and mutually disjoint. If $I_1=\{1\}$ and $q_1=p_1$, one shows that $\{I_1\}\cup\{I_e\mid e\in P_v-Z\}$ is a partition of $\{1,\ldots,l\}$ as in the previous case. Since $t_j=|p_j|+m=|q_j|+1+m$ for $j=2,\ldots,l$ and $t_1=|p_1|+m=|q_1|+m$, $b=\sum_{j=1}^l x^{|p_j|+m}h_j=x^{|q_1|+m}h_1+\sum_{j=2}^l x^{|q_j|+1+m}h_j$. The requirements on prefixes of q_j can be checked just as in the previous case. Thus, we have that $a_1\to b$ by induction hypothesis. Since $a\to_1 a_1$, we have that $a\to b$.

This concludes the proof of the case $k=1$. Assuming the induction hypothesis, let us show the claim for a with k terms in its normal decomposition. Note that if condition (2) holds, then it holds for

$a' = \sum_{i=1}^{k-1} x^{m_i} g_i$ and $b' = \sum_{i=1}^{k-1}\sum_{j\in I_i} x^{m_i+|p_{ij}|}h_j$ and for $x^{m_k}g_k$ and $\sum_{j\in I_k} x^{m_k+|p_{kj}|}h_j$. By the induction hypothesis, we have that $a' \to b'$ and that $x^{m_k}g_k \to \sum_{j\in I_k} x^{m_k+|p_{kj}|}h_h$. Hence, $a = a' + x^{m_k}g_k \to b = b' + \sum_{j\in I_k} x^{m_k+|p_{kj}|}h_j$. □

We show two corollaries of Proposition 2.2 which we use in Section 3.4. Recall that Definition 2.1 implies that $g \rightsquigarrow^p h$ implies $g \to x^{|p|}h + a$ for some a. By the first corollary, the converse also holds.

Corollary 2.4. *Let g, h be generators of F_E^Γ, $a \in F_E^\Gamma$, and m a nonnegative integer. Then $g \to x^m h + a$ holds if and only if there is a path p of length m such that $g \rightsquigarrow^p h$.*

Proof. If $g \to x^m h + a$, condition (2) of Proposition 2.2 holds by Proposition 2.2, so there is a path p of length m from g to h. The converse holds by Definition 2.1 (see the sentence following Definition 2.1). □

By the next corollary, if $a \to b$, then each monomial of b is obtained by a monomial of a. This complements the Confluence Lemma.

Corollary 2.5. *If g is a generator of F_E^Γ, $a, b \in F_E^\Gamma$, and m an integer, then $a \to x^m g + b$ implies that there is $h \in \operatorname{supp}(a)$ and $k \le m$ such that $x^k h$ is a monomial of a and that $x^k h \to x^m g + c$ for some $c \in F_E^\Gamma$.*

Proof. If $a \to x^m g + b$ holds, Proposition 2.2 guarantees the existence of a monomial $x^k h$ of a and a path p such that $k + |p| = m$ and such that $h \rightsquigarrow^p g$. Hence, $m - k = |p| \ge 0$ and $x^k h \to x^{k+|p|}g + c = x^m g + c$ for some $c \in F_E^\Gamma$ by Corollary 2.4. □

2.1. **Connectivity of the supports.** Next, we associate the relation $a \to b$ to the properties of the supports of a and b.

Definition 2.6. Let G and H be finite and nonempty sets of generators of F_E^Γ. We write $G \to H$ if there are $k \ge |G|$ and $l \ge |H|$ such that the elements of G and H can be indexed as $g_1, \ldots, g_k$ and $h_1, \ldots, h_l$ (with repetitions allowed) respectively and there is a partition $\{I_1, \ldots, I_k\}$ of $\{1, \ldots, l\}$ and finitely many paths $p_{ij}, j \in I_i, i = 1, \ldots, k$, such that $g_i \rightsquigarrow^{p_{ij}} h_j$ for all $j \in I_i, i = 1, \ldots, k$ and such that if p is a prefix of p_{ij} and P_p is as in condition (2) of Proposition 2.2 then (2)(i) and (2)(ii) of Proposition 2.2 and condition (iii) below hold.

(iii) If v is an infinite emitter and $h_j = q_Z^v$ for some Z, then $P_{p_{ij}} \subseteq Z$.

Corollary 2.7. (1) *If $a, b \in F_E^\Gamma - \{0\}$, then $a \to b$ implies $\operatorname{supp}(a) \to \operatorname{supp}(b)$.*
(2) *Let $a, b \in F_E^\Gamma - \{0\}$ be such that $\operatorname{supp}(a) \to \operatorname{supp}(b)$. Then, there is $c \in F_E^\Gamma - \{0\}$ such that $\operatorname{supp}(c) \subseteq \operatorname{supp}(b)$ and that $a \to c$.*

Proof. Definition 2.6 is really condition (2) of Proposition 2.2 stripped down from any mention of degrees. Thus, part (1) directly follows from Proposition 2.2.

To show part (2), assume that $a = \sum_{i=1}^k x^{m_i} g_i$ and $b = \sum_{j=1}^l x^{t_j} h_j$ be such that $\operatorname{supp}(a) \to \operatorname{supp}(b)$. Let m, n be the cardinalities of $\operatorname{supp}(a)$ and $\operatorname{supp}(b)$ respectively and $k' \ge m, l' \ge n$, $\{I_1, \ldots, I_{k'}\}$ and $p_{i'j'}, j' \in I_{i'}, i' = 1, \ldots, k'$ be as in Definition 2.6 for $\operatorname{supp}(a)$ and $\operatorname{supp}(b)$. Then, we let

$$a' = \sum_{i'=1}^{k'} g_{i'}, \quad b'_{i'} = \sum_{j'\in I_{i'}} x^{|p_{i'j'}|}h_{j'} \text{ for } i' = 1, \ldots, k', \quad \text{and} \quad b' = \sum_{i'=1}^{k'} b'_{i'}.$$

By construction, $\operatorname{supp}(a') = \operatorname{supp}(a)$, $\operatorname{supp}(b') = \operatorname{supp}(b)$ and $g_{i'} \to b_{i'}$ so that $a' \to b'$ holds by Proposition 2.2. For any $i = 1, \ldots, k$, there is $i' = 1, \ldots, k'$ such that $g_i = g_{i'}$. For such i, let

$$c_i = \sum_{j'\in I_{i'}} x^{m_i+|p_{i'j'}|}h_{j'} \quad \text{and let} \quad c = \sum_{i=1}^k c_i.$$

We have that $\operatorname{supp}(c) \subseteq \operatorname{supp}(b') = \operatorname{supp}(b)$ and $x^{m_i}g_i \to c_i$ so that $a = \sum_{i=1}^k x^{m_i}g_i \to c = \sum_{i=1}^k c_i$. The relation $x^{m_i}g_i \to c_i$ also implies that $c_i \neq 0$ so $c \neq 0$. □

We note that the converse of part (1) of Corollary 2.7 does not have to hold. Also, for an element c as in part (2) of Corollary 2.7, the relation $b \sim c$ does not have to hold even if $\operatorname{supp}(c) = \operatorname{supp}(b)$. Indeed, in the graph below, $v \to xw$ so $\{v\} \to \{w\}$.

$$\bullet^v \longrightarrow \bullet^w$$

However, for $a = v$ and $b = w$, we have that $\operatorname{supp}(a) \to \operatorname{supp}(b)$ but $a \to b$ fails since there are no paths of length zero from v to w. If $c = xw$, then $\operatorname{supp}(c) = \operatorname{supp}(b)$, but we do not have that $b = w \sim c = xw$ since w is a sink and the relation $w \sim d$ for some d implies $d = w$ or $d = x^{-1}v$ by the Confluence Lemma. Also, using Theorem 4.1, it is direct that $w \sim xw$ cannot hold since w is not a periodic element.

2.2. **Connecting using (A1) only.** To emphasize that $a \to b$ is such that only (A1) is used, we write $a \xrightarrow{A1} b$. If E is a row-finite graph, then $\xrightarrow{A1}$ is just the relation $\to$. If V is a finite and nonempty set of regular vertices and W a finite and nonempty set of proper vertices such that $V \to W$, we write $V \xrightarrow{A1} W$. Corollary 2.7 implies the corollary below.

Corollary 2.8. (1) *Let $a, b \in F_E^\Gamma - \{0\}$ such that $a \xrightarrow{A1} b$ and that $\operatorname{supp}(a)$ consists of regular vertices. Then $\operatorname{supp}(a) \xrightarrow{A1} \operatorname{supp}(b)$.*

(2) *Let $a, b \in F_E^\Gamma - \{0\}$ be such that $\operatorname{supp}(a) \xrightarrow{A1} \operatorname{supp}(b)$. Then, there is $c \in F_E^\Gamma - \{0\}$ such that $\operatorname{supp}(c) \subseteq \operatorname{supp}(b)$ and that $a \xrightarrow{A1} c$.*

Proof. To show (1), assume that $a \xrightarrow{A1} b$ and that $\operatorname{supp}(a)$ consists of regular vertices. By Corollary 2.7, $\operatorname{supp}(a) \to \operatorname{supp}(b)$. Since only (A1) is used in $a \xrightarrow{A1} b$, $\operatorname{supp}(b)$ does not contain any improper vertices, so $\operatorname{supp}(a) \xrightarrow{A1} \operatorname{supp}(b)$ by definition of $\xrightarrow{A1}$ for sets of vertices.

To show (2), let $\operatorname{supp}(a) \xrightarrow{A1} \operatorname{supp}(b)$. By Corollary 2.7, there is $c \in F_E^\Gamma - \{0\}$ such that $\operatorname{supp}(c) \subseteq \operatorname{supp}(b)$ and $a \to c$. Since the support of a consists of regular vertices and the support of b, thus of c as well, of proper vertices, only (A1) can be applied in a sequence for $a \to c$. Hence, $a \xrightarrow{A1} c$. □

3. Characterization of comparable elements

3.1. **Cancellative property.** First, we show that the monoid M_E^Γ is cancellative by a direct proof. This was shown in [4, Corollary 5.8] using the graph covering. Note that M_E^Δ may not be cancellative for a group $\Delta \neq \Gamma$. In particular, if E is a graph with a cycle with an exit and Δ is trivial, then M_E^Δ is not cancellative by [4, Lemma 5.5].

Proposition 3.1. *The Γ-monoid M_E^Γ is cancellative.*

Proof. Assume that $a + c \sim b + d$ holds in F_E^Γ for some $d \in F_E^\Gamma$ such that $c \sim d$. So, we have that $a + c \sim b + d \sim b + c$. We show that $a \sim b$ using induction on n for $a + c \sim^n b + c$. If $n = 1$, then either $a + c \to_1 b + c$ or $b + c \to_1 a + c$. In the first case, there is a generator g in the support of a or c such that $b + c$ is obtained by replacing a summand $x^m g$ of $a + c$ by $x^m \mathbf{r}(g)$ and keeping the rest of the monomials intact. By the nature of the three axioms, the number of monomials of the form $x^m g$ in $a + c$ is larger than in $b + c$ and each of the monomials in $x^m \mathbf{r}(g)$ appears one time less in $a + c$ than in $b + c$. Since these terms appear equal number of times in c, this means that a contains a monomial $x^m g$ and that $x^m \mathbf{r}(g)$ is a summand of b. Hence, $a = a' + x^m g$ and $b = a' + x^m \mathbf{r}(g)$ for some $a' \in F_E^\Gamma$ so that $a \to_1 b$. The case $b + c \to_1 a + c$ is similar and the induction step is analogous. □

Proposition 3.1 highlights an important difference between M_E and M_E^Γ: while M_E can be much larger than the positive cone of G_E, the monoid M_E^Γ is *equal to* the positive cone of G_E^Γ. Thus, the monoid M_E can carry some information which is lost under formation of its Grothendieck group but M_E^Γ carries *no additional information* than G_E^Γ. In other words, using the language of [9], the group G_E^Γ is equally “talented” as the monoid M_E^Γ.

3.2. **The order.** The relation $\sim$ on the monoid F_E^Γ enables one to define a relation $\precsim$ as follows.

$$a \precsim b \text{ if there is } c \in F_E^\Gamma \text{ such that } a + c \sim b$$

for all $a, b \in F_E^\Gamma$. If $a \precsim b$ and $a \nsim b$, we write $a \prec b$. Using Proposition 3.1, it is direct to show that $a \prec b$ is equivalent with $a + c \sim b$ for some nonzero c in F_E^Γ.

The relation $\precsim$ defines an order on M_E^Γ given by

$$[a] \leq [b] \quad \text{if and only if} \quad a \precsim b.$$

It is direct to show that $\leq$ is reflexive and transitive. The antisymmetry holds by Proposition 3.1. The relation $\leq$ induces an order on the Grothendieck group G_E^Γ.

In [9, Lemma 4.1], it is shown that $a \prec x^n a$ is not possible for any a and any positive n if E is row-finite. After the lemma below, we show that this statement holds for an arbitrary graph in Proposition 3.3.

Lemma 3.2. *If $a \in F_E^\Gamma - \{0\}$ is such that $a \precsim x^n a$ for some positive integer n, then the following hold.*

(1) *No vertex in the support of a is a sink.*
(2) *No vertex in the support of a is an improper vertex.*
(3) *All vertices in the support of a are regular (so a is regular).*

Proof. Since $a \precsim x^n a$, $a + b \sim x^n a$ for some $b \in F_E^\Gamma$. Then $a + b + x^n b \sim x^n a + x^n b \sim x^{2n} a$ so, by induction, $a + \sum_{i=0}^k x^k b \sim x^{(k+1)n} a$. Hence, we can find n large enough so that n is larger than the degrees of all monomials in a normal representation of a. Assume that n is such and that $a + b \sim x^n a$ for some $b \in F_E^\Gamma$. By the Confluence Lemma 1.2(2), there is $c \in F_E^\Gamma$ such that $a + b \to c$ and $x^n a \to c$.

(1) Assume that a sink v is in $\operatorname{supp}(a)$ and let $\sum_{i=1}^l x^{m_i} v$ be the sum of all monomials in a normal representation of a which contain v. By construction, $m_i < n$ for every $i = 1, \ldots, l$. Since the relation $\to_1$ cannot be applied to v, the relation $a + b \to c$ implies that $x^{m_i} v$ is a summand of c for every $i = 1, \ldots, l$. On the other hand, the relation $x^n a \to c$ implies that every monomial of c has degree larger than or equal to n so $x^{m_1} v$ cannot be a summand of c. This is a contradiction.

(2) Assume that an improper vertex q_Z^v is in $\operatorname{supp}(a)$ for some v and some Z. Let $\sum_{i=1}^l x^{m_i} q_{Z_i}$ be the sum of all monomials in a normal representation of a which contain $q_{Z_i}^v$ for some nonempty and finite $Z_i \supseteq Z$. Since an application of $\to_1$ does not change the power of a monomial with q_W^v for some $W \supseteq Z$, the relation $a + b \to c$ implies that c contains a summand of the form $\sum_{i=1}^l x^{m_i} q_{W_i}$ for some $W_i \supseteq Z_i, i = 1, \ldots, l$. On the other hand, the relation $x^n a \to c$ implies that every monomial of c has degree larger than or equal to n so $x^{m_1} q_{W_1}$ cannot be a summand of c. This is a contradiction.

(3) By part (1), to show that a vertex v in the support of a is regular, it is sufficient to show that v is not an infinite emitter. Assume that an infinite emitter v is in the support of a and let $\sum_{i=1}^l x^{m_i} v$ be the sum of all monomials in a normal representation of a which contain v. Since axioms (A1) and (A3) are not applicable to any monomials with v in them, the relation $a + b \to c$ implies that $\sum_{i=1}^l x^{m_i} g_i$, where each g_i is either v or q_Z^v for some Z, is a summand in a normal representation of c. On the other hand, the relation $x^n a \to c$ implies that every monomial of c has degree larger than or equal to n so $x^{m_1} g_1$ cannot be a summand of c which is a contradiction. $\square$

Proposition 3.3. *The relation $a \prec x^n a$ is not possible for any nonnegative n and any $a \in F_E^\Gamma$.*

Proof. Since $0 \prec 0$ is false, it is sufficient to consider $a \neq 0$. Also, since $a \prec a$ is false, it is sufficient to consider positive n. Assume that $a \prec x^n a$ for some positive n and some nonzero $a \in F_E^\Gamma$. By Lemma 3.2, all elements in the support of a are regular and proper vertices. Let m be the maximum of degrees of the monomials in a normal representation of a. If a monomial $x^l v$ in a normal representation of a is such that $l < m$, apply (A1) to $x^l v$ to replace this monomial by $\sum_{e \in \mathbf{s}^{-1}(v)} x^{l+1} \mathbf{r}(e)$. We obtain an element a_1 such that $a_1 \sim a$ so the relation $a_1 \prec x^n a_1$ also holds and, as a consequence, all vertices in the support of a_1 are regular also. Keep repeating this process until all monomials of some a_k have the same degree m so that we can write $a_k = x^m b$ where b is a sum of regular vertices. Since $x^m b \prec x^{n+m} b$ we have that $b \prec x^n b$ and so $b + c \sim x^n b$ for some nonzero $c \in F_E^\Gamma$. By the Confluence Lemma 1.2(2), there is d such that $b + c \to d$ and $x^n b \to d$. The relation $x^n b \to d$ implies that $x^{-n} d \prec d$ so $d \prec x^n d$ and all vertices in the support of d are regular by Lemma 3.2. Using the same argument as when obtaining $x^m b$ from a, we can show that there is an element f such that $d \to f$ and such that f is a sum of monomials of the same degree m'. Hence, $b + c \to d \to f$ and $x^n b \to d \to f$. Since $\to$ either increases the degree of a monomial or leaves it the same, the relation $x^n b \to f$ implies that $m' \geq n > 0$.

Let $h = x^{-n} f$ so that h is a sum of monomials of the same nonnegative degree $m' - n$ and that $b + c \to x^n h$ and $b \to h$. We use induction on the length of a sequence for $b \to h$ to show that $h + c \to x^n h$.

If $b = h$, the claim holds. Assume that it holds for length smaller than k and let $b = b_0 \to_1 b_1 \to_1 \ldots \to_1 b_k = h$. Since b is regular, $b \to_1 b_1$ is an application of (A1). Hence, $b = b' + v$ and $b_1 = b' + \sum_{e \in \mathbf{s}^{-1}(v)} x\mathbf{r}(e)$ for some regular vertex v. Since the degree of every monomial in $x^n h = f$ is strictly larger than zero, v has to be changed in the process of obtaining $x^n h$ from $b + c = b' + v + c$. Reorder the terms of the sequence for $b + c \to x^n h$ so that an application of (A1) to v is the first step. Hence,

$$b + c = b' + v + c \to b' + \sum_{e \in \mathbf{s}^{-1}(v)} x\mathbf{r}(e) + c = b_1 + c \to x^n h.$$

We can now apply the induction hypothesis to b_1 to obtain that $h + c \to x^n h$.

Lastly, we show that the relation $h + c \to x^n h$ leads to a contradiction. Indeed, since h is a sum of monomials of the same nonnegative degree and n is strictly larger than zero, we have that $h + c \neq x^n h$ so at least one of the three axioms is used. If normal representations of h and c have n_h and n_c monomials respectively, then the number of terms in the resulting $x^n h$ is larger than or equal to $n_h + n_c$. But since $x^n h$ has the same number of monomials as h, we necessarily have that $n_c = 0$ which implies that $c = 0$. This is a contradiction since c is chosen to be nonzero such that $b + c \sim x^n b$. □

3.3. **Comparable, periodic, aperiodic and incomparable elements.** Proposition 3.3 implies that there are just two possibilities for $a \in F_E^\Gamma$: either $a \gtrsim x^n a$ for some positive n or a and $x^n a$ are not comparable for any positive n. In the case when $a \gtrsim x^n a$ for some positive n we have that either $a \sim x^n a$ or $a \succ x^n a$. We introduce the following terminology.

Definition 3.4. Let $a \in F_E^\Gamma$.

(1) If $a \gtrsim x^n a$ for some positive integer n, the element a is *comparable*.
  (1i) If $a \sim x^n a$ for some positive integer n, the element a is *periodic*.
  (1ii) If $a \succ x^n a$ for some positive integer n, the element a is *aperiodic*.
(2) If a and $x^n a$ are not comparable for any positive integer n, the element a is *incomparable*.

For $[a] \in M_E^\Gamma$, we say that $[a]$ is comparable, periodic, aperiodic or incomparable if any b such that $a \sim b$ is such.

Note that 0 is periodic by this definition. An element of F_E^Γ clearly cannot be both comparable and incomparable. We also note that a comparable element of F_E^Γ cannot be both periodic and aperiodic. Indeed, if $x^m a \sim a \succ x^n a$ for some positive integers m and n, let n be the least positive integer such that $a \succ x^n a$. Since $x^m a \succ x^n a$ implies $x^{m-n} a \succ a$, $m - n$ is negative by Proposition 3.3 so $n > m$. On the other hand, the relation $x^m a \sim a \succ x^n a$ also implies that $a \sim x^{-m} a \succ x^{n-m} a$ so $n - m \geq n$ by the assumption that n is the smallest possible such that $a \succ x^n a$. The relation $n - m \geq n$ implies that $m \leq 0$ which is in contradiction with the assumption that m is positive.

3.4. **Stationary elements.** Next, we prove a series of claims which bring us to Theorem 3.17. Lemma 3.5 leads us to the notion of a stationary element introduced in Definition 3.6.

Lemma 3.5. *Let $a \in F_E^\Gamma - \{0\}$ be such that $a \sim x^n a + b$ for some positive integer n and some $b \in F_E^\Gamma$. There are $c \in F_E^\Gamma - \{0\}$ and $d \in F_E^\Gamma$ such that $c \to x^n c + d$, $a \to c$ and $b \to d$.*

Note that the assumption of the lemma is exactly that a is comparable, the case $b = 0$ corresponds exactly to the case that a is periodic, and the case $b \neq 0$ to the case that a is aperiodic.

Proof. Since $a \sim x^n a + b \sim x^{2n} a + x^n b + b \sim \ldots$, we can choose n as large as needed. Let us choose n larger than the degree of every monomial in a normal representation of a.

By the Confluence Lemma 1.2(2), $a \to f$ and $x^n a + b \to f$ and by the Refinement Lemma 1.2(1), $f = f_1 + f_2$ such that $x^n a \to f_1$ and $b \to f_2$. Let $c = x^{-n} f_1$ so that $a \to x^{-n} f_1 = c$ and that $a \to f = x^n c + f_2$.

We use induction on k for $a \to^k c$. If $k = 0$, then $a = c$. Let $d = f_2$ so that $b \to d$. Assuming the induction hypothesis, let us consider $a \to^k c$ with $a = a_0 \to_1 a_1 \to_1 \ldots \to_1 a_k = c$. Let $a = a' + x^m g$ for some generator g such that $a_1 = a' + x^m \mathbf{r}(g)$. Consider the following two cases for the relation $a \to x^n c + f_2$.

1. There is an application of the same axiom used for $a \to_1 a_1$ to $x^m g$ at some point such that $x^m g$ is not changed prior to this point. Changing the order of applications of axioms in the sequence for $a \to x^n c + f_2$, we can assume that this application of the axiom happened first. In this case $a \to a_1 \to x^n c + f_2$. Thus, we can apply the induction hypothesis to a_1 instead of a and obtain the relation $c \to x^n c + d$ for some d such that $f_2 \to d$. Hence, $b \to f_2 \to d$.
2. There is no application of the axiom used for $a \to_1 a_1$ to $x^m g$ at any point. Since n is larger than m, then $x^m g$ has to be a summand of f_2. Say $f_2 = d' + x^m g$. Then $a = a' + x^m g \to x^n c + d' + x^m g$. Replacing the terms $x^m g$ by $x^m \mathbf{r}(g)$ on both sides of the relation $\to$, we obtain that $a_1 = a' + x^m \mathbf{r}(g) \to x^n c + d' + x^m \mathbf{r}(g)$. Since we have $a_1 \to^{k-1} c$, we can apply the induction hypothesis to a_1 and obtain that $c \to x^n c + d$ for some d such that $d' + x^m \mathbf{r}(g) \to d$. Hence, $b \to f_2 = d' + x^m g \to d' + x^m \mathbf{r}(g) \to d$.

□

The properties of an element such as element c of Lemma 3.5 are significant in the characterization of a comparable element so we assign a name to such an element.

Definition 3.6. An element $a \in F_E^\Gamma - \{0\}$ is a *stationary* element if $a \to x^n a + b$ for some positive integer n and some $b \in F_E^\Gamma$.

The next lemma describes the support of a stationary element. Recall that a generator g is on a cycle if $g \rightsquigarrow^p g$ for some p with $|p| > 0$.

Lemma 3.7. *Let $a \in F_E^\Gamma$ be stationary such that $a \to x^n a + b$ for some positive integer n and some $b \in F_E^\Gamma$.*

(1) *For any positive integer k,*
$$a \to x^{kn} a + \sum_{i=0}^{k-1} x^{in} b.$$
(2) *The support of a contains an element which is on a cycle.*
(3) *Each element of the support of a which is not on a cycle is on a path exiting a cycle which contains another element of* $\operatorname{supp}(a)$. [2]
(4) *Each element of the support of a is either on a cycle or on a path exiting a cycle which contains another element of* $\operatorname{supp}(a)$.

Proof. To show (1), note that if $a \to x^n a + b$, then
$$a \to x^n a + b \to x^{2n} a + x^n b + b \to x^{3n} a + x^{2n} b + x^n b + b \to \ldots \to x^{kn} a + \sum_{i=0}^{k-1} x^{in} b.$$

To show (2), we use part (1) to choose n larger than $k - m$ for any degrees k and m of any monomials in a normal representation of a. Let l be the number of monomials in a normal representation of a.

[2] This condition can be described also in terms of the *tree* $T(g) = \{h \mid g \rightsquigarrow h\}$ of a generator g as follows: $\operatorname{supp}(a) \subseteq \bigcup \{T(g) \mid g \in \operatorname{supp}(a) \text{ and } g \text{ on a cycle}\}$.

If all generators in $\operatorname{supp}(a)$ are on cycles, there is nothing to prove. So, suppose that there is $g_1 \in \operatorname{supp}(a)$ such that $x^{m_1}g_1$ is a monomial of a and that g_1 is not on a cycle. Let $a = a_1 + x^{m_1}g_1$. By the Refinement Lemma 1.2(1), there are c_{11}, c_{12} such that

$$a_1 + x^{m_1}g_1 \to x^n a_1 + x^{n+m_1}g_1 + b = c_{11} + c_{12}, \quad a_1 \to c_{11} \text{ and } x^{m_1}g_1 \to c_{12}.$$

The monomial $x^{n+m_1}g_1$ is a summand of either c_{11} or c_{12}. In the second case, $x^{m_1}g_1 \to x^{n+m_1}g_1 + c$ for some c and Corollary 2.4 implies that there is a path of length $n > 0$ from g_1 to g_1 which means that g_1 is on a cycle. This is a contradiction with the choice of g_1. Hence, $x^{n+m_1}g_1$ is a summand of c_{11}. This implies that $c_{11} \neq 0$ and so $a_1 \neq 0$ also which means that $l > 1$ and a_1 has $l-1$ terms.

By Corollary 2.5, there is a monomial $x^{m_2}g_2$ of a_1 such that $a_1 = a_2 + x^{m_2}g_2$ (so a_2 has $l-2 \geq 0$ terms) and that $x^{m_2}g_2 \to x^{n+m_1}g_1 + c$ for some c. The choice of n guarantees that $n + m_1 - m_2 > 0$ so that there is a path of positive length from g_2 to g_1 by Corollary 2.4. If g_2 is on a cycle, we are done. If not, consider whether the term $x^{n+m_2}g_2$ is a summand of c_{11} or c_{12}. If it is a summand of c_{12}, then $x^{m_1}g_1 \to x^{n+m_2}g_2 + d$ for some d and so there is a path of positive length from g_1 to g_2. As there is a path of positive length from g_2 to g_1, g_1 is on a cycle. Since this is not the case, $x^{n+m_2}g_2$ is a summand of c_{11}.

Apply the Refinement Lemma 1.2(1) again to decompose c_{11} as $c_{21} + c_{22}$ such that $a_2 \to c_{21}$ and $x^{m_2}g_2 \to c_{22}$. Since g_2 is not on a cycle, $x^{n+m_2}g_2$ is a summand of c_{21} which implies that $c_{21} \neq 0$ and so $a_2 \neq 0$ which means that $l - 2 > 0$. By Corollary 2.5, there is a summand $x^{m_3}g_3$ of a_2 such that $a_2 = a_3 + x^{m_3}g_3$ (so that a_3 has $l - 3 \geq 0$ terms) and that $x^{m_3}g_3 \to x^{n+m_2}g_2 + d$ for some d. The choice of n guarantees that $n + m_2 - m_3$ is positive so we can conclude that there is a path of positive length from g_3 to g_2 by Corollary 2.4.

If g_3 is on a cycle, we are done. If not, the term $x^{n+m_3}g_3$ must be a summand of c_{21} as otherwise g_3 is on a cycle which is not the case. So, $x^{n+m_3}g_3$ is a summand of c_{21}, $a_3 + x^{m_3}g_3 \to c_{21}$, and we can continue the decomposition process $c_{21} = c_{31} + c_{32}$ as in the previous step.

Since l is finite, this process eventually stops. If it stops at the k-th step, g_k is on a cycle and (2) holds.

Note that the proof of part (2) implies that if g_1 is not on a cycle, then g_1 is on a path leaving a cycle which contains g_k. This is because the proof shows that there is a path from g_{i+1} to g_i for all $i = 1, \ldots, k-1$. Hence, this automatically shows part (3). Part (4) is a direct corollary of part (3). $\square$

The last part of Lemma 3.7 describes the support of a stationary element. The properties of such set are relevant and we introduce some terminology for it. First, we say that a finite and nonempty set of generators of F_E^Γ is *stationary* if every $g \in V$ is either on a cycle or on a path exiting a cycle which contains some generator $h \in V$. By Lemma 3.7, the support of every stationary element is a stationary set.

For a stationary set V, let V_c denote the set of those $g \in V$ which are on cycles (thus $V_c \neq \emptyset$). We say that V_c is the *core* of V and that $g \in V_c$ is a *core generator*. We say that the cycles which contain core generators are the *core cycles* of V. Let V_e denote $V - V_c$ (so V_e is possibly empty). We call this set the *exit set* of V and we say that $g \in V_e$ is an *exit generator*.

For a core generator $g \in V_c$, let n_g be the minimum of the set of lengths of cycles on which g is. Let n be the least common multiple of n_g for $g \in V_c$. We show that n has a special significance for a stationary set V which consists of core generators only so we call it *the core period* of such V.

If a is stationary, let $a = a_c + a_e$ such that the support of a_c is $\operatorname{supp}(a)_c$ and the support of a_e is $\operatorname{supp}(a)_e$ (thus $a_c \neq 0$ and a_e is possibly zero). We call a_c and a_e *the core part* and the *exit part* of a respectively.

If $a \in F_E^\Gamma$ is such that each $g \in \operatorname{supp}(a)$ is on a cycle, then $\operatorname{supp}(a)$ is a stationary set by definition and $a = a_c$. In the next lemma, we show that such element a is necessarily stationary.

Lemma 3.8. *Let $a \in F_E^\Gamma - \{0\}$ be such that each $g \in \operatorname{supp}(a)$ is on a cycle, and let n be the core period of $\operatorname{supp}(a)$. The following hold.*

(1) *The element a is stationary and $a \to x^n a + b$ for some $b \in F_E^\Gamma$.*
(2) *The element a is periodic if and only if the core cycles have no exits.*

Proof. If $g \in \operatorname{supp}(a)$, then $g \rightsquigarrow^{c_g} g$ where c_g is a cycle of length n_g, where n_g is the minimum of the set of lengths of cycles which contain g. Hence, $g \to x^{n_g} g + b'_g$ for some $b'_g \in F_E^\Gamma$ such that $b'_g = 0$ if and only if c_g has no exits. Since n is a multiple of n_g, $g \to x^n g + b_g$ for some b_g such that $b_g = 0$ if and only if $b'_g = 0$.

If $a = \sum_{j=1}^{l} x^{k_j} g_j$ is a normal representation of a, then we have that $x^{k_j} g_j \to x^{n+k_j} g_j + x^{k_j} b_{g_j}$. Adding these relations together produces

$$a \to \sum_{j=1}^{l} x^{n+k_j} g_j + \sum_{j=1}^{l} x^{k_j} b_{g_j} = x^n \sum_{j=1}^{l} x^{k_j} g_j + \sum_{j=1}^{l} x^{k_j} b_{g_j} = x^n a + b$$

for $b = \sum_{j=1}^{l} x^{k_j} b_{g_j}$ so (1) holds. To show (2), note that a is periodic if and only if $b = 0$ and $b = 0$ if and only if any core cycle has no exits. $\square$

We note the following corollary of Lemmas 3.5, 3.7, and 3.8.

Corollary 3.9. *The following conditions are equivalent.*

(1) *There is a comparable generator of F_E^Γ.*
(2) *There is a nonzero comparable element of F_E^Γ.*
(3) *The graph E has a cycle.*

Proof. The implication (1) $\Rightarrow$ (2) is direct. If (2) holds, there is a stationary element a by Lemma 3.5. Since $a_c \neq 0$ by Lemma 3.7, there is at least one core cycle so (3) holds. If (3) holds, any vertex of a cycle is a comparable generator of F_E^Γ by Lemma 3.8 so (1) holds. $\square$

3.5. **The Core Lemma.** The following lemma highlights an important property of a stationary element and justifies our terminology "core" – if a is stationary and $x^n a$ can be produced from a with some possible "change" b, then $x^{kn} a$, for some positive k, can be produced by using the core part a_c only with possibly some other "change" c such that $c = 0$ and $a_e = 0$ if and only if $b = 0$.

Lemma 3.10. *(The Core Lemma) Let $a \in F_E^\Gamma$ be a stationary element with the core part a_c and the exit part a_e. If $a \to x^n a + b$ for some positive integer n and some $b \in F_E^\Gamma$, then $a_c \to x^{kn} a + c$ for some positive integer k and some $c \in F_E^\Gamma$ such that $c + a_e \sim \sum_{i=0}^{k-1} x^{in} b$.*

Proof. If $a_e = 0$, the claim trivially holds with $k = 1$ and $c = b$. If $a_e \neq 0$, let $V = \operatorname{supp}(a)$ so that V_e is nonempty. Let also $V_c = V'_c \cup V''_c$ where V'_c consists of the core generators in V such that no exit generator connects to them and V''_c consists of the core generators in V such that some exit generators connect to them. By these definitions, no $g \in V''_c$ connects to any $h \in V'_c$ (otherwise h would be in V''_c). Also, note that V'_c is nonempty since otherwise some exit generator would be on a cycle which would make it a core, not an exit generator. Let also $a_c = a'_c + a''_c$ so that $\operatorname{supp}(a'_c) = V'_c$ and $\operatorname{supp}(a''_c) = V''_c$. Choose n to be larger than the difference of degrees of any two monomials in a normal representation of a by using Lemma 3.7(1) if n is not already such.

We construct a sequence of finite acyclic graphs $F_0 \supsetneq F_1 \supsetneq \ldots \supsetneq F_l \supsetneq \emptyset$ such that the sequence terminates exactly when the claim is shown.

Graph $\mathbf{F_0}$. Let us define a graph F_0 such that V_e is the set of vertices of F_0 and that there is an edge from g to h for some $g, h \in V_e$ if g connects to h in E. Since no $g \in V_e$ is on a cycle, the graph F_0 is acyclic. Since F_0 is a finite and acyclic graph, it has a source by Lemma 1.1. Let V_{e0} be the set of sources of F_0 and $a_e = a_{e0} + a'_{e0}$ such that $\operatorname{supp}(a_{e0}) = V_{e0}$ and $\operatorname{supp}(a'_{e0}) = V_e - V_{e0}$.

By the Refinement Lemma 1.2(1), there are $a_1, a_2, a_3 \in F_E^\Gamma$ such that

$$a = a'_c + (a''_c + a'_{e0}) + a_{e0} \to x^n a + b = a_1 + a_2 + a_3 \text{ and } a'_c \to a_1,\ a''_c + a'_{e0} \to a_2,\ a_{e0} \to a_3.$$

If $x^m g$ is any monomial of $x^n a_{e0}$ for $g \in V_{e0}$, then $x^m g$ is a summand of either a_1, a_2 or a_3. By Corollary 2.5 and by the choice of n, if $x^m g$ is a summand of a_3 then either g is on a cycle or there is a path from another source of F_0 to g and each of these options leads to a contradiction. If $x^m g$ is a summand of a_2, then there is either a nontrivial path from some $g' \in V_e$ to g or a nontrivial path from some $h \in V_c''$ to g also by Corollary 2.5 and by the choice of n. In the second case, there is $g' \in V_e$ and a path from g' to h and, hence, a nontrivial path from g' to g as well. Thus, both cases lead to a contradiction since g is a source of F_0. Hence, $x^m g$ has to be a summand of a_1. Since the monomial $x^m g$ was arbitrary, $x^n a_{e0}$ is a summand of a_1. In addition, if $x^m h$ is any monomial of $x^n a_c'$, $x^m h$ is a summand of a_1 also. Indeed, assuming that $x^m h$ is a summand of either a_3 or a_2 implies that h is in V_c'' not V_c'. Hence, for some $b_0 \in F_E^\Gamma$,

$$a_c' \to a_1 = x^n a_c' + x^n a_{e0} + b_0.$$

If $a_{e0}' = 0$, we claim that the process is complete. In this case, $a_e = a_{e0}$. The support of a_c'' consists of core generators so a_c'' is stationary by Lemma 3.8. Let m be the least common multiple of n and the core period of a_c'' and let $m = kn$. Let b_0'' be such that $a_c'' \to x^{kn} a_c'' + b_0''$. After repeated use of the relation $a_c' \to x^n a_c' + x^n a_e + b_0$ for k times, we have that

$$a_c' \to x^{kn} a_c' + x^{kn} a_e + \sum_{i=1}^{k-1} x^{in} a_e + \sum_{i=0}^{k-1} x^{in} b_0.$$

Thus,

$$a_c = a_c' + a_c'' \to x^{kn} a_c' + x^{kn} a_e + \sum_{i=1}^{k-1} x^{in} a_e + \sum_{i=0}^{k-1} x^{in} b_0 + x^{kn} a_c'' + b_0'' = x^{kn} a + \sum_{i=1}^{k-1} x^{in} a_e + \sum_{i=0}^{k-1} x^{in} b_0 + b_0'' = x^{kn} a + c$$

for $c = \sum_{i=1}^{k-1} x^{in} a_e + \sum_{i=0}^{k-1} x^{in} b_0 + b_0''$. Thus, $a = a_c + a_e \to x^{kn} a + c + a_e$. On the other hand, $a \to x^{kn} a + \sum_{i=0}^{k-1} x^{in} b$ holds by part (1) of Lemma 3.7. Thus,

$$x^{kn} a + c + a_e \sim x^{kn} a + \sum_{i=0}^{k-1} x^{in} b \quad \text{which implies} \quad c + a_e \sim \sum_{i=0}^{k-1} x^{in} b.$$

If $a_{e0}' \neq 0$, we construct F_1.

Graph $\mathbf{F_1}$. Let F_1 be the graph obtained by eliminating the sources and all edges they emit from F_0. Then F_1 is a finite acyclic graph which is a proper subgraph of F_0. Let V_{e1} be the set of the sources of F_1 and $a_e = a_{e0} + a_{e1} + a_{e1}'$ be such that $\operatorname{supp}(a_{e1}) = V_{e1}$ and $\operatorname{supp}(a_{e1}') = V_e - V_{e0} - V_{e1}$. Let also $a_c'' = a_{c0}'' + a_{c1}''$ such that a_{c0}'' consists of those monomials $x^m h$ of a_c'' such that $g \rightsquigarrow h$ for some $g \in V_{e0}$ and a_{c1}'' consists of all other monomials of a_c''. Using the Refinement Lemma 1.2(1) again, there are $a_1', a_2', a_3' \in F_E^\Gamma$ such that

$$a = (a_c' + a_{c0}'' + a_{e0}) + (a_{c1}'' + a_{e1}') + a_{e1} \to x^n a + b = a_1' + a_2' + a_3'$$

and that $a_c' + a_{c0}'' + a_{e0} \to a_1'$, $a_{c1}'' + a_{e1}' \to a_2'$, $a_{e1} \to a_3'$. If $x^m g$ is any summand of $x^n a_c' + x^n a_{c0}'' + x^n a_{e0} + x^n a_{e1}$, we can repeat the arguments from before to show that the assumption that $x^m g$ is a summand of a_2' or a_3' leads to a contradiction. Hence, $x^m g$ is a summand of a_1' and so

$$a_c' + a_{c0}'' + a_{e0} \to a_1' = x^n a_c' + x^n a_{c0}'' + x^n a_{e0} + x^n a_{e1} + b_1'$$

for some $b_1' \in F_E^\Gamma$. If $k_1 n$ is the least common denominator of n and the core period of a_{c0}'', there is $b_1'' \in F_E^\Gamma$ such that $a_{c0}'' \to x^{k_1 n} a_{c0}'' + b_1''$. Using the last two relations and the relation $a_c' \to x^n a_c' + x^n a_{e0} + b_0$ from the first step for k_1 times, we have that

$$a_c' + a_{c0}'' \to x^{k_1 n}(a_c' + a_{e0}) + \sum_{i=1}^{k_1-1} x^{in} a_{e0} + \sum_{i=0}^{k_1-1} x^{ni} b_0 + x^{k_1 n} a_{c0}'' + b_1'' \to$$

$$x^{(k_1+1)n}(a_c' + a_{c0}'' + a_{e0} + a_{e1}) + x^{k_1 n} b_1' + \sum_{i=1}^{k_1-1} x^{in} a_{e0} + \sum_{i=0}^{k_1-1} x^{ni} b_0 + b_1'' = x^{(k_1+1)n}(a_c' + a_{c0}'' + a_{e0} + a_{e1}) + b_1$$

for $b_1 = x^{k_1 n} b_1' + \sum_{i=1}^{k_1-1} x^{in} a_{e0} + \sum_{i=0}^{k_1-1} x^{ni} b_0 + b_1''$.

If $a'_{e1} = 0$, then $a_e = a_{e0} + a_{e1}$. Let kn be the least common multiple of $(k_1+1)n$ and the core period of a''_{c1}. Arguing as in the case $a'_{e0} = 0$, we have that $a_c \to x^{kn}a + c$ for some $c \in F_E^\Gamma$ such that $\sum_{i=0}^{k-1} x^{in}b \sim c + a_e$ and this finishes the proof. If $a'_{e1} \neq 0$, we construct F_2 and continue the process.

This process eventually terminates since V_e is a finite set. Hence, there is a positive integer l such that $a'_{el} = 0$ so that $a_c \to x^{kn}a + c$ for some k and some c. The relations $a \to x^{kn}a + \sum_{i=0}^{k-1} x^{in}b$ and $a \to x^{kn}a + c + a_e$ imply that $\sum_{i=0}^{k-1} x^{in}b \sim c + a_e$ which proves the lemma. $\square$

The Core Lemma has the following corollary, characterizing a stationary and periodic element, which we use in the proof of Theorem 4.1.

Corollary 3.11. *A stationary element a is periodic if and only if the support of a consists of regular vertices on cycles without exits.*

Proof. Let a be such that $a \to x^n a + b$ for some b and positive n. If a is periodic, then $b = 0$. By the Core Lemma 3.10, $a_c \to x^{kn}a + c$ for some k and some c such that $\sum_{i=0}^{k-1} x^{in}b \sim a_e + c$. So $b = 0$ implies that $a_e = 0$ (and $c = 0$). Hence, $a = a_c$. This enables us to use Lemma 3.8 which implies that the support of a consists of generators on cycles without exits so that these generators are regular vertices.

For the converse, assume that the support of a consists of core vertices on cycles without exits. If n is the core period, then $a \to x^n a$ so a is both stationary and periodic. $\square$

3.6. **The stationary-partition.** By Lemma 3.7, the support of a stationary element is a stationary set. By Lemma 3.8, the converse is true if a stationary set contains no exit generators. It would be convenient to have the converse of this fact in general. However, the exit generators can complicate the situation as the next example shows.

Example 3.12. Let E be the Toeplitz graph $\circlearrowleft \bullet^v \longrightarrow \bullet^w$ and $a = v + w \in F_E^\Gamma$. Since $a = v + w \to xv + xw + w = x(v+w) + w = xa + w$, a is stationary. However, $b = v + xw$ has the same stationary support as a but b is not stationary. Indeed, if $b \to c$, then $c = x^n v + x^n w + x^{n-1}w + \ldots + xw + w + xw$ for some positive n. So, if $b \to x^n b + d$ for some d, then $x^n b$ has to contain $x^{n+1}w$ which is impossible.

We note also that adding xv to b, we obtain a stationary element again since it is a sum of stationary elements $x(v+w)$ and v.

This example shows that we need additional requirements for any element a with a stationary support to be stationary. In particular, these requirements impose restrictions on powers of x which appear in a.

Let a be stationary such that $a \to x^n a + b$ holds for some positive n and some b. If $a = \sum_{i=1}^{k} x^{m_i} g_i$ is a normal representation of a, by repeated use of the Refinement Lemma 1.2(1), there are mutually disjoint subsets $I_1, \ldots, I_k$ of $\{1, 2, \ldots, k\}$ whose union is $\{1, 2, \ldots, k\}$ and there are $b_1, \ldots, b_k$ such that

$$a = \sum_{i=1}^{k} \sum_{j \in I_i} x^{m_j} g_j \quad \text{and} \quad b = \sum_{i=1}^{k} b_i$$

and that for every $i = 1, \ldots, k$

$$x^{m_i} g_i \to \sum_{j \in I_i} x^{m_j + n} g_j + b_i. \tag{Rel. 1}$$

The set I_i can be empty if i is in $I_{i'}$ for some $i' \neq i$ (see also Example 3.14 below). If $I_i \neq \emptyset$, Corollary 2.4 applied to relation (Rel. 1) ensures the existence of a path p_{ij} connecting g_i and g_j such that

$$m_i + |p_{ij}| = m_j + n. \tag{Rel. 2}$$

By Lemma 3.7(1), we can choose n such that $n > m_i - m_j$ so that $|p_{ij}| = n + m_j - m_i > 0$ for all $i, j = 1, \ldots, k$. The requirement that p_{ij} has positive length justifies the following definition and implies direction $\Rightarrow$ of Proposition 3.15.

Definition 3.13. Let $a \in F_E^\Gamma$ have a stationary support V and a normal representation $a = \sum_{i=1}^k x^{m_i} g_i$. We say that a *has a stationary-partition* if there is a positive integer n, mutually disjoint subsets $I_1, \ldots, I_k$ of $\{1, 2, \ldots, k\}$ with $\bigcup_{i=1}^k I_i = \{1, 2, \ldots, k\}$ and paths p_{ij} of *positive length* for $i = 1, \ldots, k$ and $j \in I_i$ with $\mathbf{s}(p_{ij}) = g_i$ and $\mathbf{r}(p_{ij}) = g_j$ and such that relation (Rel. 2) holds for each $i = 1, \ldots, k$ and $j \in I_i$.

The following example shows that a stationary-partition does not have to be unique.

Example 3.14. Let E be the following graph $\bullet^{v_1} \longrightarrow \bullet^{v_2}$. Then $v_1 + v_2$ is stationary since $v_1 \to xv_1 + xv_2$ and so $v_1 + v_2 \to x(v_1 + v_2) + v_2$ and $k = 2$ in this case. We can take $I_1 = \{1, 2\}$ and $I_2 = \emptyset$ since v_1 "produces" both terms of $x(v_1 + v_2)$. In this case, relations (Rel. 1) are

$$v_1 \to xv_1 + xv_2 \quad \text{and} \quad v_2 \to v_2.$$

However, $v_2 \to xv_2$ also, so the summand xv_2 can be "produced" by v_2 also. Hence, $v_1 + v_2$ is stationary also because $v_1 + v_2 \to xv_1 + xv_2 + v_2 \to xv_1 + xv_2 + xv_2 = x(v_1 + v_2) + xv_2$. So, we can also take $I_1 = \{1\}$, $I_2 = \{2\}$. In this case, relations (Rel. 1) are

$$v_1 \to xv_1 + xv_2 \quad \text{and} \quad v_2 \to xv_2.$$

We characterize a stationary element in terms of the properties of the generators in its support which is the final and key step towards Theorem 3.17.

Proposition 3.15. *Let $a \in F_E^\Gamma$ be an element such that* $\operatorname{supp}(a) = V$ *is stationary. Then a is stationary if and only if a has a stationary-partition.*

Proof. We showed that direction $\Rightarrow$ holds before Definition 3.13. To summarize, if $a = \sum_{i=1}^k x^{m_i} g_i \to x^n a + b$ holds for some n and some b, use Lemma 3.7(1) to choose $n > m_i - m_j$ for all $i, j = 1, \ldots, k$. Repeated use of the Refinement Lemma 1.2(1) produces required sets $I_1, \ldots, I_k$ such that relations (Rel. 1) hold for $i = 1, \ldots, k$. Using Corollary 2.4 produces paths p_{ij} and our choice of n ensures that the paths p_{ij} have positive length so that relations (Rel. 2) hold. Thus, a has a stationary-partition.

Conversely, let a have a stationary-partition and let n, $I_1, \ldots, I_k$ and p_{ij} be as in Definition 3.13. Starting with $x^{m_i} g_i$ and applying the axioms following the paths p_{ij} for all $i = 1, \ldots, k$ and all $j \in I_i$, we obtain $x^{m_i} g_i \to \sum_{j \in I_i} x^{m_i + |p_{ij}|} g_j + b_i$ for some $b_i \in F_E^\Gamma$ for $i = 1, \ldots, k$. By relations (Rel. 2),

$$x^{m_i} g_i \to \sum_{j \in I_i} x^{m_i + |p_{ij}|} g_j + b_i = \sum_{j \in I_i} x^{m_j + n} g_j + b_i$$

which shows that relations (Rel. 1) hold for all i. Adding these relations together produces

$$a = \sum_{i=1}^k x^{m_i} g_i \to \sum_{i=1}^k \left(\sum_{j \in I_i} x^{m_j + n} g_j + b_i \right) = x^n \sum_{i=1}^k \sum_{j \in I_i} x^{m_j} g_j + \sum_{i=1}^k b_i = x^n a + \sum_{i=1}^k b_i$$

where the last equality holds since $I_1, \ldots, I_k$ are disjoint and their union is $\{1, \ldots, k\}$. Letting $b = \sum_{i=1}^k b_i$, we have that $a \to x^n a + b$. Hence, a is stationary. $\square$

Remark 3.16. Since the relation $a \to x^n a + b$ holds for some n and b if and only if $a_c \to x^m a + c$ holds for some m and c by the Core Lemma 3.10, we can also consider a partition of $x^m a + c$ based on a normal representation of a_c instead of a. If Definition 3.13 is modified accordingly, Proposition 3.15 can be formulated to state that a is stationary if and only if a has a partition based on its core part a_c.

Proposition 3.15 also reaffirms Lemma 3.8 since if an element a has the stationary support consisting of core generators only, then a has a stationary-partition. Indeed, if $a = a_c$, one can take n to be the core period and $I_i = \{i\}$. If n_{g_i} is the minimum of the set of lengths of cycles on which g_i is and if $n = l_i n_{g_i}$, one can take p_{ii} to be the path obtaining by traversing a cycle of length n_{g_i} l_i times starting at g_i so that $|p_{ii}| = n$. Thus, relation (Rel. 2) holds trivially for each i since $m_i + n = m_i + n$ and so a has a stationary-partition.

3.7. **Characterization of comparability.** Using Propositions 2.2 and 3.15, we prove Theorem 3.17 characterizing a comparable element.

Theorem 3.17. *The following conditions are equivalent for an element $a \in F_E^\Gamma$.*

(1) *The element a is nonzero and comparable.*
(2) *There is a stationary element b such that $a \to b$.*
(3) *There is an element b with a stationary support and a stationary-partition such that condition (2) from Proposition 2.2 holds for a and b.*

Proof. The implication (1) $\Rightarrow$ (2) holds by Lemma 3.5. Conversely, if (2) holds, then b is nonzero and comparable. The relation $a \to b$ implies $a \sim b$ so a is nonzero and comparable as well.

The equivalence (2) $\Leftrightarrow$ (3) follows directly from Propositions 2.2 and 3.15. $\square$

In Theorem 3.19, we characterize when every element of F_E^Γ is comparable. First, we show the following corollary of Proposition 2.2 and Lemmas 3.5 and 3.7 which we use in the proof of Theorem 3.19.

Corollary 3.18. *Let v be an infinite emitter.*

(1) *If v connects to q_Z^v by a path of positive length, then v is on a cycle.*
(2) *If q_Z^v connects to q_W^v by a path of positive length, then q_Z^v is on a cycle.*
(3) *If q_W^v is on a cycle, then v is on a cycle and q_Z^v is on a cycle for every $\emptyset \neq Z \subseteq W$.*
(4) *If v is comparable, then v is on a cycle.*
(5) *If q_Z^v is comparable, then q_Z^v is on a cycle.*

Proof. To show (1), assume that $v \rightsquigarrow^p q_Z = q_Z^v$ for some path p of positive length n. Then $v \to x^n q_Z + a$ for some $a \in F_E^\Gamma$. By the nature of axioms (A2) and (A3), there has to be a term $x^n v$ produced at some point. Hence, $v \to x^n v + b$ for some b which implies that v is on a cycle.

To show (2), assume that $q_Z = q_Z^v \rightsquigarrow^p q_W = q_W^v$ for some path p of positive length n. Then $q_Z \to x^n q_W + a$ for some a. By the nature of axioms (A2) and (A3), there has to be a term $x^n v$ produced at some point using a cycle c based at v such that the first edge of c is not in Z. Hence, $q_Z \to x^n v + b \to x^n q_Z + c$ for some b and c by (A2) and so q_Z is on a cycle.

By Definition 2.1, if q_W^v is on a cycle, then there is a cycle based at v such that the first edge e of that cycle is not in W. So, v is on a cycle. If $Z \subseteq W$, then $e \notin Z$ and so q_Z is also on a cycle by Definition 2.1. This shows (3).

To show (4), let v be comparable. By Lemma 3.5, there is a stationary element a such that $v \to a$. If $a = v$, then v is stationary and it is necessarily on a cycle by part (2) of Lemma 3.7. So, assume that $a \neq v$. By Proposition 2.2, if $a = \sum_{j=1}^{l} x^{t_j} h_j$ is a normal representation of a, there are paths $p_j, j = 1, \ldots, l$ such that $v \rightsquigarrow^{p_j} h_j$, $t_j = |p_j|$, and at least one of h_j is q_Z^v for some Z. Reordering the terms we can assume that $j = 1$. If p_1 has positive length, then v is on a cycle by part (1). If q_Z is on a cycle, then v is on a cycle by part (3). So, let us consider the remaining case when p_1 is trivial and q_Z is not on a cycle. In this case, q_Z has to be on an exit from a core cycle by part (4) of Lemma 3.7. So, there is $j > 1$ such that h_j is on a cycle and $h_j \rightsquigarrow^p q_Z$ for some path p. Hence, $v \rightsquigarrow^{p_j} h_j \rightsquigarrow^p q_Z$. If $|p| > 0$, then v connects to q_Z by a path $p_j p$ of positive length and so v is on a cycle by part (1). If $|p| = 0$, then either $h_j = v$, in which case v is on a cycle, or $h_j = q_{Z'}$ for some $Z' \subsetneq Z$ in which case v is also on a cycle by part (3).

To show (5), let q_Z be comparable. By Lemma 3.5, there is a stationary element a such that $q_Z \to a$. If $a = q_Z$, then q_Z is stationary and it is necessarily on a cycle by part (2) of Lemma 3.7. So, assume that $a \neq q_Z$. By Proposition 2.2, if $a = \sum_{j=1}^{l} x^{t_j} h_j$ is a normal representation of a, there are paths $p_j, j = 1, \ldots, l$ such that $q_Z \rightsquigarrow^{p_j} h_j$, $t_j = |p_j|$, and at least one of h_j is q_W^v for some $W \supsetneq Z$. Reordering the terms we can assume that $j = 1$. If p_1 has positive length, then q_Z is on a cycle by part (2). If q_W is on a cycle, then q_Z is on a cycle by part (3). So, let us consider the remaining case when p_1 is trivial and q_W is not on a cycle. By part (4) of Lemma 3.7, there is $j > 1$ such that h_j is on a cycle and $h_j \rightsquigarrow^p q_W$ for some path

p. So, $q_Z \rightsquigarrow^{p_j} h_j \rightsquigarrow^p q_W$. If $|p| > 0$, q_Z connects to q_W by a path p_jp of positive length and so q_Z is on a cycle by part (2). If $|p| = 0$, then either $h_j = v$ or $h_j = q_{Z'}$ for some $Z' \subsetneq W$. In the first case, $q_Z \rightsquigarrow^{p_j} v$ so there is a cycle based at v such that its first edge e is not in Z and so q_Z is on that cycle by Definition 2.1. In the second case, $q_Z \rightsquigarrow^{p_j} q_{Z'}$. If $|p_j| > 0$, then q_Z is on a cycle by part (2). If $|p_j| = 0$, then $Z \subseteq Z'$. Since $q_{Z'}$ is on a cycle, there is a cycle based at v such that the first edge e of it is in $\mathbf{s}^{-1}(v) - Z'$. Hence, $e \notin Z$ and so q_Z is on a cycle by Definition 2.1. □

Theorem 3.19. *The following conditions are equivalent.*

1. *Every element $a \in F_E^\Gamma$ is comparable.*
2. *Every generator of F_E^Γ is comparable.*
3. *For every generator g of F_E^Γ, $g \to a$ for some stationary element a.*
4. *The following hold for every generator g of F_E^Γ.*
   - (a) *The generator g is not a sink and it connects to a cycle.*
   - (b) *If g is an infinite emitter or an improper vertex, then g is on a cycle.*
   - (c) *If g is regular, there is stationary $a \in F_E^\Gamma$ with the exit part zero such that $g \to a$.*

Proof. The implication (1) $\Rightarrow$ (2) is direct and the implication (2) $\Rightarrow$ (1) holds since a finite sum of comparable elements is comparable. The equivalence (2) $\Leftrightarrow$ (3) follows directly from Theorem 3.17. To complete the proof, we show (3) $\Leftrightarrow$ (4).

Assume that (3) holds and let g be any generator. Let a be stationary such that $g \to a$. Since g connects to all generators in the support of $a_c \neq 0$, g connects to a generator on a cycle so g is not a sink and (a) holds. Part (b) holds by parts (4) and (5) of Corollary 3.18. To show part (c), let $g = v \in E^0$ be regular. We claim that there is an element $b \in F_E^\Gamma - \{0\}$ with support containing only vertices on cycles such that $v \to b$. We prove this claim using induction on the minimum n of lengths of paths from v to cycles which exist by part (a). If this length n is zero, v is on a cycle and one can take $b = v$. Assuming the induction hypothesis, consider v with $n > 0$. For every $e \in \mathbf{s}^{-1}(v)$, either $\mathbf{r}(e)$ is on a cycle, in which case we let $b_e = \mathbf{r}(e)$ or $\mathbf{r}(e)$ is not on a cycle in which case $\mathbf{r}(e)$ is necessarily regular by parts (a) and (b). In this case, the minimum of lengths of paths from $\mathbf{r}(e)$ to cycles is less than n and we can use induction hypothesis to obtain b_e with vertices in the support on cycles and $\mathbf{r}(e) \to b_e$. Then $b = \sum_{e\in\mathbf{s}^{-1}(v)} xb_e$ has vertices in the support on cycles and

$$v \to_1 \sum_{e\in\mathbf{s}^{-1}(v)} x\mathbf{r}(e) \to \sum_{e\in\mathbf{s}^{-1}(v)} xb_e = b.$$

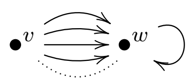
Since $\mathrm{supp}(b)$ consists of generators on cycles, b is stationary by Lemma 3.8 and its exit part is zero.

Assume that (4) holds and let g be any generator. By (a), g is not a sink. If g is an infinite emitter or an improper vertex, g is on a cycle by (b) and so it is stationary. Then (3) holds since $g \to g$. If g is regular, (3) holds by part (c). □

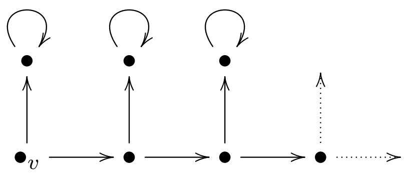
Part (4) with any of the conditions (a), (b), or (c) deleted is not equivalent with the other conditions of Theorem 3.19 as the next set of examples shows.

Example 3.20. (1) If E is the Toeplitz graph (see Example 3.12), then (b) and (c) hold. There is a sink so (a) fails and the sink w is not comparable.

(2) If E is the graph below, then (a) and (c) hold. The infinite emitter v is not on a cycle, so (b) fails and v is not comparable by Corollary 3.18(4).



(3) Let E be the graph below.



If a is any element whose support consists only of vertices on cycles, then $v \to a$ fails since there is a path originating at v which does not connect to $\mathrm{supp}(a)$ (analogous argument is used in part (3) of Example 2.3). The conditions (a) and (b) hold for E, but (c) fails and v is not comparable.

4. Characterizations of periodic, aperiodic and incomparable elements

Next, we show characterizations of periodic, aperiodic and incomparable elements as well as other properties discussed in the introduction. We start by Theorem 4.1 which characterizes a nonzero periodic element of F_E^Γ. Theorem 4.1 has already been used in [10, Theorem 3.1] to characterize Leavitt path algebras which are crossed products in terms of the properties of the underlying graphs.

Theorem 4.1. *The following conditions are equivalent for an element $a \in F_E^\Gamma - \{0\}$.*

(1) *The element a is periodic.*
(2) *There is an element b whose support consists of vertices on cycles without exits such that $a \overset{A1}{\to} b$.*
(3) *Any path originating at a generator in the support of a is a prefix of a path p ending in one of finitely many cycles with no exits and such that all vertices of p are regular. Every infinite path originating at a vertex in the support of a ends in a cycle with no exits.*

Proof. If (1) holds, then a is comparable so $a \to b$ for some stationary element b by Lemma 3.5. The relation $a \to b$ implies $a \sim b$ so b is periodic as well. Hence, the supports of both a and b consists of regular vertices only by Lemma 3.2. Thus, $a \overset{A1}{\to} b$. By Corollary 3.11, the support of b consists of regular vertices on cycles without exits which shows (2).

If (2) holds, the element b as in (2) is stationary and periodic by Lemma 3.8. Since the core cycles of b do not have exits, each generator in $\mathrm{supp}(b)$ is proper, emits exactly one edge and, hence, it is regular. As $a \overset{A1}{\to} b$, any element of $\mathrm{supp}(a)$ is proper and regular also. Let $a = \sum_{i=1}^k x^{m_i} v_i$, $b = \sum_{j=1}^l x^{t_j} w_j$, and I_i and p_{ij} be as in Proposition 2.2 for $a \to b$. Since only (A1) is used, each vertex of any path p_{ij} is regular.

If p is a path with $\mathbf{s}(p) = v_i$, we use induction on $|p|$ to show that there is a path q such that p is a prefix of q, q ends in one of the core cycles and all vertices of q are on some p_{ij} for $j \in I_i$ (thus regular) or on cycles without exits (thus also regular). If $p = v_i$, q can be taken to be p_{ij} for any $j \in I_i$. Assuming that the claim holds for p, let us consider pe for some edge e. By the induction hypothesis, all vertices of p are regular, on p_{ij} for some $j \in I_i$ or on a core cycle. If $\mathbf{r}(e)$ is on a core cycle, then it emits exactly one edge so it is regular and we can take q to be pe. So, let us consider the case that $\mathbf{r}(e)$ is not on a core cycle in which case $\mathbf{r}(p)$ is not on a core cycle also and so $\mathbf{r}(p)$ is on p_{ij} for some $j \in I_i$. Since $\mathbf{r}(p)$ is not on a cycle, there is a proper prefix r of p_{ij} which ends in $\mathbf{r}(p)$. Thus $P_r \neq \emptyset$ and so $P_r = \mathbf{s}^{-1}(\mathbf{r}(p))$ by part (2)(i) of Proposition 2.2. In particular, $e \in P_r$. Hence, there is $j' \in I_i$ such that e is in $p_{ij'}$. Let q be pe up to $\mathbf{r}(e)$ and the suffix of $p_{ij'}$ after pe. Thus, pe is a prefix of q, q ends in a core cycle and each vertex of q is on p_{ij} for some $j \in I_i$.

It remains to show the condition on the infinite path. Let $e_1e_2\ldots$ be an infinite path originating at v_i. For any n, each vertex of the path $e_1e_2\ldots e_n$ is on p_{ij} for some $j \in I_i$ or in a core cycle. Let n be strictly larger than the length of p_{ij} for all $j \in I_i$. Then $\mathbf{r}(e_n)$ must be in a core cycle and so $e_ne_{n+1}\ldots$ is on that same cycle since the cycle has no exits. This shows that (3) holds.

If condition (3) holds, then the support of a consists of regular vertices such that every path they emit connects to finitely many cycles without exits by paths which contain regular vertices only. Let $\mathrm{supp}(a) = \{v_1, \ldots, v_k\}$ and let n_i be the number of paths p from v_i to the finitely many cycles from condition (3) such that no vertex of any of the paths from (3) is on the cycle except the range of p. Index the paths originating at v_i as $p_{i1}, \ldots, p_{in_i}$ for some positive n_i and let $w_{ij} = \mathbf{r}(p_{ij})$. Let J be the set of (i,j) with $i = 1, \ldots, k$ and $j = 1, \ldots, n_i$ and let I_i be the set of those $(i', j) \in J$ such that $i' = i$. By construction, $\{I_1, \ldots, I_k\}$ is a partition of J and, by considering a bijection between J and the set $\{1, \ldots, l\}$ for $l = |J|$, this partition corresponds to a partition of $\{1, \ldots, l\}$.

If p is a prefix of p_{ij}, let us use the notation P_p in the same sense as in Proposition 2.2 and Definition 2.6. If p is a proper prefix of p_{ij}, then $\mathbf{r}(p)$ is regular and P_p is nonempty as it contains the first edge of p_{ij} not on p. If $e \in \mathbf{s}^{-1}(\mathbf{r}(p))$, then pe is a prefix of some $p_{ij'}$ by condition (3) and so $e \in P_p$. Hence, $P_p = \mathbf{s}^{-1}(\mathbf{r}(p))$. If $p = p_{ij}$, then P_p is empty by construction. Thus condition (i) of Definition 2.6 holds and conditions (ii) and (iii) are trivially satisfied. So, for $W = \{w_{ij} \mid (i,j) \in J\}$, $\operatorname{supp}(a) \to W$. Moreover, $\operatorname{supp}(a) \xrightarrow{A1} W$ since $\operatorname{supp}(a)$ and W contain regular vertices only. Hence, there is $c \neq 0$ such that $a \xrightarrow{A1} c$ and $\operatorname{supp}(c) \subseteq W$ by Corollary 2.8. The set W is stationary and, by part (1) of Lemma 3.8, every element with support contained in W is stationary and, part (2) of Lemma 3.8, periodic. Thus, c is periodic. Since $a \sim c$, a is also periodic. Hence, (1) holds. □

We note that the sources of graphs in parts (1) and (3) of Example 3.20 are such that condition (2) fails, so that these vertices are not periodic by Theorem 4.1 (and incomparable by Theorem 3.17).

In Theorem 4.2, we characterize when every element of F_E^Γ is periodic in terms of the properties of E, in terms of the form of the Leavitt path algebra, as well as in terms of the form of the Grothendieck Γ-group.

Theorem 4.2. *The following conditions are equivalent.*

(1) *Every element $a \in F_E^\Gamma$ is periodic.*
(2) *Every vertex is periodic.*
(3) *For every vertex v, $\{v\} \xrightarrow{A1} V$ for some stationary set V which contains core vertices only and every core cycle has no exits.*
(4) *Each path is a prefix of a path p ending in a cycle with no exits and such that the vertices on p are regular. Every infinite path ends in a cycle with no exits.*
(5) *E is a row-finite, no-exit graph without sinks such that every infinite path ends in a cycle.*
(6) *For any field K, the Leavitt path algebra $L_K(E)$ is graded isomorphic to an algebra of the form*
$$\bigoplus_{i \in I} \mathbb{M}_{\mu_i}(K[x^{n_i}, x^{-n_i}])(\overline{\gamma}_i)$$
where I is a set, μ_i are cardinals, n_i positive integers, and $\overline{\gamma}_i$ maps $\mu_i \to \mathbb{Z}$ for $i \in I$.
(7) *The Grothendieck Γ-group G_E^Γ is isomorphic to*
$$\bigoplus_{i \in I} \mathbb{Z}[x]/\langle x^{n_i} = 1 \rangle$$
where I is a set and n_i are positive integers for $i \in I$.

Proof. The implication (1) ⇒ (2) is direct. If (2) holds, then every vertex of E is regular by Lemma 3.2. If a vertex v is periodic, $v \to a$ for some stationary element a by Lemma 3.5. Since v is periodic, a is periodic also and the support of a consists of regular vertices on cycles without exits by Corollary 3.11. Thus, $v \to a$ implies that $v \xrightarrow{A1} a$. If $V = \operatorname{supp}(a)$, condition (3) follows by Corollary 2.8.

If (3) holds, then all vertices of E are regular. If p is any finite or infinite path, $\{\mathbf{s}(p)\} \xrightarrow{A1} V$ for some V as in condition (3). By Corollary 2.8, there is $a \in F_E^\Gamma - \{0\}$ such that $\mathbf{s}(p) \xrightarrow{A1} a$ and $\operatorname{supp}(a) \subseteq V$. Since V consists of vertices on cycles without exits, a is stationary and periodic and so $\mathbf{s}(p)$ is periodic also. Then (4) holds by Theorem 4.1.

If (4) holds, then all vertices of E are regular so E is a row-finite graph. Every vertex connects to cycles so there are no sinks. Every infinite path ends in a cycle and no cycle has an exit. So, (5) holds.

Conditions (5) and (6) are equivalent by [14, Corollary 3.6].

Condition (6) implies (7) and condition (7) directly implies that every element of G_E^Γ has a finite orbit. Hence, every element of F_E^Γ is periodic and (1) holds. □

Using Theorem 4.1, we characterize when no nonzero element of F_E^Γ is periodic.

Corollary 4.3. *The following conditions are equivalent.*

(1) *No nonzero element of F_E^Γ is periodic.*
(2) *The graph E satisfies Condition (L).*

Proof. If E has a cycle with no exits, any vertex on this cycle is periodic. Conversely, if Condition (L) holds, the core cycles of any stationary element have exits. By Theorem 4.1, no nonzero element is periodic. □

We characterize aperiodic elements next.

Theorem 4.4. *The following conditions are equivalent for an element $a \in F_E^\Gamma$.*

(1) *The element a is aperiodic.*
(2) *The element a is comparable and not periodic.*
(3) *There is a stationary element b such that $a \to b$ and at least one of the core cycles of b has an exit.*

Proof. It is direct that (1) $\Leftrightarrow$ (2). The equivalence (2) $\Leftrightarrow$ (3) holds by Theorems 3.17 and 4.1. □

We also characterize when every element of F_E^Γ is aperiodic.

Theorem 4.5. *The following conditions are equivalent.*

(1) *Every nonzero element $a \in F_E^\Gamma$ is aperiodic.*
(2) *Every generator of F_E^Γ is aperiodic.*
(3) *Every generator of F_E^Γ is comparable and every cycle has an exit.*
(4) *For every generator g of F_E^Γ, $g \to a$ for some stationary element a such that all core cycles have exits.*

Proof. The implication (1) $\Rightarrow$ (2) is direct. The converse holds since a sum of aperiodic elements is comparable and, if at least one of them is aperiodic, aperiodic.

If (2) holds and g is a generator on an arbitrary cycle (which exists by Corollary 3.9), then g is aperiodic if and only if the cycle has an exit by Lemma 3.8. Hence, (3) holds.

If (3) holds and g is an arbitrary generator, then $g \to a$ for a stationary element a. By assumption (3) all core cycle of a have exits so (4) holds.

Finally, let us assume that (4) holds and show (2). If g is an arbitrary generator and a a stationary element such that $g \to a$ and all core cycles have exit, then $a \to x^n a + b$ for some nonzero b. Hence, a is aperiodic and, since $g \to a$, g is aperiodic also. □

We also characterize when no element of F_E^Γ is aperiodic.

Corollary 4.6. *The following conditions are equivalent.*

(1) *No element of F_E^Γ is aperiodic.*
(2) *The graph E is no-exit (i.e. satisfies Condition (NE)).*

Proof. If E is not a no-exit graph, there is a cycle with an exit and any vertex on that cycle is an aperiodic element of F_E^Γ. Conversely, if a is an aperiodic element of F_E^Γ, then $a \to b$ for some stationary element b such that at least one of the core cycles of b must have an exit by Theorem 4.4. Hence, E is not no-exit. □

Since every element which is not comparable is incomparable, Theorem 3.17 implies a characterization of an incomparable element in F_E^Γ also. The following characterization of graphs such that all elements of F_E^Γ are incomparable follows directly from Corollary 3.9.

Corollary 4.7. *The following conditions are equivalent.*

(1) *Every nonzero element $a \in F_E^\Gamma$ is incomparable.*

(2) *Every generator of F_E^Γ is incomparable.*
(3) *The graph E is acyclic.*

4.1. **Strengthening results of [9].** By Proposition 3.3, a result of [9] holds without the assumption that the graph under consideration is row-finite. In this section, we show that the same assumption can be deleted from some of the main results of [9]. The second part of Corollary 4.11 shows that our results provide some further progress towards a positive answer to the Graded Classification Conjecture.

First, we show that Theorems 4.1 and 4.4 and Corollary 4.7 imply [9, Proposition 4.2] without assuming that the graph is row-finite. We formulate this in the following corollary.

Corollary 4.8. (1) *The graph E has a cycle with no exit if and only if some nonzero element of F_E^Γ is periodic.*
(2) *The graph E has a cycle with an exit if and only if some element of F_E^Γ is aperiodic.*
(3) *The graph E is acyclic if and only if every nonzero element of F_E^Γ is incomparable.*

Proof. One direction of parts (1) and (2) follows by Theorems 4.1 and 4.4. The other follows by Lemma 3.8 which implies that a vertex on a cycle is periodic if the cycle has no exits and it is aperiodic if the cycle has an exit. Part (3) directly follows from Corollary 4.7. □

By [13, Theorem 5.7], a Γ-order-ideal of M_E^Γ uniquely determines certain subset of vertices. We briefly review this construction. A subset H of E^0 is said to be *hereditary* if for any $v \in H$ and a path p with $\mathbf{s}(p) = v$, $\mathbf{r}(p)$ is in H and it is *saturated* if $\mathbf{r}(\mathbf{s}^{-1}(v)) \subseteq H$ for a regular vertex v implies that $v \in H$.

For a hereditary and saturated set H, let

$$G(H) = \{v \in E^0 - H \mid v \text{ is not regular and } \mathbf{s}^{-1}(v) \cap \mathbf{r}^{-1}(E^0 - H) \text{ is nonempty and finite}\}.$$

For $G \subseteq G(H)$, the pair (H, G) is said to be an *admissible pair*. The set of all such pairs is a lattice by

$$(H_1, G_1) \le (H_2, G_2) \text{ iff } H_1 \subseteq H_2, \ \ G_1 \subseteq G_2 \cup H_2$$

(see [13] or [3]). By [13, Theorem 5.7], this lattice is isomorphic to the lattice of graded ideals of $L_K(E)$ and by [3, Theorem 6.9], this lattice is isomorphic to the set of order-ideals of M_E. If $(H, G) \mapsto I(H, G)$ denotes this isomorphism, then $M_E/I(H, G) \cong M_{E/(H,G)}$ and both [13] and [3] contain details. By [4, Lemma 5.10], the lattices of order-ideals of M_E and of Γ-order-ideals of M_E^Γ are isomorphic. Moreover, if the assumption that E is row-finite is deleted and hereditary and saturated set replaced by an admissible pair, the proof of [9, Lemma 2.2] establishes that

$$M_E^\Gamma/I(H, G) \cong M_{E/(H,G)}^\Gamma$$

for an admissible pair (H, G).

Next, we show that the assumption that E is row-finite can be removed from [9, Corollary 4.3].

Corollary 4.9. *(1) The following conditions are equivalent.*
- (i) *The graph E satisfies Condition (L).*
- (ii) *No nonzero element of F_E^Γ is periodic.*
- (iii) *Γ acts freely on M_E^Γ.*

(2) The following conditions are equivalent.
- (i) *The graph E satisfies Condition (K).*
- (ii) *No nonzero element of M_E^Γ/I is periodic for any Γ-order-ideal I of M_E^Γ.*
- (iii) *The group Γ acts freely on M_E^Γ/I for any Γ-order-ideal I of M_E^Γ.*

Proof. Part (1) directly follows from Corollary 4.3.

By [13, Proposition 6.12], E satisfies Condition (K) if and only if $E/(H, G)$ satisfies Condition (L) for any admissible pair (H, G). Since every such pair uniquely determines a Γ-order-ideal of M_E^Γ, part (1) and Corollary 4.3 imply the equivalences of conditions in part (2). □

[9, Corollary 5.1] focuses on the monoid properties of M_E^Γ which are equivalent with various forms of simplicity of $L_K(E)$. We show these properties without requiring that E is row-finite.

Corollary 4.10. *Let K be any field.*

(1) *The following conditions are equivalent.*
  (i) *The algebra $L_K(E)$ is graded simple.*
  (ii) *The Γ-monoid M_E^Γ is simple.*
  (iii) *The Γ-group G_E^Γ is simple as an ordered Γ-group.*

(2) *The following conditions are equivalent.*
  (i) *The algebra $L_K(E)$ is simple.*
  (ii) *The Γ-monoid M_E^Γ is simple and no nonzero element of M_E^Γ is periodic.*
  (iii) *The Γ-monoid M_E^Γ is simple and every nonzero comparable element of M_E^Γ is aperiodic.*

(3) *The following conditions are equivalent.*
  (i) *The algebra $L_K(E)$ is purely infinite simple.*
  (ii) *The Γ-monoid M_E^Γ is simple, no nonzero element of M_E^Γ is periodic and some element of M_E^Γ is aperiodic.*

Proof. Part (1) directly follows from the fact that the lattices of graded ideals of $L_K(E)$, Γ-order-ideals of M_E^Γ and Γ-order-ideals of G_E^Γ are isomorphic.

By [1, Theorem 2.9.1], $L_K(E)$ is simple if and only if it is graded simple and E satisfies Condition (L). By part (1) and Corollary 4.3, this is equivalent with M_E^Γ being simple and without a nonzero periodic element. This last condition is equivalent with the requirement that every nonzero comparable element is aperiodic.

By [1, Theorem 3.1.10], $L_K(E)$ is purely infinite simple if and only if it is simple and E has a cycle with an exit. By Corollary 4.8, E has a cycle with an exit if and only if M_E^Γ has an aperiodic element. $\square$

Lastly, we show Corollary 4.11. Parts (1) and (3) show that the first part of [9, Theorem 5.7] holds without the condition that E is row-finite. Parts (4) to (8) are further corollaries of our results.

Corollary 4.11. *Let E and F be arbitrary graphs. If there is a Γ-monoid isomorphism $M_E^\Gamma \to M_F^\Gamma$, then the following hold.*

(1) *The graph E satisfies Condition (L) if and only if F satisfies Condition (L).*
(2) *The graph E satisfies Condition (K) if and only if F satisfies Condition (K).*
(3) *The lattices of graded ideals of $L_K(E)$ and $L_K(F)$ are isomorphic.*
(4) *E is acyclic if and only if F is acyclic.*
(5) *There is a cycle without an exit in E if and only if there is a cycle without an exit in F.*
(6) *There is a cycle with an exit in E if and only if there is a cycle with an exit in F.*
(7) *None of the cycles of E have exits if and only if none of the cycles of F have an exit.*
(8) *E satisfies the condition below if and only if F satisfies the condition below.*
The graph is row-finite, no-exit, has no sinks and it is such that every infinite path ends in a cycle.

Proof. Parts (1) and (2) directly follow from Corollary 4.9. To show part (3), note that a Γ-monoid isomorphism $M_E^\Gamma \to M_F^\Gamma$ induces a lattice isomorphism on the lattices of Γ-order-ideals. Since these lattices are isomorphic to lattices of graded ideals of $L_K(E)$ and $L_K(F)$, part (3) holds.

Part (4) holds since E has a cycle if and only if there is a nonzero comparable element in M_E^Γ by Corollary 3.9. Part (5) holds since E has a cycle with no exit if and only if there is a nonzero periodic element in M_E^Γ by Corollary 4.8(1). Part (6) holds since E has a cycle with an exit if and only if there is an aperiodic element in M_E^Γ by Corollary 4.8(2).

Part (7) holds by Corollary 4.6 and part (8) by Theorem 4.2. $\square$

References

[1] G. Abrams, P. Ara, M. Siles Molina, Leavitt path algebras, Lecture Notes in Mathematics 2191, Springer, London, 2017. 5, 6, 10, 30

[2] G. Abrams, P.N. Ánh, A. Louly, E. Pardo, *The classification question for Leavitt path algebras*, J. Algebra **320 (5)** (2008), 1983–2026. 9

[3] P. Ara, K. R. Goodearl, *Leavitt path algebras of separated graphs,* J. Reine Angew. Math. **669** (2012), 165–224. 1, 5, 6, 8, 29

[4] P. Ara, R. Hazrat, H. Li, A. Sims, *Graded Steinberg algebras and their representations*, Algebra Number Theory **12 (1)** (2018), 131–172. 2, 6, 7, 8, 15, 29

[5] P. Ara, M. A. Moreno, E. Pardo, *Nonstable K-theory for graph algebras,* Algebr. Represent. Theory **10 (2)** (2007), 157–178. 1, 5

[6] K. R. Goodearl, Partially ordered abelian groups with interpolation, Mathematical Surveys and Monographs **20**, American Mathematical Society, 1986. 4

[7] R. Hazrat, *The graded Grothendieck group and classification of Leavitt path algebras,* Math. Annalen **355 (1)** (2013), 273–325. 2

[8] R. Hazrat, Graded rings and graded Grothendieck groups, London Math. Soc. Lecture Note Ser. 435, Cambridge Univ. Press, 2016. 4, 5

[9] R. Hazrat, H. Li, *The talented monoid of a Leavitt path algebra*, J. Algebra, **547** (2020) 430–455. 1, 3, 6, 15, 16, 29, 30

[10] R. Hazrat, L. Vaš, *Crossed product Leavitt path algebras*, submitted for publication, preprint `arXiv: 2002.11230 [math.RA]`3, 26

[11] P. S. Muhly, M. Tomforde, *Adding tails to C^*-correspondences*, Doc. Math. **9** (2004), 79–106. 8

[12] C. Năstăsescu, F. van Oystaeyen, Methods of graded rings, Lecture Notes in Mathematics 1836, Springer-Verlag, Berlin, 2004. 4

[13] M. Tomforde, *Uniqueness theorems and ideal structure for Leavitt path algebras*, J. Algebra **318** (2007), no. 1, 270–299. 29

[14] L. Vaš, *Graded chain conditions and Leavitt path algebras of no-exit graphs*, J. Aust. Math. Soc. **105 (2)** (2018), 229 – 256. 8, 27

[15] L. Vaš, *Simplicial and dimension groups with group action and their realization*, submitted for publication, preprint `arXiv: 1805.07636 [math.KT]`. 5

Centre for Research in Mathematics and Data Science, Western Sydney University, Australia

Email address: `r.hazrat@westernsydney.edu.au`

Department of Mathematics, Physics and Statistics, University of the Sciences, Philadelphia, PA 19104, USA

Email address: `l.vas@usciences.edu`