THE GRADED CLASSIFICATION CONJECTURE HOLDS FOR GRAPHS WITH DISJOINT CYCLES

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Abstract. The Graded Classification Conjecture (GCC) states that the pointed K_0^{gr}-group is a complete invariant of the Leavitt path algebras of finite graphs when these algebras are considered with their natural grading by $\mathbb{Z}$. The conjecture has previously been shown to hold in some rather special cases. The main result of the paper shows that the GCC holds for a significantly more general class of graphs – countable graphs with disjoint cycles, with only finitely many infinite emitters, finitely many sinks and cycles and such that every infinite path ends in a cycle. In particular, our result holds for finite graphs with disjoint cycles (the Toeplitz graph is such, for example). We formulate and show the main result also for graph C^*-algebras. As a consequence, the graded version of the Isomorphism Conjecture holds for the class of graphs we consider.

Besides showing the conjecture for the class of graphs we consider, we realize the Grothendieck $\mathbb{Z}$-group isomorphism by a specific graded $*$-isomorphism. In particular, we introduce a series of graph operations which preserve the graded $*$-isomorphism class of their algebras. After performing these operations on a graph, we obtain well-behaved "representative" graphs, which we call canonical forms. We consider an equivalence relation $\approx$ on graphs such that $E \approx F$ holds when there is a graph isomorphism between some of the canonical forms of E and F. In the main result, we show that the relation $E \approx F$ is equivalent to the existence of an isomorphism f of the Grothendieck $\mathbb{Z}$-groups of the algebras of E and F in the appropriate category. As $E \approx F$ can be realized by a finite series of specific graph operations, any such isomorphism f can be realized by *an explicit graded $*$-algebra isomorphism*. Because of this, our main result describes the graded ($*$-)isomorphism classes of the algebras of graphs we consider. Besides its possible relevance in symbolic dynamics, such a description is relevant for the active program of classification of graph C^*-algebras.

1. Introduction

If E is a directed graph and K a field, the Leavitt path algebra $L_K(E)$ and its operator theory counterpart, the graph C^*-algebra $C^*(E)$, are naturally graded by the group of integers $\mathbb{Z}$. It is often advantageous to consider the algebras with this grading. For example, it is easy to come up with examples of graphs with isomorphic pointed K_0-groups of their Leavitt path algebras while the algebras themselves are not isomorphic. No such examples are known if the algebras are considered with the grading and their K_0-groups are adjusted accordingly. The Graded Classification Conjecture, formulated for finite graphs by Roozbeh Hazrat in [10], states that such examples do not exist.

1.1. **The conjecture and its state.** In the unital case (when E has finitely many vertices), the $\mathbb{Z}$-grading of $L_K(E)$ induces an action of the infinite cyclic group $\Gamma = \langle t \rangle \cong \mathbb{Z}$ on the set of the graded isomorphism classes of finitely generated graded projective $L_K(E)$-modules. This action makes the Grothendieck group, formed using the finitely generated *graded* projective modules and

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their *graded* isomorphism classes, into a pre-ordered Γ-group. Although the notation $K_0^{\mathrm{gr}}(L_K(E))$ has often been used for this group, we use $K_0^\Gamma(L_K(E))$ in order to emphasize that the Grothendieck group itself is not graded by Γ but that Γ acts on it. If $L_K(E)$ is unital, the graded isomorphism class $[L_K(E)]$ is an order-unit of the group $K_0^\Gamma(L_K(E))$. If $L_K(E)$ is not unital, $K_0^\Gamma(L_K(E))$ can be defined via a unitization of $L_K(E)$ and a certain generating interval can be considered instead of $[L_K(E)]$ (see section 2.7 or [13] for more details). A widely accepted formulation (e.g. in [1] and in [3]) of The Graded Classification Conjecture (GCC) states the equivalence of the two conditions below for finite graphs E and F and a field K.

(1^u) There is an order-preserving Γ-group isomorphism of $K_0^\Gamma(L_K(E))$ and $K_0^\Gamma(L_K(F))$ which maps the order-unit $[L_K(E)]$ to the order-unit $[L_K(F)]$.
(2^u) The algebras $L_K(E)$ and $L_K(F)$ are graded isomorphic.

By allowing infinite emitters to be present and by considering the generating intervals instead of the order-units, one can also formulate the GCC for the non-row-finite and the non-unital cases and thus extend its scope to all graphs. This generalized version, which we also refer to as the GCC, is stating the equivalence of conditions (1) and (2) below. Any involution on K equips $L_K(E)$ with an involution and we also consider the equivalence of condition (3) with (1) and (2).

(1) There is an order-preserving Γ-group isomorphism of $K_0^\Gamma(L_K(E))$ and $K_0^\Gamma(L_K(F))$ which maps the generating interval $D_{L_K(E)}$ onto the generating interval $D_{L_K(F)}$.
(2) The algebras $L_K(E)$ and $L_K(F)$ are graded isomorphic.
(3) The algebras $L_K(E)$ and $L_K(F)$ are graded $*$-isomorphic.

In [10], the GCC is shown to hold for polycephalic graphs. These are finite graphs in which every vertex connects to a sink, a cycle without exits, or to a vertex emitting no other edges but finitely many loops and the graph is such that when these loops as well as an edge of each cycle with no exits are removed, the resulting graph is a finite acyclic graph. In [3], a weaker version of the GCC is shown to hold for finite graphs without sources or sinks. In [13], the GCC, generalized to include the non-unital case, is shown to hold for countable graphs in which no cycle has an exit and in which every infinite path ends in a (finite or infinite) sink or in a cycle if the involution on the underlying field is reasonably well-behaved (as the complex-conjugation on $\mathbb{C}$ is). By [9], the GCC holds for countable graphs for which whenever there is an edge from a vertex v to a vertex w, there are infinitely many edges from v to w.

The strong version of this conjecture states that the functor K_0^Γ is full and faithful when considered on the category of Leavitt path algebras of finite graphs and their graded homomorphisms modulo conjugations by invertible elements of the zero components. Recently and almost simultaneously, in [5] and in [22], it was shown that the functor K_0^Γ is full, in [22] for countable graphs with finitely many vertices and, in [5] for finite graphs. In [3], it is shown that K_0^Γ is not faithful.

We note that [6] contains a survey of the present state of the GCC. In recent work [7], the term "Graded Classification Conjecture" is defined differently, as a statement that condition (1) *without the condition on the generating intervals* is equivalent to the condition that the Leavitt path algebras are graded Morita equivalent. In our present work, we use the term GCC to refer only to its *original* form.

1.2. **The composition S-NE graphs.** Composition series of graphs were introduced in [23]. Since this concept is pivotal for our approach to the proof, we review it in section 2, together with some material on porcupine-quotients, the graph Γ-monoid and some other results of [23]. We recall that each graded ideal of $L_K(E)$ uniquely corresponds to a pair (H, S) of two subsets of the set of vertices

E^0 of a graph E called an *admissible pair*. The admissible pairs can be ordered in such a way that the lattice of graded ideals is isomorphic to the lattice of admissible pairs. The *porcupine-quotient* graph $(G,T)/(H,S)$ of two such pairs $(H,S) \leq (G,T)$, also introduced in [23], has its Leavitt path algebra graded $*$-isomorphic to the quotient $I(G,T)/I(H,S)$ of the two corresponding ideals. This correspondence enables us to transfer the consideration of a graded composition series of $L_K(E)$ to the consideration of a composition series of E. In particular, E has a *composition series of length* n if there is a finite chain of admissible pairs

$$(\emptyset,\emptyset) = (H_0,S_0) \leq (H_1,S_1) \leq \ldots \leq (H_n,S_n) = (E^0,\emptyset)$$

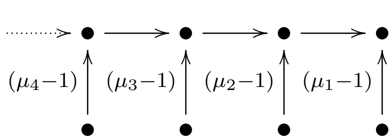
such that the porcupine-quotient graph $(H_{i+1},S_{i+1})/(H_i,S_i)$ is cofinal for all $i = 0,\ldots,n-1$ (cofinality of a graph corresponds to graded simplicity of its Leavitt path algebra, see section 2.2 for a review of this concept).

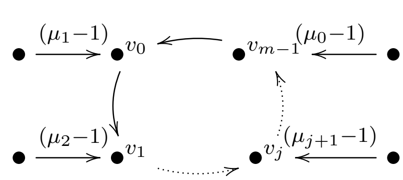
By [23, Theorem 5.7] (stated here as Theorem 2.3), a cofinal graph has exactly one of four types of certain *terminal* vertices. Only two of the four types are relevant in this paper: a sink or the set of vertices on a cycle without exits. If only those two types appear for every cofinal porcupine-quotient of E, we say that E is an *S-NE* graph. Here S is used for "sinks" and NE for "no-exits". If E is an S-NE graph and it has a composition series, we say that E is a *composition S-NE graph*. If the composition length of such E is n, E is an *n-S-NE graph* (Definition 3.1).

If E is a composition S-NE graph, each of its (finitely many) infinite emitters is a sink of exactly one composition factor. Each of the (disjoint and finite in number) cycles of E is a cycle without exits of exactly one composition factor. Our choice to work with composition S-NE graphs with possibly infinitely many vertices is not a vacuous exercise in generalization, but an absolute necessity: a porcupine graph of a graph with finitely many vertices can have infinitely many vertices as the example of the Toeplitz graph in section 1.4 illustrates.

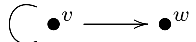
1.3. **1-S-NE graphs.** To prove our main result, Theorem 5.3, we start by considering 1-S-NE graphs (i.e., cofinal graphs with either a sink or a cycle without exits) in section 3. It is known that the GCC holds for such finite graphs ([10, Theorem 4.8]) and such general graphs under some assumption on K ([13, Theorem 5.5]). We show that GCC holds for all 1-S-NE graphs in Proposition 3.4. We introduce a canonical form of a 1-S-NE graph (unique up to a graph isomorphism) and show that the conditions from the GCC are equivalent to the canonical forms of the graphs being isomorphic (as graphs). In graphical representations of canonical forms of 1-S-NE graphs below, we abbreviate the graphs as follows: if v receives k edges originating at sources, no matter whether k is a finite or an infinite cardinal, we depict this as $\bullet \xrightarrow{(k)} \bullet^v$.

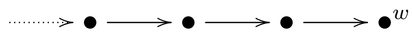
The first graph below is a canonical form of a 1-S-NE graph with a sink. Its algebra is graded isomorphic to $\mathbb{M}_\kappa(1,\mu_1,\mu_2,\ldots,\mu_k)$, where k can be finite or infinite cardinal and κ is the (cardinal) sum $1+\sum_{i=1}^k \mu_i$. The second graph below is a canonical form of a 1-S-NE graph with a cycle of length m. Its algebra is graded isomorphic to $\mathbb{M}_\kappa[x^m,x^{-m}](\mu_0,\mu_1,\ldots,\mu_{m-1})$, where $\kappa=\sum_{i=0}^{m-1}\mu_i$.

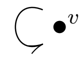




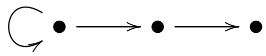
We refer to the horizontal line of length k in the first graph as the *spine* of the graph and to the edges not on the spine which end in the spine as the *tails*. In the second graph, we also refer to the edges ending in the cycle but not on the cycle as the *tails*.

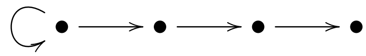
1.4. **2-S-NE graphs.** The work on the 2-S-NE graphs in section 4 contains crucial arguments for the general case. We introduce *canonical* graphs (Definition 4.10) of 2-S-NE graphs and we develop graph operations needed to transform a given 2-S-NE graph into its canonical form. For example, if E is the Toeplitz graph , then it is a 2-S-NE graph since its composition series is $(\emptyset, \emptyset) \leq (\{w\}, \emptyset) \leq (E^0, \emptyset)$. The composition factors are the porcupine graph $P_{\{w\}}$, listed first, and the quotient $E/\{w\}$ listed second below.





It turns out that the Toeplitz graph E is its own canonical form and that it is a canonical form of any of the graphs below.





$\ldots$

The order-unit is not available to us for graphs with infinitely many vertices. So, for such graphs, we use cardinality arguments instead of considering the order units – *we count the tails.* The next example illustrates how this is achieved in a simple case.

Let E_1, E_2, E_3, and E_4 be the four graphs below.

By Definition 4.10, E_1 and E_2 are canonical and the two graphs below, E_3' and E_4', are canonical forms of E_3 and E_4.

If the types of exits from cycles differ (like, for example, they do for E_1 and E_2), we show that there is no isomorphism (in the appropriate category) of the Γ-groups of the graphs. If the types of exits are the same but the number of tails is different (like, for example, it is the case for E_1 and E_3'), we also we eliminate the possibility that the Γ-groups are isomorphic. Thus, the four graphs have Γ-groups in different isomorphism classes and, consequently, their algebras are in different graded $*$-isomorphism classes.

A canonical form is obtained by a series of graph operations we introduce. Some of these operations are out-splits and out-amalgamations and, as such, are the "graph moves" of symbolic dynamics. However, some of the operations we consider are not any of the previously considered graph moves. Each of the operations $\phi : E \to F$ we consider is defined on the set of vertices and edges of E and its values are elements of $L_K(F)$ which satisfy the axioms (V) and (E1) of the Leavitt path algebra axioms (we review them in section 2.3). The map ϕ is such that $\phi(v) \neq 0$ for $v \in E^0$, such that $\phi(v)$ is a homogeneous element of degree zero, and such that $\phi(e)$ is a homogeneous element of degree one for $e \in E^1$. The map ϕ is also such that if the values on the ghost edges are defined by $\phi(e^*) = \phi(e)^*$ for $e \in E^1$, then ϕ is such that (E2), (CK1), and (CK2) hold. By the Universal Property of Leavitt path algebras, ϕ extends to a homomorphism which we typically continue to call ϕ. The property $\phi(e^*) = \phi(e)^*$ ensures that the extension is a $*$-homomorphism. The extension is a graded homomorphism by the requirement on the degrees of $\phi(v)$ and $\phi(e)$ for $v \in E^0$ and $e \in E^1$ and it is injective by the Graded Uniqueness Theorem. For every such map ϕ we consider, we check that there is an operation $\psi : F \to E$ having analogous properties as ϕ and such that ϕ and ψ compose to the identity maps on the sets of vertices and edges. This requirement implies that the extension of ϕ is a *graded $*$-algebra isomorphism.* In particular, the process of obtaining a canonical form of a graph yields a specific graded $*$-isomorphism of algebras of E and its canonical form.

We also introduce an equivalence relation $\approx$ on n-S-NE graphs so that $E \approx F$ holds if there are canonical forms of E and F which are isomorphic. In the main result on countable 2-S-NE graphs, Theorem 4.13, we show that the relation

(4) $E \approx F$.

holds exactly when conditions (1) to (3) hold. In addition, if $f : K_0^\Gamma(E) \to K_0^\Gamma(F)$ is an isomorphism from condition (1), we *realize* f by a specific graded $*$-algebra isomorphism obtained from a graph operation on two isomorphic canonical forms of E and F. By "realize f", we mean that we exhibit a graph operation which extends to a graded $*$-isomorphism ϕ of the two algebras such that $K_0^\Gamma(\phi) = f$. Since every canonical form is obtained by a finite series of specific graph operations, any isomorphism of two Γ-groups of any two graphs, not necessarily canonical, is realizable. We generalize this result to countable n-S-NE graphs. Thus, canonical forms can be used to explicitly describe the graded ($*$-)isomorphism class of Leavitt path and graph C^*-algebras.

1.5. **The main result and its corollaries.** In section 5.2, we introduce the graph operations on n-S-NE graphs and define canonical forms of such graphs. We show that suitably selected out-splits transform an n-S-NE graph to a graph without breaking vertices and with hereditary and saturated sets $H_j, j = 1, \ldots, n$ such that $\emptyset \subseteq H_1 \subseteq H_2 \subseteq \ldots \subseteq H_n$ is a composition series of E.

We can use induction and assume that a canonical form of the quotient E/H_1 is defined and then consider only the H_j-to-H_1 paths. In addition, by the $n = 2$ case, we can transform the H_2-to-H_1 part to its canonical form without impacting the quotient E/H_1, only possibly impacting the H_j-to-H_1 paths for $j > 2$. Then, we move on to consider the H_3-to-H_1 paths and the 3-S-NE graph P_{H_3} and continue the process of making H_j-to-H_1 part canonical for $j = 2, 3, \ldots, n$. Eventually, we arrive to a canonical form of E. The main result of the paper, Theorem 5.3, states that the conditions (1) to (4) are equivalent for countable composition S-NE graphs.

A direct corollary of Theorem 5.3 is that the GCC holds for countable graphs with finitely many vertices and disjoint cycles (Corollary 5.4). In Corollary 5.5, we establish the graph C^*-algebra version of our main result: there is a gauge-invariant isomorphism of the graph C^*-algebras of two countable composition S-NE graphs if and only if the K_0^Γ-groups of the two C^*-algebras, considered with their generating intervals, are isomorphic. This implies that the graded version of the *Strong Isomorphism Conjecture* holds for the class of graphs we consider. The Isomorphism Conjecture, posed in [2], states that two Leavitt path algebras over $\mathbb{C}$ are isomorphic as algebras precisely when they are isomorphic as $*$-algebras, which is known as the Isomorphism Conjecture, and that any such algebras are isomorphic as rings precisely when they are isomorphic as $*$-algebras, which is known as the Strong Isomorphism Conjecture. By [2, Propositions 7.4 and 8.5], the latter conjecture holds for Leavitt path algebras of countable acyclic graphs as well as row-finite cofinal graphs with a cluster of extreme cycles. By [8, Theorem 14.7], the Strong Isomorphism Conjecture holds for all graphs with finitely many vertices. The graded versions of these conjectures have every instance of "isomorphism" replaced by "graded isomorphism". Corollary 5.6 states that the following conditions are equivalent with conditions (1) to (4) above.

(5) The algebras $L_K(E)$ and $L_K(F)$ are graded isomorphic as rings.
(6) The algebras $L_K(E)$ and $L_K(F)$ are graded isomorphic as $*$-rings.
(7) The algebras $C^*(E)$ and $C^*(F)$ are graded isomorphic.

The equivalence of conditions (2) to (4) indicates that Theorem 5.3 does more than prove the GCC for the class of graphs we consider – it describes the graded isomorphism class (and graded $*$-isomorphism class) of the algebra (Leavitt path as well as graph C^*). This is relevant for an ongoing initiative to classify all graph C^*-algebra (the introduction to [8] contains a comprehensive overview of such initiative and some important results towards obtaining a complete classification).

We briefly reflect on possible future directions. The methods of our proof indicate that the GCC should generally be considered along with a condition on graphs and that using the length of a composition series for induction is promising.

Every graph with finitely many vertices has a composition series where each composition factor has either a sink, a cycle without exits, or a cluster of extreme cycles. When considering composition graphs whose factors may have any of these three types of terminal elements, some of the methods of our proof may still be applicable. The remaining roadblock to proving the GCC for all graphs with finitely many vertices seems to be proving it for cofinal graphs with extreme cycles. If this turns out to be possible, some of our arguments on the length of a composition series could be applicable to establishing the equivalence of conditions (1) to (7) for graphs with finitely many vertices.

2. Prerequisites

In the rest of the paper, we use $\mathbb{Z}^+$ to denote the set of nonnegative integers and ω for the same set but when considered as the countably infinite cardinal (the smallest infinite ordinal). When considering $n \in \omega$ as a finite cardinal, it is equal to the set containing all of the finite cardinals smaller than it, so $n = \{0, 1, \ldots, n-1\}$. Writing $i \in n$ means that i is an element of the set $\{0, 1, \ldots, n-1\}$. When working with cardinals, we assume the usual cardinal arithmetic laws. We also let $+_n$ denote the addition modulo n (i.e. the addition in $\mathbb{Z}/n\mathbb{Z}$).

We let $\Gamma = \langle t \rangle$ be the infinite cyclic group generated by t and K be a field trivially graded by $\mathbb{Z}$.

While the sets of vertices and edges of graphs in section 3 have arbitrary cardinalities, all graphs in subsequent sections have *countably* many vertices and edges.

2.1. **Graded rings.** A ring R (not necessarily unital) is *graded* by a group Γ if $R = \bigoplus_{\gamma\in\Gamma} R_\gamma$ for additive subgroups R_γ and if $R_\gamma R_\delta \subseteq R_{\gamma\delta}$ for all $\gamma, \delta \in \Gamma$. The elements of the set $\bigcup_{\gamma\in\Gamma} R_\gamma$ are said to be *homogeneous*. A left ideal I of a graded ring R is *graded* if $I = \bigoplus_{\gamma\in\Gamma} I \cap R_\gamma$. Graded right ideals and graded ideals are defined similarly. A graded ring is *graded simple* if there are no nontrivial and proper two-sided graded ideals (note that we do not require it to be graded Artinian).

A ring R is an involutive ring, or a $*$-ring, if there is an anti-automorphism $* : R \to R$ of order two. If R is also a K-algebra for some commutative $*$-ring K, then R is a $*$-algebra if $(kx)^* = k^*x^*$ for all $k \in K$ and $x \in R$. If R is a Γ-graded ring with involution, it is a *graded $*$-ring* if $R^*_\gamma \subseteq R_{\gamma^{-1}}$.

We use the standard definitions of graded right and left R-modules, graded module homomorphisms and we use $\cong_{\text{gr}}$ for a graded module or a graded ring isomorphism. If M is a graded right R-module and $\gamma \in \Gamma$, the γ-*shifted* graded right R-module $(\gamma)M$ is defined as the module M with the Γ-grading given by $(\gamma)M_\delta = M_{\gamma\delta}$ for all $\delta \in \Gamma$. If N is a graded left module, the γ-shift of N is the graded module N with the Γ-grading given by $M(\gamma)_\delta = M_{\delta\gamma}$ for all $\delta \in \Gamma$.

Any finitely generated graded free right R-module has the form $(\gamma_1)R \oplus \ldots \oplus (\gamma_n)R$ and any finitely generated graded free left R-module has the form $R(\gamma_1) \oplus \ldots \oplus R(\gamma_n)$ for $\gamma_1, \ldots, \gamma_n \in \Gamma$ ([11, Section 1.2.4] contains more details). A finitely generated graded projective module is a direct summand of a finitely generated graded free module.

The presence of the shifts $\gamma_1, \ldots, \gamma_n$ in the above form of a finitely generated graded free module explains the presence of $\gamma_1, \ldots, \gamma_n \in \Gamma$ in the graded matrix ring over a graded ring R. In [11], $\mathbb{M}_n(R)(\gamma_1, \ldots, \gamma_n)$ denotes the ring of matrices $\mathbb{M}_n(R)$ with the Γ-grading given by

$$(r_{ij}) \in \mathbb{M}_n(R)(\gamma_1, \ldots, \gamma_n)_\delta \quad \text{if} \quad r_{ij} \in R_{\gamma_i^{-1}\delta\gamma_j} \text{ for } i, j = 1, \ldots, n.$$

The definition of $\mathbb{M}_n(R)(\gamma_1, \ldots, \gamma_n)$ in [15] is different: $\mathbb{M}_n(R)(\gamma_1, \ldots, \gamma_n)$ in [15] corresponds to $\mathbb{M}_n(R)(\gamma_1^{-1}, \ldots, \gamma_n^{-1})$ in [11]. More details on the relations between the two definitions can be found in [19, Section 1]. Although the definition from [15] has been in circulation longer, some matricial representations of Leavitt path algebras involve positive integers instead of negative integers making the definition from [11] more convenient for us. Since we deal almost extensively with Leavitt path algebras, we opt to use the definition from [11]. With this definition, if F is the graded free right module $(\gamma_1^{-1})R \oplus \cdots \oplus (\gamma_n^{-1})R$, then $\mathrm{Hom}_R(F, F) \cong_{\mathrm{gr}} \mathbb{M}_n(R)(\gamma_1, \ldots, \gamma_n)$ as graded rings.

We also recall [15, Remark 2.10.6] stating the first two parts in Lemma 2.1 and [11, Theorem 1.3.3] stating part (3) for Γ abelian. Although we use the lemma below only in the case $\Gamma \cong \mathbb{Z}$, we note that it generalizes to arbitrary Γ. The part on the isomorphism being $*$-isomorphisms follows by [13, Proposition 1.3].

Lemma 2.1. [15, Remark 2.10.6], [11, Proposition 1.4.4, Theorems 1.3.3 and 1.4.5], [13, Proposition 1.3]. *Let R be a Γ-graded $*$-ring, $\gamma_1, \ldots, \gamma_n \in \Gamma$, and let e_{ij} denote the standard matrix unit for $i, j \in \{1, \ldots, n\}$.*

(1) If π a permutation of the set $\{1, \ldots, n\}$, the K-linear extension of the map $e_{ij} \mapsto e_{\pi^{-1}(i)\pi^{-1}(j)}$ is a graded $*$-isomorphism
$$\mathbb{M}_n(R)(\gamma_1, \gamma_2, \ldots, \gamma_n) \cong_{\mathrm{gr}} \mathbb{M}_n(R)(\gamma_{\pi(1)}, \gamma_{\pi(2)} \ldots, \gamma_{\pi(n)}).$$
(2) If δ in the center of Γ, then e_{ij}, which is in the $\gamma_i\gamma_j^{-1}$-component, can be considered as an element of the $\gamma_i\delta(\gamma_j\delta)^{-1}$-component, so the identity becomes a graded $*$-isomorphism $\mathbb{M}_n(R)(\gamma_1, \gamma_2, \ldots, \gamma_n) = \mathbb{M}_n(R)(\gamma_1\delta, \gamma_2\delta, \ldots, \gamma_n\delta)$.
(3) If $\delta_i \in \Gamma$ is such that there is an invertible element a_{δ_i} in R_{δ_i} for $i = 1, \ldots, n$, then the K-linear extension ϕ of the map $e_{ij} \mapsto a_{\delta_i}^{-1} a_{\delta_j} e_{ij}$ is a graded isomorphism
$$\mathbb{M}_n(R)(\gamma_1, \gamma_2, \ldots, \gamma_n) \cong_{\mathrm{gr}} \mathbb{M}_n(R)(\gamma_1\delta_1, \gamma_2\delta_2 \ldots, \gamma_n\delta_n).$$
If the elements a_{δ_i} can be found so that $a_{\delta_i}^{-1} = a_{\delta_i}^*$, then ϕ is a graded $*$-isomorphism.

If Γ is abelian and R and S are Γ-graded division rings, then
$$\mathbb{M}_n(R)(\gamma_1, \gamma_2, \ldots, \gamma_n) \cong_{\mathrm{gr}} \mathbb{M}_m(S)(\delta_1, \delta_2, \ldots, \delta_m)$$
implies that $R \cong_{\mathrm{gr}} S$, that $m = n$, and that the list $(\delta_1, \delta_2, \ldots, \delta_m)$ is obtained from the list $(\gamma_1, \gamma_2, \ldots, \gamma_n)$ by applying finitely many operations of the lists as in parts (1) to (3).

We are going to be interested exclusively in the case when $\Gamma = \mathbb{Z}$ and the matrices are formed over either a field K graded trivially by $\mathbb{Z}$ or the ring $K[x^m, x^{-m}]$ of Laurent polynomials $\mathbb{Z}$-graded by $K[x^m, x^{-m}]_{mk} = Kx^{mk}$ and $K[x^m, x^{-m}]_n = 0$ if m does not divide n. Note that $K[x^m, x^{-m}]$, graded as above, is a graded field.

It turns out that dealing with graphs having infinitely many paths ending in a sink or a cycle requires us to generalize previously noted results to matrices of arbitrary size. Let Γ be any group, K be a Γ-graded division ring, and let κ be an infinite cardinal. We let $\mathbb{M}_\kappa(K)$ denote the ring of infinite matrices over K, having rows and columns indexed by κ, with only finitely many nonzero entries. If $\overline{\gamma}$ is any function $\kappa \to \Gamma$, we let $\mathbb{M}_\kappa(K)(\overline{\gamma})$ denote the Γ-graded ring $\mathbb{M}_\kappa(K)$ with the δ-component consisting of the matrices $(a_{\alpha\beta})$, $\alpha, \beta \in \kappa$, such that $a_{\alpha\beta} \in K_{\overline{\gamma}(\alpha)^{-1}\delta\overline{\gamma}(\beta)}$. If K is a graded $*$-ring, $\mathbb{M}_\kappa(K)(\overline{\gamma})$ is a graded $*$-ring with the $*$-transpose involution.

The first part of Lemma 2.1 has been generalized in [13, Proposition 4.12] as the lemma below.

Lemma 2.2. [13, Proposition 4.12] *Let Γ be an abelian group, R be a Γ-graded $*$-ring, κ a cardinal, $\overline{\gamma} \in \Gamma^\kappa$, and $e_{\alpha\beta}$ denotes the standard matrix units for $\alpha, \beta \in \kappa$.*

(1) *If π is a bijection $\kappa \to \kappa$ and $\overline{\gamma}\pi$ denotes the composition of $\overline{\gamma}$ and π, then the R-linear extension of the map $e_{\alpha\beta} \mapsto e_{\pi^{-1}(\alpha)\pi^{-1}(\beta)}$ is a graded $*$-isomorphism $\mathbb{M}_\kappa(R)(\overline{\gamma}) \cong_{\text{gr}} M_\kappa(R)(\overline{\gamma}\pi)$.*
(2) *If δ is in the center of Γ and $\overline{\gamma}\delta$ denotes the map $(\overline{\gamma}\delta)(\alpha) = \overline{\gamma}(\alpha)\delta$, then $e_{\alpha\beta}$, which is in the $\overline{\gamma}(\alpha)\overline{\gamma}(\beta)^{-1}$-component, can be considered as an element of the $\overline{\gamma}(\alpha)\delta(\overline{\gamma}(\beta)\delta)^{-1}$-component, so the identity becomes a graded isomorphism $\mathbb{M}_\kappa(R)(\overline{\gamma}) \cong_{\text{gr}} M_\kappa(R)(\overline{\gamma}\delta)$.*
(3) *If $\overline{\delta} : \kappa \to \Gamma$ is such that the component $R_{\overline{\delta}(\alpha)}$ contains an invertible element a_α for every $\alpha \in \kappa$, then the R-linear extension of the map $e_{\alpha\beta} \mapsto a_\alpha^{-1} a_\beta e_{\alpha\beta}$ is a graded isomorphism ϕ : $\mathbb{M}_\kappa(R)(\overline{\gamma}) \cong_{\text{gr}} \mathbb{M}_\kappa(R)(\overline{\gamma}\overline{\delta})$. If the elements a_α are unitary (i.e. $a_\alpha^{-1} = a_\alpha^*$), then ϕ is a graded $*$-isomorphism.*

2.2. **Graphs and properties of vertex sets.** If E is a directed graph, we let E^0 denote the set of vertices, E^1 denote the set of edges, and $\mathbf{s}$ and $\mathbf{r}$ denote the source and the range maps of E. If κ is any cardinal, we use $\bullet^v \xrightarrow{\kappa} \bullet^w$ to depict that v emits κ edges to w. If $V \subseteq E^0$, a subgraph *generated by* V is the graph with V as its vertex set and with its edge set consisting of the edges of E which have both their source and their range in V. The graphs E and F are *isomorphic*, written as $E \cong F$, if there is a bijection f of their vertices such that there is an edge from v to w if and only if there is an edge from $f(v)$ to $f(w)$ for all $v, w \in E^0$.

A *sink* of E is a vertex which emits no edges and an *infinite emitter* is a vertex which emits infinitely many edges. A vertex of E is *regular* if it is neither a sink nor an infinite emitter. A graph E is *row-finite* if it has no infinite emitters and E is *finite* if it has finitely many vertices and edges.

A *path* is a single vertex or a sequence of edges $e_1 e_2 \ldots e_n$ for some positive integer n such that $\mathbf{r}(e_i) = \mathbf{s}(e_{i+1})$ for $i = 1, \ldots, n-1$. The length $|p|$ of a path p is zero if p is a vertex and it is n if p is a sequence of n edges. The set of vertices on a path p is denoted by p^0.

The functions $\mathbf{s}$ and $\mathbf{r}$ extend to paths naturally. A path p is *closed* if $\mathbf{s}(p) = \mathbf{r}(p)$. A *cycle* is a closed path such that different edges in the path have different sources. A cycle has *an exit* if a vertex on the cycle emits an edge which is not an edge of the cycle. A cycle c is *extreme* if c has exits and for every path p with $\mathbf{s}(p) \in c^0$, there is a path q such that $\mathbf{r}(p) = \mathbf{s}(q)$ and $\mathbf{r}(q) \in c^0$. Two cycles c and d of any graph are *disjoint* if $c^0 \cap d^0 = \emptyset$.

If E is $\bullet \underset{f}{\overset{e}{\rightleftarrows}} \bullet$, it is not uncommon to hear a reference to the "single cycle" of E. However, E has two cycles: ef and fe. In our work, if $c_0 = e_0 \ldots e_{n-1}$ is a cycle of length n, we use $[c_0]$ for the set of n elements $c_i = e_i e_{i+_n 1} \ldots e_{i-_n 1}$ for $i = 0, \ldots, n-1$. The relation $c \sim d$ if $[c] = [d]$ is an equivalence relation of the set of all cycles of a graph. Thus, while a cycle in the above graph with two vertices is not unique, there is a unique equivalence class of the relation $\sim$ on this graph.

If $u, v \in E^0$ are such that there is a path p with $\mathbf{s}(p) = u$ and $\mathbf{r}(p) = v$, we write $u \geq v$. For $V \subseteq E^0$, the set $T(V) = \{u \in E^0 \mid v \geq u$ for some $v \in V\}$ is called the *tree* of V, and, following [21], we use $R(V)$ to denote the set $\{u \in E^0 \mid u \geq v$ for some $v \in V\}$ called the *root* of V. If $V = \{v\}$, we use $T(v)$ for $T(\{v\})$ and $R(v)$ for $R(\{v\})$.

An *infinite path* is a sequence of edges $e_1e_2\ldots$ such that $\mathbf{r}(e_n) = \mathbf{s}(e_{n+1})$ for $n = 1, 2\ldots$. Just as for finite paths, we use p^0 for the set of vertices of an infinite path p. In [23, Definition 5.3], an infinite path p of a graph E is said to be *terminal* if no element of $T(p^0)$ is an infinite emitter or on a cycle and if every infinite path q originating at a vertex of p is such that $T(q^0) \subseteq R(q^0)$. Note that the property $T(p^0) \subseteq R(p^0)$ is shared for any path (finite or infinite) which contains a vertex which is on an extreme cycle, or on a cycle without exits or which is a sink and the relevance of these types of vertices is evident from [23, Theorem 5.7] which we review in Theorem 2.3. A vertex is *terminal* if it is a sink, on a cycle without exits, on an extreme cycle or on a terminal path.

Let $E^{\leq\infty}$ be the set of infinite paths or finite paths ending in a sink or an infinite emitter. A vertex v is *cofinal* if for each $p \in E^{\leq\infty}$ there is $w \in p^0$ such that $v \geq w$ and E is *cofinal* if each vertex is cofinal. This property matches the existence of a unique equivalence class of terminal vertices such that $w \approx v$ if and only if v and w are on the elements of $E^{\leq\infty}$ which have the same root (see [23] for more details). In [23], this equivalence class of terminal vertices is called a *terminal cluster*, or a *cluster* for short. For example, the cluster of a sink v is $\{v\}$, the cluster of a cycle c without exists is c^0, and the cluster of a vertex on an extreme cycle c is $T(c^0)$.

A subset H of E^0 is said to be *hereditary* if $T(H) \subseteq H$. The set H is *saturated* if $v \in H$ for any regular vertex v such that $\mathbf{r}(\mathbf{s}^{-1}(v)) \subseteq H$. For every $V \subseteq E^0$, the intersection of all saturated sets of vertices which contain V is the smallest saturated set which contains V. This set is the *saturated closure of* V. The saturated closure $\overline{V}$ of $T(V)$ is both hereditary and saturated and it is the smallest hereditary and saturated set which contains V.

2.3. **Leavitt path algebras.** If K is a field, the *Leavitt path algebra* $L_K(E)$ of E over K is a free K-algebra generated by the set $E^0 \cup E^1 \cup \{e^* \mid e \in E^1\}$ such that

(V) $vw = 0$ if $v \neq w$ and $vv = v$,
(E1) $\mathbf{s}(e)e = e\mathbf{r}(e) = e$,
(E2) $\mathbf{r}(e)e^* = e^*\mathbf{s}(e) = e^*$,
(CK1) $e^*f = 0$ if $e \neq f$ and $e^*e = \mathbf{r}(e)$,
(CK2) $v = \sum_{e\in\mathbf{s}^{-1}(v)} ee^*$ for each regular vertex v

hold for for $v, w \in E^0$ and $e, f \in E^1$. Let $v^* = v$ for $v \in E^0$ and $p^* = e_n^* \ldots e_1^*$ for a path $p = e_1 \ldots e_n$. The set of elements pq^* where p and q are paths of E with $\mathbf{r}(p) = \mathbf{r}(q)$ generates $L_K(E)$ as a K-algebra and $(\sum_{i=1}^n k_ip_iq_i^*)^* = \sum_{i=1}^n k_i^*q_ip_i^*$, where $k_i \mapsto k_i^*$ is any involution on K, is an involution of $L_K(E)$. In addition, $L_K(E)$ is graded locally unital (with the finite sums of vertices as the local units, the elements u such that for every finite set of elements F, $xu = ux = x$ for every $x \in F$). The algebra $L_K(E)$ is unital if and only if E^0 is finite in which case $\sum_{v\in E^0} v$ is the identity.

If we consider K to be trivially graded by $\mathbb{Z}$, $L_K(E)$ is naturally graded by $\mathbb{Z}$ so that the n-component $L_K(E)_n$ is the K-linear span of the elements pq^* for paths p, q with $|p| - |q| = n$. This grading and the involutive structure make $L_K(E)$ into a graded $*$-algebra.

If R is a K-algebra which contains elements p_v for $v \in E^0$, and x_e and y_e for $e \in E^1$ such that the five axioms hold for these elements, the Universal Property of $L_K(E)$ states that there is a unique algebra homomorphism $\phi : L_K(E) \to R$ such that $\phi(v) = p_v, \phi(e) = x_e$, and $\phi(e^*) = y_e$ (see [1, Remark 1.2.5]). If R is $\mathbb{Z}$-graded and $p_v \in R_0$ for $v \in E^0$, $x_e \in R_1$ and $y_e \in R_{-1}$ for $e \in E^1$, then ϕ is graded. By the Graded Uniqueness Theorem ([1, Theorem 2.2.15]), such graded map ϕ is injective if $p_v \neq 0$ for $v \in E^0$. If R is involutive and ϕ is such that $y_e = x_e^*$, then ϕ is a $*$-homomorphism (i.e., $\phi(x^*) = \phi(x)^*$ for every $x \in L_K(E)$).

We recall [23, Theorem 5.7] characterizing graded simplicity of $L_K(E)$ in terms of the existence of exactly one type of four types of terminal vertices.

Theorem 2.3. [23, Theorem 5.7] *Let E be a graph and K be a field. The following conditions are equivalent.*

(1) *$L_K(E)$ is graded simple (equivalently, E is cofinal).*
(2) *The set of terminal vertices is nonempty and it consists of a single cluster C such that E^0 is the (hereditary and) saturated closure of C.*
(3) *Exactly one of the following holds.*
 (a) *The set E^0 is the (hereditary and) saturated closure of a sink. In this case, E is row-finite and acyclic and $E^0 = R(v)$ for a sink v.*
 (b) *The set E^0 is the (hereditary and) saturated closure of c^0 for a cycle c without exits. In this case, E is row-finite, $E^0 = R(c^0)$, and c is the only cycle in E.*
 (c) *The set E^0 is the hereditary and saturated closure of c^0 for an extreme cycle c. In this case, every cycle of E is extreme, every infinite emitter is on a cycle, and $E^0 = R(c^0)$.*
 (d) *The set E^0 is the hereditary and saturated closure of α^0 for a terminal path α. In this case, E is acyclic and row-finite and $E^0 = R(\alpha^0)$.*

2.4. **Porcupine-quotient graphs.** If H is hereditary and saturated, a *breaking vertex* of H is an element of the set

$$B_H = \{v \in E^0 - H \,|\, v \text{ is an infinite emitter and } 0 < |\mathbf{s}^{-1}(v) \cap \mathbf{r}^{-1}(E^0 - H)| < \omega\}.$$

For each $v \in B_H$, let v^H stands for $v - \sum ee^*$ where the sum is taken over $e \in \mathbf{s}^{-1}(v) \cap \mathbf{r}^{-1}(E^0 - H)$.

An *admissible pair* is a pair (H, S) where $H \subseteq E^0$ is hereditary and saturated and $S \subseteq B_H$. For an admissible pair (H, S), the ideal $I(H, S)$ generated by $H \cup \{v^H \,|\, v \in S\}$ is graded since it is generated by homogeneous elements. It is the K-linear span of the elements pq^* for paths p, q with $\mathbf{r}(p) = \mathbf{r}(q) \in H$ and the elements $pv^H q^*$ for paths p, q with $\mathbf{r}(p) = \mathbf{r}(q) = v \in S$ (see [17, Lemma 5.6]). Conversely, for a graded ideal I, $H = I \cap E^0$ is hereditary and saturated and for $S = \{v \in B_H \mid v^H \in I\}$, $I = I(H, S)$ ([17, Theorem 5.7], also [1, Theorem 2.5.8]). If $S = \emptyset$, we shorten $(H, \emptyset)$ to H and $I(H, \emptyset)$ to $I(H)$.

The set of admissible pairs is a lattice with respect to the relation

$$(H, S) \leq (G, T) \quad \text{if } H \subseteq K \text{ and } S \subseteq G \cup T$$

(see [1, Proposition 2.5.6] for the meet and the join of this lattice). The correspondence $(H, S) \mapsto I(H, S)$ is a lattice isomorphism of this lattice and the lattice of graded ideals.

An admissible pair (H, S) gives rise to the *quotient graph* $E/(H, S)$ and to the *porcupine graph* $P_{(H,S)}$ so that the algebras $L_K(E)/I(H, S)$ and $L_K(E/(H, S))$ are graded isomorphic and that the algebras $L_K(P_{(H,S)})$ and $I(H, S)$ are also graded isomorphic (by [17, Theorem 5.7] and [20, Theorem 3.3]). Recently, the two constructions have been generalized by a single construction and the *porcupine-quotient graph* corresponding to the quotient of one admissible pair with respect to another admissible pair was introduced in [23]. Below are the relevant definitions.

If $H \subseteq G$ are two sets of vertices of E, let

$$B_H^G = \{v \in E^0 - H \mid v \text{ is an infinite emitter and } 0 < |\mathbf{s}^{-1}(v) \cap \mathbf{r}^{-1}(G - H)| < \omega\}.$$

If (H,S) and (G,T) are two admissible pairs of a graph E such that $(H,S)\leq(G,T)$, we let

$F_1(G-H,T-S)=\{e_1e_2\dots e_n$ is a path of $E\mid \mathbf{r}(e_n)\in G-H,\mathbf{s}(e_n)\notin(G-H)\cup(T-S)\}$ and
$F_2(G-H,T-S)=\{p$ is a path of $E\mid \mathbf{r}(p)\in T-S,|p|>0\}$.

The *porcupine-quotient graph* $(G,T)/(H,S)$ is defined by letting the set of its vertices be

$$(G-H)\cup(T-S)\cup\{w^p\mid p\in F_1(G-H,T-S)\cup F_2(G-H,T-S)\}\cup\{v'\mid v\in((G\cup T)-S)\cap B_H^G\}$$

and the set of its edges be

$$\{e\in E^1\mid \mathbf{r}(e)\in G-H \text{ and either } \mathbf{s}(e)\in G-H \text{ or } \mathbf{s}(e)\in T-S\}\cup$$

$$\{f^p\mid p\in F_1(G-H,T-S)\cup F_2(G-H,T-S)\}\cup\{e'\mid \mathbf{r}(e)\in((G\cup T)-S)\cap B_H^G\}.$$

The source and range of an edge of $(G,T)/(H,S)$ which is also in E^1 are the same as in E.

If $e\in E^1\cap(F_1(G-H,T-S)\cup F_2(G-H,T-S))$, we let $\mathbf{s}(f^e)=w^e$ and $\mathbf{r}(f^e)=\mathbf{r}(e)$. If $p=eq$ where $e\in E^1$, $q\in F_1(G-H,T-S)\cup F_2(G-H,T-S)$, and $|q|>0$, let $\mathbf{s}(f^p)=w^p$ and $\mathbf{r}(f^p)=w^q$.

If $\mathbf{r}(e)\in((G\cup T)-S)\cap B_H^G$, we let $\mathbf{r}(e')=\mathbf{r}(e)'$. If $\mathbf{r}(e)\in(G-S)\cap B_H^G$ and if $\mathbf{s}(e)\in(G-H)\cup(T-S)$, we let $\mathbf{s}(e')=\mathbf{s}(e)$. If $\mathbf{s}(e)\notin(G-H)\cup(T-S)$, then either $\mathbf{s}(e)\notin G\cup T$ or $\mathbf{s}(e)\in S\cap T$. In either case, $e\in F_1(G-H,T-S)$ and we let $\mathbf{s}(e')=w^e$. If $\mathbf{r}(e)\in(T-S)\cap B_H^G$, then $e\in F_2(G-H,T-S)$ and we let $\mathbf{s}(e')=w^e$.

If $S=T=\emptyset$, we write $(G,\emptyset)/(H,\emptyset)$ shorter as G/H.

If $(G,T)=(E^0,\emptyset)$, the porcupine-quotient graph is exactly the quotient graph $E/(H,S)$. If $(H,S)=(\emptyset,\emptyset)$, the porcupine-quotient graph is exactly the porcupine graph $P_{(G,T)}$.

By [23, Theorem 3.6], the Leavitt path algebra of a porcupine-quotient $(G,T)/(H,S)$ is graded isomorphic to the quotient $I(G,T)/I(H,S)$ of two corresponding graded ideals.

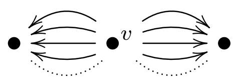
2.5. **The maximal and the total out-splits.** If E is a graph and v a vertex which emits edges, let $\mathcal{E}_1,\dots\mathcal{E}_n$ be a partition $\mathcal{P}$ of $\mathbf{s}^{-1}(v)$. We review the definition of the *out-split graph* $E_{v,\mathcal{P}}$ ([1, Definition 6.3.23]). If $\mathbf{s}^{-1}(v)=\mathcal{E}_1\cup\dots\cup\mathcal{E}_n$, the sets of vertices and edges of $E_{v,\mathcal{P}}$ are

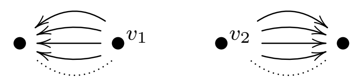
$$E^0_v=E^0-\{v\}\cup\{v_1,\dots,v_n\},\quad E^1_v=\{f_1,\dots,f_n\mid f\in E^1,\mathbf{r}(f)=v\}\cup\{f\in E^1\mid \mathbf{r}(f)\neq v\},$$

the range of f_i in $E_{v,\mathcal{P}}$ is v_i and the range of f in $E_{v,\mathcal{P}}$ is $\mathbf{r}(f)$, the source of $g\in E^1_{v,\mathcal{P}}$ is

- v_i if $g=f_j\in\mathcal{E}_i$ (so $\mathbf{s}(f)=\mathbf{r}(f)=v$)
- v_i if $g=f\in\mathcal{E}_i$ (so $\mathbf{s}(f)=v$ and $\mathbf{r}(f)\neq v$)
- $\mathbf{s}(f)$ if $g=f_j$ and $\mathbf{s}(f)\neq v$ (so $\mathbf{r}(f)=v$)
- $\mathbf{s}(f)$ if $g=f$ and $\mathbf{s}(f)\neq v$ (so $\mathbf{r}(f)\neq v$).

The map on the vertices and edges of E given by $\phi(v)=\sum_{i=1}^n v_i$, $\phi(w)=w$ for $w\in E^0-\{v\}$, $\phi(f)=\sum_{i=1}^n f_i$ if $\mathbf{r}(f)=v$ and $\phi(f)=f$ otherwise, extends to a graded $*$-monomorphism $L_K(E)\to L_K(E_{v,\mathcal{P}})$. This map is a graded $*$-isomorphism if v is regular or if it is an infinite emitter with all but one set $\mathcal{E}_1,\dots\mathcal{E}_n$ finite. In this last case, and if $\mathcal{E}_n$ is the infinite set, the inverse of ϕ can be defined on the vertices and edges by $\psi(v_i)=\sum_{e\in\mathcal{E}_i}ee^*$ for $i=1,\dots,n-1$ $\psi(v_n)=v-\sum_{i=1}^{n-1}\sum_{e\in\mathcal{E}_i}ee^*$, $\psi(w)=w$ for $w\in E^0-\{v_1,\dots,v_n\}$, and $\psi(f)=f\psi(\mathbf{r}(f))$. If v is an infinite emitter and more than one partition set is infinite, ϕ may not be onto. For example, the first graph below has a composition series length 3 (see section 2.9). The second graph is an out-split of the first with respect to the two-element partition consisting of the sets of all edges with equal range. It has a composition series of length 4, so the algebras of the two graphs are not graded isomorphic.





All out-splits we consider are with respect to either a partition of the set of edges emitted from a regular vertex or the set of edges emitted from an infinite emitter such that only one partition set is infinite. The operation of taking the *out-amalgamation* of a graph with respect to some v and $\mathcal{P}$ is inverse to the operation of taking the out-split. So, the graph E is the out-amalgamation of $E_{v,\mathcal{P}}$.

We often deal with the case when v is regular and $\mathcal{E}_i$ is a single element set for $i = 1, \ldots, |\mathbf{s}^{-1}(v)|$. We call the out-split graph with respect to this partition the *maximal out-split with respect to* v. If $V \subseteq E^0$ is a set of regular vertices, the graph obtained by performing the maximal out-splits with respect to every $v \in V$ is said to be the *maximal out-split with respect to* V and the term *the maximal out-split* is used in the case when V is the set of all regular vertices. We are interested in the case when V is the set of all regular vertices not on any cycle. We say that the maximal out-split with respect to this set is the *total out-split* of E and denote it by E_{tot}. So, E_{tot} is a graph in which every regular vertex which is not in a cycle emits exactly one edge.

2.6. **Pre-order monoids and their order-ideals.** If G is an abelian group with a left action of a group Γ compatible with the group operation, then G is a Γ-*group*. If M is an abelian monoid with a left action of a group Γ compatible with the monoid operation, then M is a Γ-*monoid*. Let $\geq$ be a reflexive and transitive relation (a pre-order) on a Γ-monoid M (Γ-group G) such that $g_1 \geq g_2$ implies $g_1+h \geq g_2+h$ and $\gamma g_1 \geq \gamma g_2$ for all g_1, g_2, h in M (in G) and $\gamma \in \Gamma$. We say that such monoid M is a *pre-ordered* Γ-*monoid* and that such a group G is a *pre-ordered* Γ-*group*. Let **POM** denote the category whose objects are pre-ordered Γ-monoids and whose morphisms are order-preserving Γ-monoid homomorphisms and **POG** denote the category whose objects are pre-ordered Γ-groups and whose morphisms are order-preserving Γ-group homomorphisms ($\mathbb{Z}[\Gamma]$-homomorphisms).

If G is a pre-ordered Γ-group, $G^+ = \{x \in G \mid x \geq 0\}$ is a pre-ordered Γ-monoid. If a Γ-group homomorphism $f : G \to H$ is order-preserving, then $f(G^+) \subseteq H^+$. Conversely, if M is a cancellative pre-ordered Γ-monoid, then its Grothendieck group is a pre-ordered Γ-group such that $G^+ = M$ and if $f : M \to N$ is an order-preserving Γ-monoid homomorphism of cancellative pre-ordered Γ-monoids, then f induces an order-preserving homomorphism of the Grothendieck groups of M and N. The objects of **POM** we consider are cancellative, so the formulation of our main result in terms of the elements of **POM** is equivalent to the formulation in terms of the elements of **POG**.

If $\mathbb{Z}^+$ denotes set of nonnegative integers, the set of finite sums of the $\mathbb{Z}^+$-multiples of the elements of Γ is a pre-ordered Γ-monoid which we denote by $\mathbb{Z}^+[\Gamma]$. An element u of a pre-ordered Γ-monoid M is an *order-unit* if for any $x \in M$, there is a nonzero $a \in \mathbb{Z}^+[\Gamma]$ such that $x \leq au$. If M and N are pre-ordered Γ-monoids with order-units u and v respectively, we refer to the pairs (M, u) and (N, v) as the *pointed monoids*. An order-preserving $\mathbb{Z}[\Gamma]$-module homomorphism $f : M \to N$ is a *homomorphism of pointed monoids* if $f(u) = v$. Let $\mathbf{POM}^u$ denote the category whose objects are pointed monoids and whose morphisms are homomorphisms of pointed monoids.

An element u of a pre-ordered Γ-group G is an *order-unit* if $u \in G^+$ and for any $x \in G$, there is a nonzero $a \in \mathbb{Z}^+[\Gamma]$ such that $x \leq au$. Let $\mathbf{POG}^u$ denote the category whose objects are pairs (G, u) where G is an object of **POG** and u is an order-unit of G and whose morphisms are morphisms of **POG** which are order-unit-preserving. If G is an upwards directed pre-ordered Γ-group, then

u is an order-unit of G if and only if u is an order-unit of G^+. Since all the objects of **POG** we consider are going to be upwards directed, the formulation of our main result in terms of objects and morphisms of $\mathbf{POM}^u$ is equivalent to the formulation in terms of their counterparts in $\mathbf{POG}^u$.

A submonoid I of a pre-ordered monoid M is an *order-ideal* of M if $x+y \in I$ implies $x \in I$ and $y \in I$ (equivalently $x \geq y$ and $x \in I$ implies $y \in I$). A Γ-submonoid I of a pre-ordered Γ-monoid M which is an order-ideal is a Γ-*order-ideal*. If M is a pre-ordered Γ-monoid, a subset D of M is a *generating interval* if D is an order-ideal (thus upwards directed and convex in the sense used in [19]) such that $\mathbb{Z}^+[\Gamma]D = M$. Let $\mathbf{POM}^D$ denote the category whose objects are pairs (M, D) where M is an object of **POM** and D is a generating interval of M, and whose morphisms are morphisms of **POM** which map the generating interval into the generating interval. Thus, a **POM**-homomorphism $f : (M, D) \to (N, F)$ is a $\mathbf{POM}^D$ homomorphism if and only if $f(D) \subseteq F$. If f is an **POM**-isomorphism, then $F = f(f^{-1}(F)) \subseteq f(D) \subseteq F$, so $f(D) = F$.

If G is a pre-ordered Γ-group, a subset D of G^+ is a *generating interval* of G if D is a generating interval of G^+. Let $\mathbf{POG}^D$ denote the category of **POG** objects considered with their generating intervals and **POG**-morphisms mapping the generating interval into the generating interval. For the Γ-monoids and Γ-groups we are considering, the categories $\mathbf{POM}^D$ and $\mathbf{POG}^D$ are equivalent.

2.7. **The Γ-monoid and the Grothendieck Γ-group of a graded ring.** If R is a unital Γ-graded ring, let $\mathcal{V}^\Gamma(R)$ denote the monoid of the graded isomorphism classes $[P]$ of finitely generated graded projective right R-modules P with the addition given by $[P]+[Q] = [P \oplus Q]$ and the left Γ-action given by $(\gamma, [P]) \mapsto [(\gamma^{-1})P]$. The monoid $\mathcal{V}^\Gamma(R)$ can be represented using the classes of left modules in which case the Γ-action is $(\gamma, [P]) \mapsto [P(\gamma)]$. The two representations are equivalent (see [11, Section 1.2.3] or [19, Section 2.3]). Note that the action in [11] is given by $(\gamma, [P]) \mapsto [(\gamma)P]$, not as above. For abelian groups, this is equivalent, but for non-abelian groups, this map would not be an action, so we consider the action with $(\gamma, [P]) \mapsto [(\gamma^{-1})P]$. This causes some formulas to be different than in [11]. The monoid $\mathcal{V}^\Gamma(R)$ can also be defined via homogeneous matrices (see [11, section 3.2]). Although we focus on the case when Γ is $\mathbb{Z}$, we note that the definitions and results of [11, Section 3.2] carry to the case when Γ is not necessarily abelian by [19, Section 1.3].

The *Grothendieck* Γ-*group* $K_0^\Gamma(R)$ is the group completion of the Γ-monoid $\mathcal{V}^\Gamma(R)$ with the action of Γ inherited from $\mathcal{V}^\Gamma(R)$. If Γ is the trivial group, $K_0^\Gamma(R)$ is the usual K_0-group. The pointed monoid $(\mathcal{V}^\Gamma(R), [R])$ is an object of $\mathbf{POM}^u$ and the image of $\mathcal{V}^\Gamma(R)$ under the natural map $\mathcal{V}^\Gamma(R) \to K_0^\Gamma(R)$ makes $(K_0^\Gamma(R), [R])$ into an object of $\mathbf{POG}^u$. If ϕ is a graded ring homomorphism, then $\mathcal{V}^\Gamma(\phi)$ is a morphism of **POM** and $K_0^\Gamma(\phi)$ is a morphism of **POG**. If ϕ is unital (i.e. ϕ maps the identity onto identity), $\mathcal{V}^\Gamma(\phi)$ is a morphism of $\mathbf{POM}^u$ and $K_0^\Gamma(\phi)$ is a morphism of $\mathbf{POG}^u$.

If K is a Γ-graded division ring, recall that the support $\Gamma_K = \{\gamma \in \Gamma \mid K_\gamma \neq \{0\}\}$ is a subgroup of Γ. If Γ is abelian, the map $\mathcal{V}^\Gamma(K) \to \mathbb{Z}^+[\Gamma/\Gamma_K]$ given by $\left[(\gamma_1^{-1})K^{p_1} \oplus \ldots \oplus (\gamma_n^{-1})K^{p_n}\right] \mapsto \sum_{i=1}^n p_i(\gamma_i\Gamma_K)$ is a canonical isomorphism of Γ-monoids by [11, Proposition 3.7.1] (note that the difference of signs in the formula in [11, Proposition 3.7.1] is present because the action of Γ on $\mathcal{V}^\Gamma(R)$ is different than the action we consider here).

If the grade group is the group of integers and K is a trivially graded field, $K_0^\Gamma(K)$ is a $\mathbb{Z}[\mathbb{Z}]$-module. To avoid confusion of working with $\mathbb{Z}$-modules which also have an additional $\mathbb{Z}$-action, we consider the infinite group $\Gamma = \langle t \rangle$ on a single generator t to act on the monoids and their Grothendieck groups. Hence, if n is a positive integer, $\gamma_1, \gamma_2, \ldots, \gamma_n \in \mathbb{Z}$, and $R = \mathbb{M}_n(K)(\gamma_1, \ldots, \gamma_n)$, then R is graded by $\mathbb{Z}$, but its Γ-monoid $\mathcal{V}^\Gamma(R)$ and the Γ-group $K_0^\Gamma(R)$ are considered with the action of $\Gamma = \langle t \rangle$ not the action of $\mathbb{Z}$. By Lemma 2.1, we can assume that

$\gamma_1, \gamma_2, \ldots, \gamma_n \in \mathbb{Z}^+$ when considering R. Let k be the maximum of $\gamma_1, \gamma_2, \ldots, \gamma_n$ and $l_0, \ldots, l_k$ be such that l_i is the number of times i appears on the list $\gamma_1, \gamma_2, \ldots, \gamma_n$ for $i = 0, \ldots, k$. There is a canonical $\mathbf{POM}^u$-isomorphism

$$f_{n,\overline{\gamma}} : (\mathcal{V}^\Gamma(\mathbb{M}_n(K)(\overline{\gamma})), [\mathbb{M}_n(K)(\overline{\gamma})]) \cong (\mathbb{Z}^+[t, t^{-1}], \sum_{i=0}^{k} l_i t^i).$$

(both [19, Section 3.1] and [13, Section 1.5] have more details). The choice to consider $\mathbf{POM}^u$ instead of $\mathbf{POM}$ is relevant: if $\Gamma = \langle t \rangle \cong \mathbb{Z}$, $R = K = K(0)$, and $S = \mathbb{M}_2(K)(0, 1)$, then both $\mathcal{V}^\Gamma(R)$ and $\mathcal{V}^\Gamma(S)$ are isomorphic to $\mathbb{Z}^+[t, t^{-1}]$, so only their order-units, corresponding to 1 and $1 + t$ respectively, carry the information on the size of R and S and their shifts.

Let m and n be positive integers, $\gamma_1, \gamma_2, \ldots, \gamma_n \in \mathbb{Z}$ and $R = \mathbb{M}_n(K[x^m, x^{-m}])(\overline{\gamma})$. By using Lemma 2.1 when considering R, we can consider the shift γ_i modulo m so we can assume $\gamma_1, \gamma_2, \ldots, \gamma_n \in \{0, 1, \ldots, m-1\}$. Let $l_0, \ldots, l_{m-1}$ be such that l_i is the number of times i appears on the list $\gamma_1, \gamma_2, \ldots, \gamma_n$ for $i = 0, \ldots, m-1$. There is a a canonical $\mathbf{POM}^u$-isomorphism

$$f_{n,m,\overline{\gamma}} : (\mathcal{V}^\Gamma(\mathbb{M}_n(K[x^m, x^{-m}])(\overline{\gamma})), [\mathbb{M}_n(K[x^m, x^{-m}])(\overline{\gamma})]) \cong (\mathbb{Z}^+[t, t^{-1}]/(t^m = 1), \sum_{i=0}^{m-1} l_i t^i).$$

Let κ be any cardinal now. Let $\overline{\gamma} : \kappa \to \mathbb{Z}^+$ be a map and μ_n be the cardinality of $\overline{\gamma}^{-1}(n)$ for any $n \in \mathbb{Z}^+$. We use $e_{\alpha\beta}$ for a standard matrix unit and we denote the standard matrix unit with one in the first row and the first column by e_{00} not e_{11} so that this agrees with the fact that 0, not 1, is the smallest element of a finite ordinal n. With this convention, we note that the standard generating interval $D_{\overline{\gamma}}$ of $\mathcal{V}^\Gamma(\mathbb{M}_\kappa(K)(\overline{\gamma}))$ consists of the finite sums of the elements of the form $l_n t^n [e_{00} \mathbb{M}_\kappa(K)(\overline{\gamma})]$ where l_n is the cardinality of a finite subset of μ_n.

If $\overline{\gamma}(\alpha)$ is considered modulo m for every $\alpha \in \kappa$, and μ_n is the cardinality of $\overline{\gamma}^{-1}(n)$ for any $n \in \{0, \ldots, m-1\}$, the standard generating interval $D_{\overline{\gamma}}$ of $\mathcal{V}^\Gamma(\mathbb{M}_\kappa(K[x^m, x^{-m}])(\overline{\gamma}))$ consists of the finite sums of the elements of the form $l_n t^n [e_{00} \mathbb{M}_\kappa(K[x^m, x^{-m}])(\overline{\gamma})]$ where l_n is the cardinality of a finite subset of μ_n.

2.8. **The talented monoid.** For any infinite emitter v of a graph E and any finite and nonempty $Z \subseteq \mathbf{s}^{-1}(v)$, let $q_Z^v = v - \sum_{e \in Z} ee^*$. If it is clear which infinite emitter we consider, we use q_Z for q_Z^v. If $\Gamma = \langle t \rangle$ is the infinite cyclic group on t, the *talented monoid* or the *graph* Γ*-monoid* M_E^Γ is the free abelian Γ-monoid on generators $[v]$ for $v \in E^0$ and $[q_Z^v]$ for infinite emitters v and nonempty and finite sets $Z \subseteq \mathbf{s}^{-1}(v)$ subject to the relations

$$[v] = \sum_{e \in \mathbf{s}^{-1}(v)} t[\mathbf{r}(e)], \quad [v] = [q_Z^v] + \sum_{e \in Z} t[\mathbf{r}(e)], \text{ and } [q_Z^v] = [q_W^v] + \sum_{e \in W - Z} t[\mathbf{r}(e)]$$

where v is a vertex which is regular for the first relation and an infinite emitter for the second two relations in which $Z \subsetneq W$ are finite and nonempty subsets of $\mathbf{s}^{-1}(v)$. The map $[v] \mapsto [vL_K(E)]$ and $[q_Z^v] \mapsto [q_Z^v L_K(E)]$ extends to an isomorphism γ_E^Γ of M_E^Γ and $\mathcal{V}^\Gamma(L_K(E))$ (see [4, Proposition 5.7]).

If M_E is the monoid obtained analogously but by considering the trivial group instead of the group Γ, the monoid M_E registers only whether two vertices are connected, while the "talent" of M_E^Γ is to register the lengths of paths between vertices: if p is a path of length n, the relation $[\mathbf{s}(p)] = t^n[\mathbf{r}(p)] + x$ holds in M_E^Γ for some $x \in M_E^\Gamma$.

If E has finitely many vertices, the finite sum $u_E = \sum_{v \in E^0}[v]$ is an order-unit of M_E^Γ. If E^0 is infinite, $D_E = \{x \geq 0 \mid x \leq \sum_{v \in F}[v]$ for some finite $F \subseteq E^0\}$ is a generating interval of M_E^Γ (one can indeed check that D_E is convex, upwards directed, and that $\mathbb{Z}^+[\Gamma]D$ generates M_E^Γ). The interval D_E coincides with the "standard" generating interval of $\mathcal{V}^\Gamma(L_K(E))$ obtained using unitization of $L_K(E)$ (see [11, Section 3.5], [13, Section 4.1], and [19, Section 4.3]). If F is another graph and $\phi : L_K(E) \to L_K(F)$ is a graded homomorphism, then $\mathcal{V}^\Gamma(\phi)$ is a morphism of $\mathbf{POM}^D$ and $K_0^\Gamma(\phi)$ is a morphism of $\mathbf{POG}^D$. We use $\overline{\phi}$ to shorten the notation of both morphisms. If $\iota : E \to F$ is a graph isomorphism, it extends to a unique graded $*$-algebra isomorphism which we continue to denote by ι and a unique $\mathbf{POM}^D$-isomorphism which we denote by $\bar{\iota}$.

If (H, S) is an admissible pair of a graph E and $I(H, S)$ the graded ideal of $L_K(E)$ generated by $\{v \mid v \in H\} \cup \{v^H \mid v \in S\}$, then the Γ-order-ideal $J^\Gamma(H, S)$ of M_E^Γ generated by $\{[v] \mid v \in H\} \cup \{[v^H] \mid v \in S\}$ is such that (H, S), $I(H, S)$ and $J^\Gamma(H, S)$ correspond to each other in the isomorphisms of the three lattices: the lattice of admissible pairs of E, the lattice of graded ideals of $L_K(E)$, and the lattice of Γ-order-ideals of M_E^Γ (by [1, Theorem 2.5.8] and [4, Theorem 5.11]). By [23, Proposition 3.7], if (H, S) and (G, T) admissible pairs of a graph E such that $(H, S) \le (G, T)$, then there is a pre-ordered Γ-monoid isomorphism $M^\Gamma_{(G,T)/(H,S)} \cong J^\Gamma(G, T)/J^\Gamma(H, S)$.

We briefly review an alternative construction of M_E^Γ which we also use in one of the proofs. Let $\mathcal{F}_E^\Gamma$ be a free commutative Γ-monoid on generators $v \in E^0$ and q_Z^v for v infinite emitter and Z finite and nonempty subset of $\mathbf{s}^{-1}(v)$. The monoid M_E^Γ can be obtained as a quotient of $\mathcal{F}_E^\Gamma$ subject to the congruence closure $\sim$ of the relation $\to_1$ on $\mathcal{F}_E^\Gamma - \{0\}$ defined by (A1), (A2) and (A3) below for any $n \in \mathbb{Z}$ and $a \in \mathcal{F}_E^\Gamma$. In (A1), v is a regular vertex and in (A2) and (A3), v is an infinite emitter and Z and W are finite and nonempty subsets of $\mathbf{s}^{-1}(v)$ such that $Z \subsetneq W$.

(A1) $a + t^n v \to_1 a + \sum_{e \in \mathbf{s}^{-1}(v)} t^{n+1}\mathbf{r}(e)$.
(A2) $a + t^n v \to_1 a + t^n q_Z + \sum_{e \in Z} t^{n+1}\mathbf{r}(e)$.
(A3) $a + t^n q_Z \to_1 a + t^n q_W + \sum_{e \in W-Z} t^{n+1}\mathbf{r}(e)$.

If $\to$ is the reflexive and transitive closure of $\to_1$ on $\mathcal{F}_E^\Gamma$, then $\sim$ is the congruence on $\mathcal{F}_E^\Gamma$ generated by the relation $\to$. We review the Confluence Lemma.

Lemma 2.4. The Confluence Lemma. [4, Lemma 5.9] *Let E be a graph and $a, b \in \mathcal{F}_E^\Gamma - \{0\}$. The relation $a \sim b$ holds if and only if $a \to c$ and $b \to c$ for some $c \in \mathcal{F}_E - \{0\}$.*

The monoid M_E^Γ is cancellative (by [4, Corollary 5.8]) so the natural pre-order is, in fact, an order. By [14, Proposition 3.4], the relation $x < t^n x$ is impossible for any $x \in M_E^\Gamma$ and any positive integer n. The remaining possibilities give rise to the following types.

(1) If $x = t^n x$ for some positive integer n, we say that x is *periodic*.
(2) If $x > t^n x$ for some positive integer n, we say that x is *aperiodic*.
(3) If x and $t^n x$ are incomparable for any positive integer n, we say that x is *incomparable*.

If x is periodic or aperiodic, x is *comparable*. This terminology matches the one used in [14]. We note that [12] uses "cyclic" for "periodic" and "non-comparable" for "incomparable". In our terminology, the authors of [12] define a Γ-order-ideal I of M_E^Γ to be *periodic* (respectively, *comparable, incomparable*) if its every nonzero element is periodic (respectively, comparable, incomparable). We also say that I is *aperiodic* if its every nonzero element is aperiodic.

We summarize some results of [23] which we use.

Theorem 2.5. [23, Theorems 5.7, 7.4 and 7.5] *Let E be any graph.*

(1) *If E is cofinal, exactly one of the following holds.*
   (a) *E has a sink, has no infinite paths, and every $x \in M_E^\Gamma, x \neq 0$, is incomparable.*

(b) *E has a cycle c without exits, every infinite path ends in c, and every $x \in M_E^\Gamma$ is periodic.*
(c) *E has an extreme cycle and two non-disjoint cycles and every $x \in M_E^\Gamma, x \neq 0$, is aperiodic.*
(d) *E has a terminal path and an infinite path which does not end in a cycle and every $x \in M_E^\Gamma, x \neq 0$, is incomparable.*

(2) *The following two conditions are equivalent for any graph E.*
(a) *If (H,S) and (G,T) are admissible pairs of E such that $(G,T)/(H,S)$ is cofinal, then $M^\Gamma_{(G,T)/(H,S)}$ is either periodic or incomparable.*
(b) *The cycles of E are mutually disjoint.*

2.9. **Composition series.** By [23, Corollary 4.3], the following are equivalent for any graph E.

(1) There is a chain of admissible pairs
$$(\emptyset,\emptyset) = (H_0,S_0) \leq (H_1,S_1) \leq \ldots \leq (H_n,S_n) = (E^0,\emptyset)$$
such that the porcupine-quotient graph $(H_{i+1},S_{i+1})/(H_i,S_i)$ is cofinal for all $i = 0,\ldots,n-1$. If $S_i = \emptyset$ for all i, we write the above chain shorter as $\emptyset = H_0 \leq H_1 \leq \ldots \leq H_n = E^0$.
(2) There is a chain of (graded) ideals $\{0\} = I_0 \leq I_1 \leq \ldots \leq I_n = L_K(E)$ such that the (graded) algebra I_{i+1}/I_i is (graded) simple for all $i = 0,\ldots,n-1$.
(3) There is a chain of Γ-order-ideals $\{0\} = J_0 \leq J_1 \leq \ldots \leq J_n = M_E^\Gamma$ such that the Γ-monoid J_{i+1}/J_i is simple (i.e., without any nontrivial and improper Γ-order-ideals) for all $i = 0,\ldots,n-1$.

If the above conditions hold, we say that E (respectively $L_K(E)$, M_E^Γ) has *composition length* n. The graph E (respectively $L_K(E)$, M_E^Γ) has a *composition series* if E (respectively $L_K(E)$, M_E^Γ) has composition length n for some positive integer n.

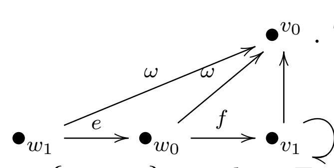
Let $\operatorname{Ter}(E)$ denote the saturated closure of the (hereditary) set of terminal vertices. In [23], the *composition quotients* of a graph E are graphs defined by
$$E_0 = E \;\text{ and }\; E_{n+1} = E_n/(\operatorname{Ter}(E_n, B_{\operatorname{Ter}(E_n)})) \text{ if } \operatorname{Ter}(E_n) \subsetneq E_n^0 \;\text{ and }\; E_{n+1} = \emptyset \text{ otherwise.}$$
These quotients are used to characterize graphs with composition series in the result below.

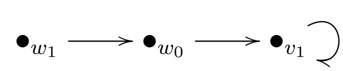
Theorem 2.6. [23, Theorem 6.5] *The following conditions are equivalent for a graph E.*

(1) *The graph E has a composition series.*
(2) *There is a nonnegative integer n such that $E_{n+1} = \emptyset$ and $E_n \neq \emptyset$ and for each n for which the composition quotient E_n is nonempty,* $\operatorname{Ter}(E_n)$ *is nonempty, the set of terminal vertices of E_n contains finitely many clusters, and the set of breaking vertices of* $\operatorname{Ter}(E_n)$ *is finite.*

By [23, Corollary 6.6]), every graph with finitely many vertices has a composition series.

One can use Theorem 2.6 to obtain a specific composition series. We illustrate the construction in the example below.

Example 2.7. Let E be the graph  . This graph has composition length 4.

We have that $\operatorname{Ter}(E) = \{v_0\}$, $B_{\operatorname{Ter}(E)} = \{w_0, w_1\}$, so that E_1 is the graph  with a unique cluster containing v_1, so $\operatorname{Ter}(E_1) = E_1^0$ and
$$(\emptyset,\emptyset) \leq (\{v_0\},\emptyset) \leq (\{v_0\},\{w_0\}) \leq (\{v_0\},\{w_0,w_1\}) \leq (E^0,\emptyset)$$

is a composition series of E. The composition factors are the porcupine graph of the pair $(\{v_0\}, \emptyset)$, and the graphs $\bullet_{w^{ee^*}} \longrightarrow \bullet_{w_0}$, $\bullet_{w_1}$, and $\bullet_{w_1} \longrightarrow \bullet_{w_0} \longrightarrow \bullet_{v_1} \circlearrowright$.

3. 1-S-NE graphs

We recall that we use $\mathbb{Z}^+$ to denote the set of nonnegative integers, $\Gamma = \langle t \rangle$ for the infinite cyclic group generated by t, and K for a field trivially graded by $\mathbb{Z}$. For any positive integer n, we consider $K[x^n, x^{-n}]$ to be graded as in section 2.1, we let $+_n$ denote the addition modulo n (i.e. the addition in $\mathbb{Z}/n\mathbb{Z}$), and $i \in n$ means that i is an element of the set $\{0, 1, \dots, n-1\}$. If ϕ is a $\mathbb{Z}$-graded algebra homomorphism, then $\overline{\phi}$ denotes the map induced by ϕ on the Γ-monoids and on the Γ-groups.

3.1. **S-NE graphs.** We recall that S shortens "sinks", that NE stands for "no-exits" and indicates cycles without exits, and we recall the definitions from the introduction.

Definition 3.1. A graph E is an *S-NE graph* if $(G, T)/(H, S)$ has either a sink or a cycle without exits for every two admissible pairs $(H, S) \leq (G, T)$ such that $(G, T)/(H, S)$ is cofinal. An S-NE graph E is an *n-S-NE graph* for a positive integer n if E has composition length n. An S-NE graph E is a *composition S-NE graph* if it is an n-S-NE graph for some positive integer n.

By Theorem 2.3, E is a 1-S-NE graph if and only if E is a cofinal graph with a sink or a cycle without exits. A finite acyclic graph with n sinks is an n-S-NE graph. The Toeplitz graph (see the introduction) is a 2-S-NE graph and the graph from Example 2.7 is a 4-S-NE graph. By Theorem 2.5, an S-NE graph has disjoint cycles. If E has finitely many vertices, the converse holds by Theorem 2.6 as well as by [23, Corollary 6.6]. By Theorem 2.6 and [23, Lemma 6.3 and Corollary 6.6], E is a composition S-NE graphs if and only if the cycles of E are disjoint, E has finitely many cycles, sinks and infinite emitters, and every infinite path ends in a cycle. The first graph below is an S-NE graph which is not a composition S-NE graph because it has infinitely many cycles. The second graph below is an acyclic graph of composition length two, but it is not an S-NE graph because there is an infinite path which does not end in a cycle.

By [23, Proposition 4.2], E has a composition series if and only if $P_{(H,S)}$ and $E/(H, S)$ have composition series for each admissible pair (H, S) of E. By this result and by Definition 3.1, E is a composition S-NE graph if and only if $P_{(H,S)}$ and $E/(H, S)$ are composition S-NE graphs.

3.2. **The graded isomorphism classes of matrix algebras.** In Proposition 3.2, we generalize the second part of Lemma 2.1, show that κ and the cardinals μ_k for $k \in \mathbb{Z}^+$, considered in section 2.7, are unique for the graded isomorphism class of $\mathbb{M}_\kappa(K)(\overline{\gamma})$ and for the graded isomorphism class of $\mathbb{M}_\kappa(K[x^m, x^{-m}])(\overline{\gamma})$ for positive integer m. If κ is infinite, the proof of Lemma 2.1 as in [11] does not generalize to the matrix algebras of infinite size because the endomorphism ring of a κ-dimensional K-vector space, when represented via $\kappa \times \kappa$ matrices, contains matrices with infinitely many nonzero entries (most prominently, the identity has infinitely many nonzero entries on the diagonal). Hence, such an endomorphism ring is not isomorphic to the ring $\mathbb{M}_\kappa(K)$ if κ is infinite. So, to prove the proposition, we use a different strategy which involves the Grothendieck Γ-monoids.

Proposition 3.2. *Let κ and κ' be cardinals, $\overline{\gamma}:\kappa\to\mathbb{Z}^+$ and $\overline{\gamma}':\kappa'\to\mathbb{Z}^+$ be any functions, and let $e_{\alpha\beta}$ to denote the standard matrix units of a matrix algebra of $\kappa\times\kappa$ size.*

(1) *Let μ_k be the cardinality of $\overline{\gamma}^{-1}(k)$ and μ'_k be the cardinality of $(\overline{\gamma}')^{-1}(k)$ for any $k\in\mathbb{Z}^+$. If $f:(\mathcal{V}^\Gamma(\mathbb{M}_\kappa(K)(\overline{\gamma})),D_{\overline{\gamma}})\cong(\mathcal{V}^\Gamma(\mathbb{M}_{\kappa'}(K)(\overline{\gamma}')),D_{\overline{\gamma}'})$ is a $\mathbf{POM}^D$-isomorphism, then*

$$\kappa=\kappa' \text{ and } \quad \mu_k=\mu'_k \text{ for each } k\in\mathbb{Z}^+$$

and there is a bijection $\rho:\kappa\to\kappa$ and a graded algebra $*$-isomorphism $\phi:\mathbb{M}_\kappa(K)(\overline{\gamma})\to\mathbb{M}_{\kappa'}(K)(\overline{\gamma}')$ such that $\phi(e_{\alpha\beta})=e_{\rho(\alpha)\rho(\beta)}$ for every $\alpha,\beta\in\kappa$ and such that $\mathcal{V}^\Gamma(\phi)=f$.

(2) *For positive integers m and m', $\alpha\in\kappa$, $k\in m$, $\alpha'\in\kappa'$, and $k'\in m'$, let μ_k be the cardinality of $\overline{\gamma}^{-1}(k)$ when $\overline{\gamma}(\alpha)$ is considered modulo m and let μ'_k be the cardinality of $(\overline{\gamma}')^{-1}(k)$ when $\overline{\gamma}'(\alpha)$ is considered modulo m'. If $f:(\mathcal{V}^\Gamma(\mathbb{M}_\kappa(K[x^m,x^{-m}])(\overline{\gamma})),D_{\overline{\gamma}})\to(\mathcal{V}^\Gamma(\mathbb{M}_{\kappa'}(K[x^{m'},x^{-m'}])(\overline{\gamma}')),D_{\overline{\gamma}'})$ is a $\mathbf{POM}^D$-isomorphism, then*

$$\kappa=\kappa',\ m=m', \text{ there is } i\in m \text{ such that } \mu_k=\mu'_{k+_m i} \text{ for each } k\in m,$$

and there is a bijection $\rho:\kappa\to\kappa$ and a graded algebra $*$-isomorphism $\phi:\mathbb{M}_\kappa(K[x^m,x^{-m}])(\overline{\gamma})\to\mathbb{M}_\kappa(K[x^m,x^{-m}])(\overline{\gamma}')$ such that ϕ maps a standard matrix unit $e_{\alpha\beta}$ to the element of the form $x^{lm}e_{\rho(\alpha)\rho(\beta)}$ for some $l\in\mathbb{Z}$ determined by α and β using $\overline{\gamma}$ and $\overline{\gamma}'$ and such that $\mathcal{V}^\Gamma(\phi)=f$.

Proof. To show part (1), let $f_{\kappa,\overline{\gamma}}$ be the $\mathbf{POM}^D$-isomorphism $\mathcal{V}^\Gamma(\mathbb{M}_\kappa(K)(\overline{\gamma}))\to\mathbb{Z}^+[t,t^{-1}]$ induced by the map $[e_{00}]\mapsto 1$ and let $f_{\kappa',\overline{\gamma}'}$ be analogous such isomorphism for κ' and $\overline{\gamma}'$. The assumption of part (1) implies that $g=f_{\kappa',\overline{\gamma}'}ff_{\kappa,\overline{\gamma}}^{-1}$ is a $\mathbf{POM}^D$-isomorphism $(\mathbb{Z}^+[t,t^{-1}],D_\mu)\to(\mathbb{Z}^+[t,t^{-1}],D_{\mu'})$ where the generating intervals D_μ and $D_{\mu'}$ are as in section 2.7. As $g(D_\mu)\subseteq D_{\mu'}$ and $g^{-1}(D_{\mu'})\subseteq D_\mu$, D_μ and $D_{\mu'}$ have the same cardinality, so $\kappa=\kappa'$.

The isomorphism g is uniquely determined by the image of 1. Assume that $g(1)=p(t)$ for some Laurent polynomial $p(t)\in\mathbb{Z}^+[t,t^{-1}]$. As $p(t)\in D_{\mu'}$, $p(t)$ contains no negative powers of t. Analogously, if $q(t)$ is the polynomial $g^{-1}(1)$, then $q(t)$ does not contain any negative powers of t as $q\in D_\mu$. Since $1=g(g^{-1}(1))=g(q(t))=q(t)g(1)=q(t)p(t)$, the degrees of p and q add up to zero which implies that $p(t)=a$ and $q(t)=b$ are constant polynomials for some $a,b\in\mathbb{Z}^+$. Since $ab=1$, we have that $a=b=1$ which shows that $g(1)=1$.

If κ is finite, equating the order-units $\sum_k\mu_kt^k$ and $\sum_k\mu'_kt^k$ implies the relation $\mu_k=\mu'_k$. If κ is infinite, g maps the set of monomials of the form l_kt^k, where l_k is the cardinality of a finite subset of μ_k, to the set of monomials of the form l'_kt^k where l'_k is the cardinality of a finite subset of μ'_k. As $g(1)=1$, this implies that the cardinality of the set of all finite sets of μ_k is equal to the cardinality of the set of all finite sets of μ'_k. If μ_k is finite, this implies that μ'_k is finite and that $\mu_k=\mu'_k$. If μ_k is infinite, this implies that μ_k is infinite. As the cardinality of the set of all finite subsets of an infinite cardinal is equal to that cardinal, we have that $\mu_k=\mu'_k$. Thus, $\mu_k=\mu'_k$ for any $k\in\mathbb{Z}^+$.

Let $\pi:\kappa\to\kappa$ be a bijection which permutes the images of $\overline{\gamma}$ so they are listed in the non-decreasing order and let π' be analogous such bijection for $\overline{\gamma}'$. Let ϕ_π and $\phi_{\pi'}$ be graded $*$-isomorphisms induced by the maps $e_{\alpha 0}\mapsto e_{\pi(\alpha)0}$ and $e_{\alpha 0}\mapsto e_{\pi'(\alpha)0}$ respectively. If $\overline{\mu}:\kappa\to\kappa$ denotes the map $\overline{\gamma}\pi$, we have that $f_{\kappa,\overline{\gamma}}=f_{\kappa,\overline{\mu}}\overline{\phi}_\pi$ and that $f_{\kappa,\overline{\gamma}'}=f_{\kappa,\overline{\mu}}\overline{\phi}_{\pi'}$. Hence, $g=f_{\kappa,\overline{\mu}}\overline{\phi}_{\pi'}f\overline{\phi}_\pi^{-1}f_{\kappa,\overline{\mu}}^{-1}$ is the identity which implies that $\overline{\phi}_{\pi'}f\overline{\phi}_\pi^{-1}$ is the identity so that $f=\overline{\phi}_{\pi'}^{-1}\overline{\phi}_\pi=\overline{\phi_{\pi'}^{-1}\phi_\pi}$. This shows that if we take $\rho={\pi'}^{-1}\pi$ and $\phi=\phi_{\pi'}^{-1}\phi_\pi$, then ρ is as required in the statement of (1) and ϕ is a graded $*$-isomorphism such that $f=\overline{\phi}$. The isomorphism ϕ is obtained by compositions of two maps as in part (1) of Lemma 2.2, so it maps a standard matrix unit to a standard matrix unit.

To show part (2), let $f_{\kappa,m,\overline{\gamma}}$ be the $\mathbf{POM}^D$-isomorphism $\mathcal{V}^\Gamma(\mathbb{M}_\kappa(K[x^m,x^{-m}])(\overline{\gamma}))\to\mathbb{Z}^+[t,t^{-1}]/(t^m=1)$ induced by the map $[e_{00}]\mapsto 1$ and let $f_{\kappa',m',\overline{\gamma}'}$ be analogous such isomorphism for κ', m', and $\overline{\gamma}'$. The assumption of part (2) implies that $g = f_{\kappa',m',\overline{\gamma}'}ff^{-1}_{\kappa,m,\overline{\gamma}}$ is a $\mathbf{POM}^D$-isomorphism $(\mathbb{Z}^+[t,t^{-1}]/(t^m=1),D_\mu)\to(\mathbb{Z}^+[t,t^{-1}]/(t^{m'}=1),D_{\mu'})$. As $g(D_\mu)\subseteq D_{\mu'}$ and $g^{-1}(D_{\mu'})\subseteq D_\mu$, D_μ and $D_{\mu'}$ have the same cardinality, so $\kappa=\kappa'$. As $t^m=t^mg(g^{-1}(1))=g(t^mg^{-1}(1))=g(g^{-1}(1))=1$, the relation $t^m=1$ holds in $\mathbb{Z}^+[t,t^{-1}]/(t^{m'}=1)$. Thus, $m\geq m'$. Analogously, $m'\geq m$, so $m=m'$.

The map g is uniquely determined by the image of 1. Assume that $g(1)=p(t)$ for some $p(t)\in\mathbb{Z}^+[t,t^{-1}]/(t^m=1)$. As $p(t)\in D_{\mu'}$, $p(t)$ contains no negative powers of t. Analogously, if $q(t)\in\mathbb{Z}^+[t,t^{-1}]/(t^m=1)$ is $g^{-1}(1)$, then $q(t)$ does not contain any negative powers of t. As $1=g(g^{-1}(1))=g(q(t))=q(t)g(1)=q(t)p(t)$, $p=t^i$ and $q=t^{m-i}$ for some $i\in m$. Hence, $g(1)=t^i$.

Let σ be the cyclic permutation given by $k\mapsto k+_m i$ (recall that $+_m$ denotes the addition modulo m). If κ is finite, equating the order-units $\sum_{k=0}^{m-1}\mu_kt^k$ and $\sum_{k=0}^{m-1}\mu'_kt^{k+i}$ implies $\mu_k=\mu'_{\sigma(k)}$. If κ is infinite, g maps the set of monomials of the form l_kt^k for l_k the cardinality of any finite subset of μ_k to the set of monomials of the form l'_kt^{k+i} for l'_k the cardinality of any finite subset of μ'_{k+_mi}. The first set has cardinality μ_k and the second set has cardinality μ'_{k+_mi}, so $\mu_k=\mu'_{k+_mi}$ for any $k\in\mathbb{Z}^+$.

Let $\overline{\mu}$ be the list obtained by permuting the images of $\overline{\gamma}$ so that when they are considered modulo m, they are listed in a non-decreasing order. Let $\pi:\kappa\to\kappa$ be the corresponding bijection and let π' be the analogous bijection for $\overline{\gamma}'$. Let $\overline{\mu}=\overline{\gamma}\pi$ and $\overline{\mu}'=\overline{\gamma}'\pi'$. Let $\rho_1:\kappa\to\kappa$ denote a permutation so that $\sigma\overline{\mu}\rho_1=\overline{\mu}'$. Let $\phi_{\rho_1,\sigma}$ be the graded algebra isomorphism $\phi_{\rho_1,\sigma}:\mathbb{M}_\kappa(K[x^m,x^{-m}])(\overline{\mu})\to\mathbb{M}_\kappa(K[x^m,x^{-m}])(\overline{\mu}')$ determined by first permuting the shifts so that the list $\mu_k=\mu'_{\sigma(k)}$ moves from the k-th place to the $(k-_m i)$-th place and then adding i modulo m to each of the shifts producing exactly the $\overline{\mu}'$ list. Note that this second operation can be obtained by considering the graded $*$-isomorphism induced by the map $e_{\alpha\beta}\mapsto x^{(\sigma(\overline{\mu}(\beta))-\sigma(\overline{\mu}(\alpha)))m}e_{\alpha\beta}$. As any element of the form $x^{lm}, l\in\mathbb{Z}$ is unitary, the map $\phi_{\rho_1,\sigma}$ is a $*$-isomorphism.

If ϕ_π and $\phi_{\pi'}$ denote the graded $*$-isomorphisms induced by the bijections π and π', we have that $f_{\kappa,m,\overline{\mu}}\overline{\phi}_\pi=f_{\kappa,m,\overline{\gamma}}$ and $f_{\kappa,m,\overline{\mu}'}\overline{\phi}_{\pi'}=f_{\kappa,m,\overline{\gamma}'}$ so that $g=f_{\kappa,m,\overline{\mu}}\overline{\phi}_{\pi'}f\overline{\phi}_\pi^{-1}f^{-1}_{\kappa,m,\overline{\mu}}=f_{\kappa,m,\overline{\mu}'}\overline{\phi}_{\rho_1,\sigma}f^{-1}_{\kappa,m,\overline{\mu}}$. Thus,

$$\overline{\phi}_{\pi'}f\overline{\phi}_\pi^{-1}=\overline{\phi}_{\rho_1,\sigma}\text{ which implies that }f=\overline{\phi}_{\pi'}^{-1}\overline{\phi}_{\rho_1,\sigma}\overline{\phi}_\pi.$$

This shows that for $\phi=\phi_{\pi'}^{-1}\phi_{\rho_1,\sigma}\phi_\pi$, we have that $f=\overline{\phi}$. Letting $\rho=\pi'^{-1}\rho_1\pi$, we obtain a bijection as needed and we have that $\phi(e_{\alpha\beta})=x^{(\sigma(\overline{\mu}(\beta))-\sigma(\overline{\mu}(\alpha)))m}e_{\rho(\alpha)\rho(\beta)}$. □

3.3. **1-S-NE graphs and their canonical forms.** Recall that E is a 1-S-NE graph if and only if E is a cofinal graph with a sink or a cycle without exits. If a cofinal graph E has a sink v_0 and if P^{v_0} is the set of paths ending in v_0, let κ be the cardinality of P^{v_0} so that we can index the elements of P^{v_0} by the elements of κ and write P^{v_0} as $\{p_\alpha\mid\alpha\in\kappa\}$. We can chose a bijection $\kappa\to P^{v_0}$ indexing the elements of P^{v_0} so that 0 corresponds to the zero-length path v_0. Let $\overline{\gamma}:\kappa\to\mathbb{Z}^+$ map α to the length of the path p_α. If $e_{\alpha\beta},\alpha,\beta\in\kappa$ are the standard matrix units of $\mathbb{M}_\kappa(K)(\overline{\gamma})$, the correspondence $p_\alpha p_\beta^*\mapsto e_{\alpha\beta}$ extends to a graded $*$-isomorphism $L_K(E)\cong_{\text{gr}}\mathbb{M}_\kappa(K)(\overline{\gamma})$ ([13, Proposition 5.1] has more details). The paths from P^{v_0} corresponds to the standard matrix units in the first column and v_0 corresponds to e_{00}.

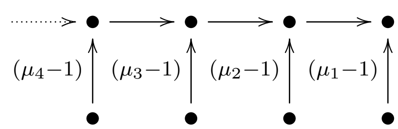
If μ_k is the cardinality of $\overline{\gamma}^{-1}(k)$, the list of cardinals $\mu_0,\mu_1,\mu_2,\ldots,$ is such that $\mu_0=1$ (because $p_0=v_0$ is the only element of P^{v_0} of length zero) and if $\mu_k\neq 0$ for some $k>0$, then $\mu_i\neq 0$ for all $i\leq k$ (because the suffices of a path of length k have lengths $i\leq k$). One can generalize the proof of [18, Proposition 3.2] (given for κ finite) to show the converse: for every κ and a list of cardinals

$\mu_k, k \in \mathbb{Z}^+$ with the above two properties, there is a cofinal graph, which we denote by E_{can}, such that that $L_K(E_{\text{can}}) \cong_{\text{gr}} \mathbb{M}_\kappa(K)(\overline{\gamma})$ where γ is such that $|\overline{\gamma}^{-1}(k)| = \mu_k$ for any $k \in \mathbb{Z}^+$.

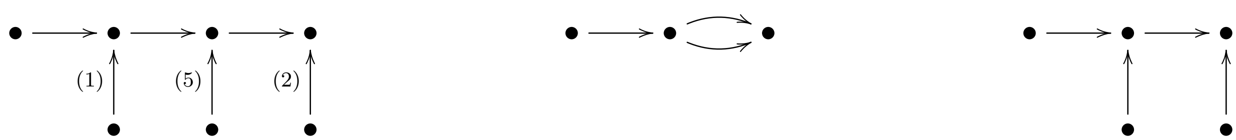
The graph E_{can} can be constructed by the following inductive process. Let E_0 be a single vertex v_{00}, let E_1 be obtained by adding μ_1 new vertices $v_{1\alpha}, \alpha \in \mu_1$ and μ_1 new edges $e_{1\alpha}$ where $\mathbf{s}(e_{1\alpha}) = v_{1\alpha}$ and $\mathbf{r}(e_{1\alpha}) = v_{00}$ for all $\alpha \in \mu_1$. Let E_2 be obtained by adding μ_2 new vertices $v_{2\alpha}, \alpha \in \mu_2$ and μ_2 new edges originating at the new vertices and ending at v_{10}. If $E_k, k \in \omega$, is the graph obtained by continuing this process, let E_{can} be the directed union of $E_k, k \in \omega$. Note that all vertices of E_{can} except the sink v_{00} are regular and they emit exactly one edge.

If there is k such that $\mu_i = 0$ for all $i > k$ and $\mu_k \neq 0$, then k is the *spine length* and the path $e_{k0} \ldots e_{20}e_{10}$ is the *spine* of E_{can}. Otherwise, E_{can} has the spine of infinite length and the left-infinite path $\ldots e_{20}e_{10}$ is the spine of E_{can}. The $\mu_i - 1$ edges $e_{i\alpha}$, $\alpha \in \mu_i - \{0\}$ are referred to as the $(i-1)$-*tails*. This terminology reflects the fact that the i-tails end at v_{i0}.

To make graphical representation of tails clearer, we introduce the following abbreviation: if v receives k edges originating at sources, no matter whether k is finite or infinite cardinal, we depict this as $\bullet \xrightarrow{(k)} \bullet^v$. So, a canonical form with infinite spine length can be represented by the graph below.



If E is a 1-S-NE graph and if its algebra has the matrix representation $\mathbb{M}_{12}(1, 1+2, 1+5, 1+1)$, then our abbreviated graphical representation of E_{can} is the first graph below. Its spine length is three. As another example, the third graph below is a canonical form of the second graph.



The notation $\bullet \xrightarrow{(k)} \bullet$ is not to be confused with $\bullet \xrightarrow{k} \bullet$ since the first notation indicates that there are k sources each emitting a single edge to the terminal vertex and the second indicates that there are two vertices and the source vertex emits k edges into the terminal vertex.

This construction enables one to come up with a way to associate a graph, unique up to a graph isomorphism, to all graphs with the algebras in the same graded $*$-isomorphism class: if E is such that $L_K(E)$ is graded $*$-isomorphic to $\mathbb{M}_\kappa(\overline{\gamma})$, then we can repeat this construction using possibly different bijection $\overline{\gamma}^{-1}(k)$ and paths of length k in E and obtain some E'_{can}. However, the only difference between E_{can} and E'_{can} is the labeling of the edges and vertices and, by construction, $E'_{\text{can}} \cong E_{\text{can}}$. Because of this, a *canonical form* E_{can} is unique up to a graph isomorphism.

If a cofinal graph E has a cycle c of length $m > 0$ without exits and $v_0 = \mathbf{s}(c)$, let P^{v_0} be the set of paths ending at v_0 and let

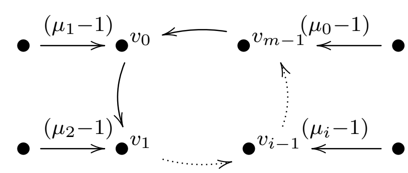
$$P^{v_0}_{\not c} \text{ be the set of paths ending in } v_0 \text{ which do not contain the cycle } c.$$

If κ is the cardinality of $P^v_{\not c}$, we index the elements of $P^v_{\not c}$ by the elements of κ so that $p_0 = v_0$ is the path of length zero. Then, let $\overline{\gamma} : \kappa \to \mathbb{Z}^+$ be the map $\alpha \mapsto |p_\alpha|$. The correspondence mapping $p_\alpha p_\beta^* \mapsto e_{\alpha\beta}$ extends to a graded $*$-isomorphism $L_K(E) \cong_{\text{gr}} \mathbb{M}_\kappa(K[x^m, x^{-m}])(\overline{\gamma})$ ([13, Proposition

5.1] has more details) which maps v_0 to e_{00}. The paths from $P^v_{\not c}$ corresponds to the standard matrix units in the first column.

Let μ_k be the cardinality of $\overline{\gamma}^{-1}(k)$ for $k \in m$. The list of cardinals $\mu_0, \mu_1, \ldots, \mu_{m-1}$ is such that $\mu_k > 0$ for each $k \in m$ since there is a paths of length k within the cycle c. By generalizing the proof of [18, Proposition 3.4] to infinite cardinals, one can show the converse: for any κ, any integer $m > 0$ and a list of nonzero cardinals $\mu_0, \mu_1, \ldots, \mu_{m-1}$ there is a cofinal graph E_{can} with a cycle without exits of length m such that $L_K(E_{\text{can}}) \cong_{\text{gr}} \mathbb{M}_\kappa(K[x^m, x^{-m}])(\mu_0, \ldots, \mu_{m-1})$.

The graph E_{can} can be obtained as follows. Let E_0 be an isolated cycle c of length m and let $v_0, v_1, \ldots v_{m-1}$ be the vertices of c listed in the order they appear in c. The graph E_{can} is obtained by adding $\mu_i - 1$ new vertices $v_{i\alpha}$ and $\mu_i - 1$ new edges $e_{i\alpha}$, $\alpha \in \mu_i$ such that $\mathbf{s}(e_{i\alpha}) = v_{i\alpha}$ and $\mathbf{r}(e_{i\alpha}) = v_{i-_m 1}$ for all $i \in m$. Such graph E_{can} is a *canonical form* of any graph whose algebra is graded $*$-isomorphic to the matrix algebra $\mathbb{M}_\kappa(K[x^m, x^{-m}])(\mu_0, \ldots, \mu_{m-1})$. It is unique up to a graph isomorphism and it can be represented as the graph below.

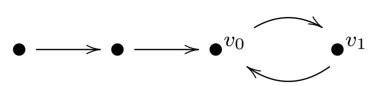


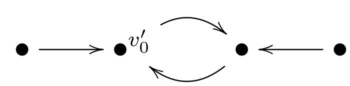
For example, the second graph in Example 3.3 below is a canonical form of the first graph.

If $m > 1$, let $v_0, v_1, \ldots, v_{m-1}$ be the vertices listed in the order they appear in c and let $c_i, i \in m$, be the element of the equivalence class $[c]$ with $\mathbf{s}(c_i) = v_i$. Consideration of c_i instead of c impacts the values of μ_k only up to the cyclic permutation given by $k \mapsto k +_m i$. The change from c to c_i corresponds to the graded isomorphism on the matricial level mapping e_{00} onto $e_{\alpha\alpha}$ for some $\alpha \in \kappa$ such that $\overline{\gamma}(\alpha) = i$. The induced map on $\mathbb{Z}^+[t, t^{-1}]/(t^m = 1)$ corresponds to the map with $1 \mapsto t^i$.

Let us consider an example illustrating the switch from $c = c_0$ to another $c_i \in [c]$.

Example 3.3. Let E and F be the two graphs below and let c and c' be their cycles considering starting at v_0 and v_0', respectively.





The paths in $P^{v_0}_{\not c}$ have lengths 0, 1, 1, and 2. If c_1 is the cycle based in v_1, the paths in $P^{v_1}_{\not c_1}$ have the lengths, 0, 1, 2 and 3. Hence, both $\mathbb{M}_4(K[x^2, x^{-2}])(0, 1, 1, 2)$ and $\mathbb{M}_4(K[x^2, x^{-2}])(0, 1, 2, 3)$ are matricial representations of $L_K(E)$. The two representations are graded $*$-isomorphic by Lemma 2.1 since, when the shifts are considered modulo 2 and listed in non-decreasing order, the resulting list is $0, 0, 1, 1$ in each case. The change from c to c_1 corresponds to mapping the unit e_{00} to e_{11}. This change induces $[e_{00}] \mapsto [e_{11}] = t[e_{00}]$ on the Γ-monoid level, so this corresponds to the automorphism of $\mathbb{Z}^+[t, t^{-1}]/(t^2 = 1)$ mapping 1 to t.

For the second graph, the lengths of paths in $P^{v_0'}_{\not c'}$ are $0, 1, 1, 2$ and, considered modulo 2 and listed in non-decreasing order, we obtain $0, 0, 1, 1$. As this is the same list as the list for E, we have that $L_K(E) \cong_{\text{gr}} L_K(F)$ by Lemma 2.1. Mapping $P^{v_0}_{\not c}$ to $P^{v_0'}_{\not c'}$ bijectively and such that the lengths of paths remain the same modulo 2, induces such a graded isomorphism.

This example also illustrates that the Leavitt path algebras of two graphs can be graded isomorphic without the graphs being out-split equivalent (i.e obtained one from the other by finitely many out-splits and out-amalgamations).

To integrate the terminology, we would like to unify the cases when a 1-S-NE graph has a sink and when it has a cycle by treating a path of trivial length as a cycle of length zero. We say that such a cycle is a *trivial cycle.* If E is a 1-S-NE graph with the terminal cluster c^0 for a cycle c (possibly trivial) and if $m = |c|$, then $m = 0$ indicates that E has a sink and $m > 0$ indicates that E has a cycle with no exits. If $v_0 = \mathbf{s}(c)$ and $m = 0$, we let $P^{v_0}_{\not c} = P^{v_0}$.

We say that 1-S-NE graphs E and F are *1-S-NE equivalent* and write $E \approx F$ if $E_{\text{can}} \cong F_{\text{can}}$.

Next, we show that the GCC holds for 1-S-NE graphs. This is known to hold for some special types of 1-S-NE graphs (the introduction has more details) but not for all 1-S-NE graphs and all underlying fields. In addition, condition (2) of the proposition below has not been previously considered together with (1) and (3).

Proposition 3.4. The GCC holds for 1-S-NE graphs. *The following conditions are equivalent for 1-S-NE graphs E and F.*

(1) *There is a $\mathbf{POM}^D$-isomorphism $f : M^\Gamma_E \to M^\Gamma_F$.*
(2) $E \approx F$.
(3) *There is a graded $*$-isomorphism $\phi : L_K(E) \to L_K(F)$.*

If (1) *holds, the isomorphism ϕ from condition* (3) *can be found so that $f = \overline{\phi}$.*

Proof. Since both $L_K(E)$ and $L_K(E_{\text{can}})$ are graded $*$-isomorphic to the same graded matrix algebra by the definition of E_{can}, we have that (2) $\Rightarrow$ (3). The implication (3) $\Rightarrow$ (1) is direct, so it remains to show (1) $\Rightarrow$ (2) and the last sentence of the theorem.

Assume that (1) holds and let m be the length of the terminal cycle of E. If $m = 0$, then every nonzero element of M^Γ_E is incomparable. So, every element of M^Γ_F is incomparable, which implies that F has a sink. If $m > 0$, then every nonzero element of M^Γ_E is periodic with the period m. The existence of f implies that every nonzero element of M^Γ_F is periodic with the period m. Hence, F has a cycle without exits of length m.

Let $\overline{\mu}$ and $\overline{\mu}'$ be the maps corresponding to the shifts of the matrix representations $\mathbb{M}_E$ and $\mathbb{M}_F$ and let ϕ_E and ϕ_F be graded $*$-isomorphisms of the Leavitt path algebras and their matrix representations. The map $\overline{\phi_F} f \overline{\phi_E}^{-1}$ is a $\mathbf{POM}^D$-isomorphism. By Proposition 3.2, the cardinalities κ and $\mu_k, k \in \mathbb{Z}^+$ from Proposition 3.2 match and there is $i \in m$ such that μ_k and $\mu'_{k+_m i}$ for $k \in \mathbb{Z}^+$ have the same values modulo m. If $i \neq 0$, we can choose a different matrix representation of $L_K(F)$ so that the images of ϕ_E and ϕ_F is the same matrix algebra. If id is the identity map on this matrix algebra, then $\overline{\phi_F} f \overline{\phi_E}^{-1} = \overline{\text{id}}$. Thus, $f = \overline{\phi_F^{-1} \,\text{id}\, \phi_E}$, so we realized f by a graded $*$-isomorphism. Since the two graph algebras have the same matrix algebra representation, we have that $E_{\text{can}} \cong F_{\text{can}}$ and so $E \approx F$ holds. □

3.4. **Relative canonical form.** If E is a 1-S-NE graph, let $v_0, \ldots, v_{m-1}$ be the vertices of a terminal cycle c. In addition to the sets P^{v_0} and $P^{v_0}_{\not c}$, we also consider the following set for $j \in m$ and a positive integer k.

$$\mathcal{P}^{v_j}_k \text{ is the set of paths of length } k \text{ ending in } v_j \text{ which share no edge with the cycle } c.$$

$$\text{Let } \mathcal{P}^{v_j} = \bigcup_{0<k\in\omega} \mathcal{P}^{v_j}_k, \text{ and let } E^{v_j} \text{ be the subgraph generated by the vertices of the paths in } \mathcal{P}^{v_j}.$$

We use calligraphic $\mathcal{P}$ instead of P to highlight the difference between P^{v_j} and $\mathcal{P}^{v_j}$: the first set contains all paths ending at v_j while the second can be strictly smaller. The sets $\mathcal{P}^{v_j}$ and $P^{v_j}_{\not c}$ can also be different. The graph E^{v_j} is a 1-S-NE graph and v_j is its sink.

Let $E = E_{\text{tot}}$ be a 1-S-NE graph and $V \subseteq E^0$. We define a graph $E_{\text{can},V}$ which is canonical only *relative* to the root of V and we call it the *canonical form relative to* V. If $V = \emptyset$, $E_{\text{can},V} = E_{\text{can}}$ so this construction also presents a specific operation transforming a 1-S-NE graph E to E_{can}. If V contains a vertex of the terminal cycle c or the sources of $\mathcal{P}^{v_j}_1$ paths for all $j \in m$, then $E_{\text{can},V} = E$.

Our main application of this construction is the porcupine graph P_H of a 2-S-NE graph E in which case V is the set of vertices of P_H which are not in H. In this case, the outcome of the construction is the direct-exits form of E.

We let E_V be the subgraph of E generated by $c^0 \cup R(V)$. We define $E_{\text{can},V}$ as a graph which contains E_V as well as some new vertices and edges. We introduce these new elements simultaneously with creating a bijection σ of the set $P^{v_0}_{\not c}$ of E and $P^{v_0}_{\not c}$ of $E_{\text{can},V}$.

The case $\mathbf{m > 0}$. For any $k \geq 0$, we let $c_{(m-_mk)0}$ be the part of c from v_{m-_mk} to v_0 if $m -_m k \neq 0$ and $c_{(m-_mk)0} = v_0$ if $m -_m k = 0$. With this notation, we let $\sigma(c_{j0}) = c_{j0}$ for $j \in m$ and $\sigma(hc_{j0}) = hc_{j0}$ for $h \in \mathcal{P}^{v_j}_1$ such that $\mathbf{s}(h) \in R(V)$.

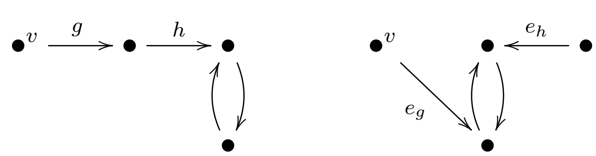
For $k > 0$, and $p = gq \in \mathcal{P}^{v_j}_k$ where g is an edge and $q \in \mathcal{P}^{v_j}_{k-1}$ is such that $\mathbf{s}(q) \notin R(V)$, we consider the cases $\mathbf{s}(g) \notin R(V)$ and $\mathbf{s}(g) \in R(V)$.

If $\mathbf{s}(g) \notin R(V)$, let v_g be a new vertex and e_g be a new edge with the source v_g and range v_{j-_mk}. Thus, one can consider that the path p of E is replaced by the tail e_g in $E_{\text{can},V}$, so we let $\sigma(pc_{j0}) = e_g c_{(j-_m|q|)0}$. Note that the length of pc_{j0} is $1 + |q| + m -_m j$ and the length of $e_g c_{(j-_m|q|)0}$ is $1 + m -_m (j -_m |q|) = 1 + m -_m j +_m |q|$. So, $|\sigma(pc_{j0})| = |pc_{j0}| (\operatorname{mod} m)$.

If $\mathbf{s}(g) \in R(V)$, let e_g be a new edge with the source $\mathbf{s}(g)$ and range v_{j-_mk} and let $\sigma(p)$ be defined as in the previous case. Requiring that $\mathbf{s}(g) = \mathbf{s}(e_h)$ ensures that (E1) holds for the images of the map ϕ from the proof of Proposition 3.5.

If $p = rgq$ is such that all vertices of r are in $R(V)$, $g \in E^1$ is such that $\mathbf{r}(g) \notin R(V)$ and $q \in \mathcal{P}^{v_j}_k$ for some k and $j \in m$, we let $\sigma(rgq)$ be $r\sigma(gq)$. With these definitions, the map σ becomes defined on $P^{v_0}_{\not c}$ and it bijectively maps it to the set $P^{v_0}_{\not c}$ of $E_{\text{can},V}$.

For example, if E is the first graph below and $V = \{v\}$, $E_{\text{can},V}$ is the second graph below.



The case $\mathbf{m = 0}$. If there is a maximum of lengths of paths from a vertex in V to v_0, let us denote it with k. If there is no such maximum k, let $k = \omega$.

We let $e_{k-1} \ldots e_0$ ($\ldots e_1 e_0$ if $k = \omega$) be a new path containing new edges and new vertices except its range v_0. We refer to such path as the *spine of E relative to V*. Pick a path p_1 in the set of paths in $\mathcal{P}^{v_0}_1$ with sources not in $R(V)$ and let P_1 be the remaining set of paths in $\mathcal{P}^{v_0}_1$ with sources not in $R(V)$, if any. For each $p \in P_1$, we add a tail e_p to v_0 (i.e. a new edge e_p ending in v_0 and a new vertex v_p as the source of e_p) and we let $\sigma(p) = e_p$. We also let $\sigma(p_1) = e_0$. If p is a path in $\mathcal{P}^{v_0}_1$ such that $\mathbf{s}(p) \in R(V)$, then this path is also a path of E_V and we let $\sigma(p) = p$.

If $k > 1$ we continue this process by considering the set of paths $p = gh$ in $\mathcal{P}^{v_0}_2$ such that $\mathbf{s}(h) \notin R(V)$, we again consider the cases $\mathbf{s}(g) \notin R(V)$ and $\mathbf{s}(g) \in R(V)$.

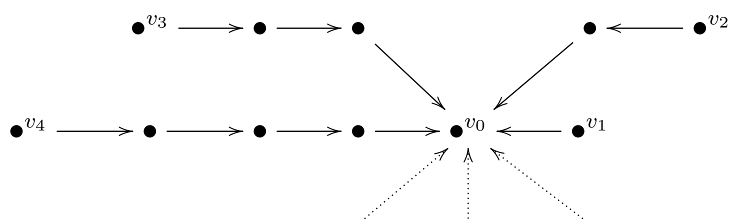
If $\mathbf{s}(g) \notin R(V)$, then the set P_2 of all paths in $\mathcal{P}_2^{v_0}$ such that neither the source or the range of the first edge is in $R(H)$ is nonempty. Let p_2 be arbitrary element of P_2 and let $P_2' = P_2 - \{p_2\}$. If P_2' is nonempty and if $p \in P_2'$, we add a new edge e_g and its new source $v_g = \mathbf{s}(e_g)$ to be a tail to $\mathbf{s}(e_0)$, we let $\sigma(p) = e_g e_0$, and we let $\sigma(p_2) = e_1 e_0$.

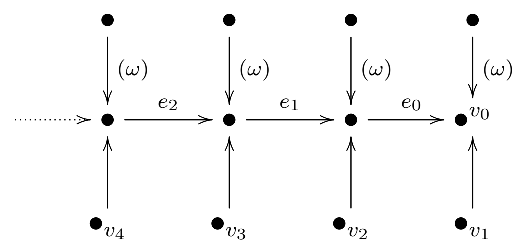
If $\mathbf{s}(g) \in R(V)$, we add a new edge e_g with $\mathbf{s}(g)$ as its source and $\mathbf{s}(e_0)$ as its range (recall that e_0 is the last edge of our newly added spine). In addition, we let $\sigma(p) = e_g e_0$.

If $k = 1$, then there are no paths $p = gh \in \mathcal{P}_2^{v_0}$ such that $\mathbf{s}(g) \in R(V)$ and $\mathbf{r}(g) \notin R(V)$. So, in this case, it is sufficient to consider the case when $\mathbf{s}(g) \notin R(V)$ and this case is similar to the consideration of the same case if $k > 1$: the condition $\mathbf{s}(g) \notin R(V)$ rules out the possibility $\mathbf{s}(h) \in R(V)$, so we have that $\mathbf{s}(h) \notin R(V)$. In this case, we let $\sigma(p) = e_g e_0$ for a new edge e_g starting at a new source v_g and ending in $\mathbf{s}(e_0)$.

We continue this process by considering $\mathcal{P}_3^{v_0}$ if needed. Eventually, the map σ becomes defined on the entire set P^{v_0} and it bijectively maps it to the set P^{v_0} of $E_{\text{can},V}$.

For example, let E be the first graph below so that E consists of paths p_n of length n ending at v_0 for every n such that p_n and p_l have no common vertices or edges except their range v_0 if $n \neq l$. If $\mathbf{s}(p_n) = v_n$ for $n \in \omega$, let $V = \{v_1, v_2, \dots\}$ be the set of the sources of E. Then, $E_{\text{can},V}$ is the second graph below.





We show that the algebras of E and $E_{\text{can},V}$ are graded $*$-isomorphic.

Proposition 3.5. *If $E = E_{\text{tot}}$ is a 1-S-NE graph and $V \subseteq E^0$, then there is an operation $E \to E_{\text{can},V}$ which extends to a graded $*$-isomorphism $L_K(E) \to L_K(E_{\text{can},V})$.*

If E, F, and f are as in Proposition 3.4, there are operations $\phi_E : E \to E_{\text{can}}$, $\iota : E_{\text{can}} \cong F_{\text{can}}$, and $\phi_F : F \to F_{\text{can}}$, and f can be realized as $\overline{\phi_F^{-1}\iota\phi_E = f}$.

Proof. Let us consider the case $m > 0$ first. Keeping our previous notation, let σ be the bijection introduced along with $E_{\text{can},V}$. We define a map ϕ on $E^0 \cup E^1$ by mapping the vertices and edges of E_V identically onto themselves. For $p = gq \in \mathcal{P}_k^{v_j}$, $g \in E^1$, and $q \in \mathcal{P}_{k-1}^{v_j}$ such that $\mathbf{s}(q) \notin R(V)$, let

$$\phi(g) = e_g c_{(j-_m k)0} \sigma(qc_{j0})^*. \text{ If } \mathbf{s}(g) \notin R(V), \text{ we also let } \phi(\mathbf{s}(g)) = v_g.$$

Defining $\phi(g^*)$ as $\phi(g)^*$ ensures that the algebra extension of ϕ will be a $*$-homomorphism. It is direct to check that (V) holds. To check (E1), let g be an edge as above such that $\mathbf{s}(g) \notin R(V)$. If $k > 2$ and $q = hr$ for $h \in E^1$, we have that

$$\phi(g)\phi(\mathbf{r}(g)) = e_g c_{(j-_m k)0} \sigma(qc_{j0})^* v_h = e_g c_{(j-_m k)0} \sigma(qc_{j0})^* = \phi(g)$$

where the middle equality holds since $\sigma(qc_{j0})$ is a path which originates at the source of the new edge with v_h as its source, so $\sigma(qc_{j0})^* v_h = \sigma(qc_{j0})^*$. If $k = 2$, then $q = h$, $\sigma(qc_{j0}) = qc_{j0} = hc_{j0}$ and

$$\phi(g)\phi(\mathbf{r}(g)) = e_g c_{(j-_m k)0} c_{j0}^* h^* \phi(\mathbf{s}(h)) = e_g c_{(j-_m k)0} c_{j0}^* h^* \mathbf{s}(h) = e_g c_{(j-_m k)0} \sigma(hc_{j0})^* = \phi(g)$$

If $\mathbf{s}(g) \in R(V)$, the argument is very similar. The relation $\phi(\mathbf{s}(g))\phi(g) = \phi(g)$ is direct to check.

By the definition of ϕ on the ghost edges, (E2) holds since (E1) holds. If g', k', and j' are analogous to g, k, and j above, then (CK1) holds since

$$\sigma(q'c_{j'0})c^*_{(j'-_m k')0}e^*_{g'}e_g c_{(j-_m k)0}\sigma(qc_{j0})^*$$

is zero unless $g = g', j = j', k = k'$ and $q = q'$ and, in that case, the above expression is

$$\sigma(qc_{j0})c^*_{(j-_m k)0}e^*_g e_g c_{(j-_m k)0}\sigma(qc_{j0})^* = \sigma(qc_{j0})c^*_{(j-_m k)0}c_{(j-_m k)0}\sigma(qc_{j0})^* =$$

$$\sigma(qc_{j0})v_0\sigma(qc_{j0})^* = \mathbf{s}(\sigma(qc_{j0})) = \phi(\mathbf{s}(q)) = \phi(\mathbf{r}(g)).$$

If $v \in E^0$ is in $c^0 \cup R(V) - V$ or it is a source of a path in $\mathcal{P}_1^{v_j}$, then it is direct to check that (CK2) holds. Otherwise, v emits a single edge, say g with its range not in $R(V)$. There is unique k and j and unique path $q \in \mathcal{P}_{k-1}^{v_j}$ with $\mathbf{s}(q) = \mathbf{r}(g)$. If $\mathbf{s}(g) \in R(V)$ then $\phi(v) = v$ and $\phi(v) = v_g$ otherwise. In either case, $\phi(v)$ is a vertex of $E_{\text{can},V}$ which emits a single edge e_g, and $\phi(v) = \phi(g)\phi(g)^*$ since

$$e_g c_{(j-_m k)0}\sigma(qc_{j0})^*\sigma(qc_{j0})c^*_{(j-_m k)0}e^*_g = e_g c_{(j-_m k)0}v_0 c^*_{(j-_m k)0}e^*_g = e_g v_{j-_m k}e^*_g = e_g e^*_g.$$

By the Universal Property, ϕ extends to a homomorphism which is graded and a $*$-homomorphism by the definition of ϕ on vertices, edges and ghost edges. Using the inverse of σ, we can obtain the inverse of ϕ, so ϕ is a graded $*$-isomorphism.

If $m = 0$, we define a map ϕ on $E^0 \cup E^1$ by mapping the vertices and edges of E_V identically on themselves. So, for $p = gq \in \mathcal{P}_l^{v_0}$ with $0 < l \leq k$, $g \in E^1$, and $q \in \mathcal{P}_{l-1}^{v_0}$, it is sufficient to consider the case $\mathbf{s}(q) \notin R(V)$. In this case, we let

$$\phi(g) = e_g e_{l-1}e_{l-2}\dots e_0\sigma(q)^*. \text{ If } \mathbf{s}(g) \notin R(V), \text{ we also let } \phi(\mathbf{s}(g)) = v_g.$$

By this definition, $\phi(g)\phi(\mathbf{r}(g)) = e_g e_{l-1}e_{l-2}\dots e_0\sigma(q)^*v_q = e_g e_{l-1}e_{l-2}\dots e_0\sigma(q)^* = \phi(g)$ and checking that (E1) holds in other cases is direct. With $\phi(g^*) = \phi(g)^*$ and (E1) holding, (E2) holds. Checking (CK1) and (CK2) is similar to the $m > 0$ case. The argument that ϕ extends to a graded $*$-isomorphism is also the same as in the $m > 0$ case.

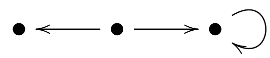
It remains to show the last sentence of the proposition. If E, F, and f are as specified, the previous part of the proposition establishes the existence of graph operations $\phi_E : E \to E_{\text{can}}$ and $\phi_F : F \to F_{\text{can}}$ which extend to graded $*$-isomorphisms. The existence of f and the proof of Proposition 3.4 implies that there is $\iota : E_{\text{can}} \cong F_{\text{can}}$ such that $\overline{\phi_F}f\overline{\phi_E}^{-1} = \overline{\iota}$. Hence, $f = \overline{\phi_F^{-1}\iota\phi_E}$ □

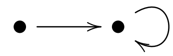
4. 2-S-NE graphs

In this section, we define a canonical form of a countable 2-S-NE graph and prove Theorem 4.13, the main result for this class of graphs.

If E is a 2-S-NE graph, there is an admissible pair (H, S) such that $(\emptyset, \emptyset) \leq (H, S) \leq (E^0, \emptyset)$ is a composition series of E. Since $E/(H, S)$ is cofinal, $S = B_H$. As H is nontrivial and $P_{(H,S)}$ is cofinal, $S = \emptyset$ and so $B_H = \emptyset$. The set H contains no infinite emitters as H is the saturated closure of a sink or a cycle without exits. Since E/H is cofinal, it has at most one sink, so $E^0 - H$ contains at most one infinite emitter v. Since E/H is row-finite, v does not emit infinitely many edges to $E^0 - H$. As $B_H = \emptyset$, v emits zero edges to $E^0 - H$, so v emits all the edges it emits to H. As both P_H and E/H are cofinal, E cannot have more than one infinite emitter, more than two cycles, or more than an infinite emitter and a cycle (the existence of any such elements would imply that the composition series is longer than two). The graph E has either one or two terminal clusters.

4.1. **2-S-NE graphs with two terminal clusters – the easy case.** If there are two terminal clusters c_1^0 and c_2^0, then E does not have an infinite emitter or a cycle with exits. The algebra $L_K(E)$ is graded $*$-isomorphic to the direct sum of $I(\overline{c_1^0})$ and $I(\overline{c_2^0})$. Say that $H = \overline{c_1^0}$ so that $I(\overline{c_2^0}) \cong_{\text{gr}} L_K(E/H)$. The total out-split E_{tot} in this case consists of two disconnected 1-S-NE graphs whose algebras are graded $*$-isomorphic to $I(\overline{c_1^0})$ and $I(\overline{c_2^0})$, so we can consider E_{tot} instead of E. For example, if E is the graph



then E_{tot} consists of two connected components and  .

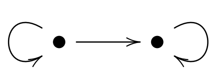
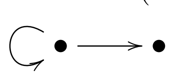
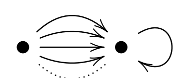
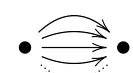
If $E = E_{\text{tot}}$ is a 2-S-NE graph with two terminal clusters and E_1 and E_2 are the two disconnected graphs forming E, we can consider the 1-S-NE canonical forms of E_1 and E_2 and we let E_{can} be the union of these two canonical forms. With this definition, if E and F are two 2-S-NE graphs with two terminal clusters and if $F_{\text{can}} = F_1 \cup F_2$ and $F_1 \cap F_2 = \emptyset$, we define the relation $E \approx F$ by $E_1 \approx F_1$ and $E_2 \approx F_2$ or $E_1 \approx F_2$ and $E_2 \approx F_1$. Thus, we have that

$$E \approx F \quad \Longleftrightarrow \quad E_{\text{can}} \cong F_{\text{can}}.$$

4.2. **2-S-NE graphs with a unique terminal cluster.** Having the easy case out of the way, we remain primarily interested in the case when E has only one terminal cluster. In this case, H is the saturated closure of that cluster and $E^0 - H$ contains an infinite emitter or a cycle with exits to H. We introduce some notation which we use for this type of graphs. Let c^0 be the terminal cluster of E/H and d^0 be the terminal cluster of E (so $H = \overline{d^0}$). Recall that we allow the case that c and d are trivial cycles so that this scenario encompasses the situation that c^0 is an infinite emitter or that d^0 is a sink. Since d^0 is the only terminal cluster of E, $c^0 \subseteq R(H)$ (otherwise c^0 would be the terminal cluster of E also), so there are paths from c^0 to d^0. We say that p is a *c-to-d path* if the source of p is in c^0, the range in d^0 and no edge of p is on c or d. The examples with the Toeplitz graph from the introduction illustrates that graphs whose algebras are graded $*$-isomorphic may have different number of c-to-d paths of certain length.

Let $n = |d|$ and $m = |c|$. If $m > 0$, then c is a proper cycle with at least one exit to H and such that all its exits have ranges in H. The cycle c is without exits in E/H and E is row-finite. In this case, E is a *cycle-to-cycle* graph if $n > 0$ and a *cycle-to-sink* graph if $n = 0$. If $m = 0$, c is a trivial cycle and $v_0 = \mathbf{s}(c)$ is an infinite emitter of E which emits all of its edges to H and which is a sink in E/H. In this case, E is an *infinite-emitter-to-cycle* graph if $n > 0$ and an *infinite-emitter-to-sink* graph if $n = 0$. If $m = 0$, we say that the edges which v_0 emits are *exits*. This enables us to unify the terminology in the $m = 0$ and $m > 0$ cases.

For example, the first graph below is cycle-to-cycle ($m > 0$ and $n > 0$), the second (the Toeplitz graph) is cycle-to-sink ($m > 0$ and $n = 0$), the third is infinite-emitter-to-cycle ($m = 0$ and $n > 0$), and the fourth is infinite-emitter-to-sink ($m = n = 0$).

For the rest of this section, let $E = E_{\text{tot}}$ be a 2-S-NE graph with a single terminal cluster, H be its nonempty and proper hereditary and saturated set, $m = |c|$ be the length of the terminal cycle c of the 1-S-NE invariant of E/H, and $n = |d|$ be the length of the terminal cycle d of the 1-S-NE graph P_H. Since $E = E_{\text{tot}}$, every vertex which not terminal for neither E nor E/H emits exactly one edge and any non-terminal vertex of E/H (if any) emits its only emitted edge to $E^0 - H$.

We start by a series of operations on 2-S-NE graphs. Some of these operations are out-splits and out-amalgamations, so they belong to the class characterized as the "graph moves" of symbolic dynamics. However, some of the operations we consider are not "moves" in the sense used in the current literature. All of these operations have properties described in section 1.4, so they induce a graded $*$-isomorphism of the graph algebras. The process from section 3.4 of transforming a 1-S-NE graph into its canonical form is an example of such an operation for 1-S-NE graphs.

4.3. **Direct-exit forms and connecting matrix.** Let $E = E_{\text{tot}}$ be such that d is a proper cycle (so $n > 0$). Consider the porcupine graph P_H and let V be the set of all vertices of P_H which are not in H. Thus, $V = R(V)$. We define a 2-S-NE graph E_{dir} which we call the *direct-exit form* of E. We let E_{dir} consists of the part of $(P_H)_{\text{can},V}$ (defined in section 3.4) outside of $R(c^0)$ and we let the rest of E_{dir} be E/H. If ϕ_H is the graded $*$-isomorphism of P_H and $(P_H)_{\text{can},V}$ (see section 3.4 and Proposition 3.5), we define a map ϕ on the vertices and edges of E by

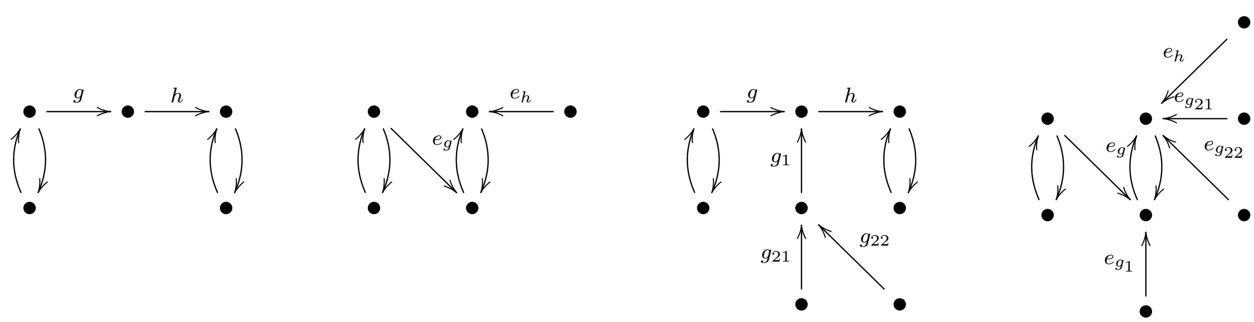
$$\phi(v) = \phi_H(v) \text{ if } v \in H \quad \text{and} \quad \phi(v) = v \text{ if } v \in E^0 - H \text{ and}$$

$$\phi(e) = \phi_H(e) \text{ if } \mathbf{r}(e) \in H \quad \text{and} \quad \phi(e) = e \text{ if } \mathbf{r}(e) \in E^0 - H.$$

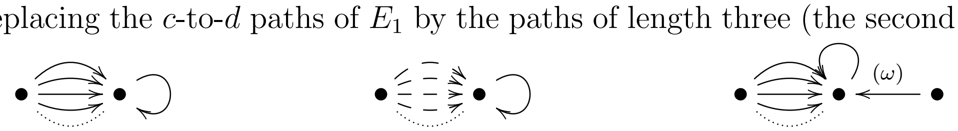
It is direct to check that the axioms hold for these images by the definition of $(P_H)_{\text{can},V}$ and ϕ_H. Thus, ϕ extends to a graded $*$-homomorphism which is invertible since ϕ_H is invertible.

We point out some properties of E_{dir}. First, all c-to-d paths have length one and the notation "dir" for "direct" emphasized that the c-to-d paths are as direct as possible. Second, all edges with sources in $H - d^0$ end at vertices of d. We refer to such edges ending at w_i as the *i-tails* of E_{dir}.

For example, the second graph is the direct-exit form of the first and the fourth graph is such form of the third graph below.



Let us consider an example with $m = 0$. Let E_1 be the first graph below and let E_3 be the graph obtained by replacing the c-to-d paths of E_1 by the paths of length three (the second graph below).



We have that $E_1 = (E_1)_{\text{dir}}$ and the last graph is the direct-exit form of E_3. If E_k is obtained by replacing the c-to-d paths of E_1 by the paths of length $k > 1$, the last graph is the direct-exit form of E_k also. The Leavitt path algebras of E_1 and of $E_k, k > 1$ are not graded $*$-isomorphic because the first algebra is unital and the second is not. A direct argument for the algebras of E_3 and E_k being graded $*$-isomorphic is that the two graphs have the same direct-exit form.

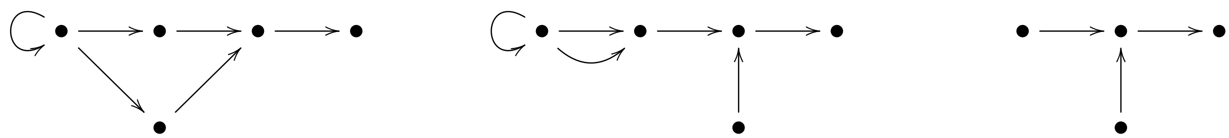
If E is a direct-exit graph with $n > 0$, we let a_{ji}, $j \in m, i \in n$, be the number of edges v_j emits to w_i. If $m > 0$ this number is finite and we refer to the $m \times n$ matrix $[a_{ji}]$ as the c-to-d *connecting matrix*. This matrix is dependent on the choice of $d \in [d]$ and $c \in [c]$, so the reference to c and

d is needed. If $c' \in [c]$ is a different element, the c'-to-d connecting matrix can be obtained by applying a degree m cyclic permutation of rows of the original matrix. Similarly, if $d' \in [d]$ is a different element then the c-to-d' connecting matrix is obtained by permuting the columns of the c-to-d matrix by a cyclic permutation of length n.

If $m = 0$, we allow ω to be the entry of a connecting matrix. In this case, the connecting matrix is a $1 \times n$ matrix over $\omega \cup \{\omega\}$ such that a_{0i} is the cardinality of the exits ending at w_i.

Let us move on to the $n = 0$ case. If $E = E_{\text{tot}}$ has a sink, we define E_{dir} analogously by letting $V = R(V)$ be the set of vertices of P_H which are not in H and pasting the graphs $(P_H)_{\text{can},V}$ and E/H together using Proposition 3.5 just as in the $n > 0$ case.

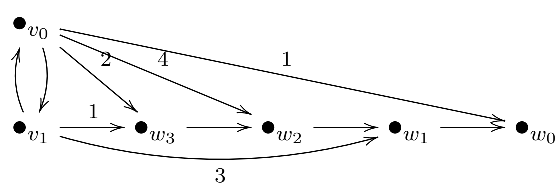
We also let H_t (where t is for "tails") be the subgraph of E_{dir} generated by H and k_t be its spine length. The graph H_t is a 1-S-NE graph and w_0 is its sink. We say that H_t is the *tail graph.* For example, if E is the first graph below, the second graph is E_{dir} and the third graph is H_t.



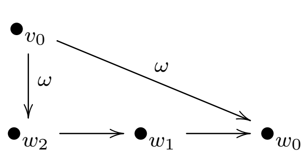
Let w_i be the vertex on the spine of H_t which is at length i from w_0. If the set of i-values such that w_i receives an exit has the largest element k, we say that k is the *spine length of* E. The path from w_k to w_0 is the *spine of* E. In the example above, $k = k_t = 2$. Some of the following examples exhibit graphs with $k < k_t$.

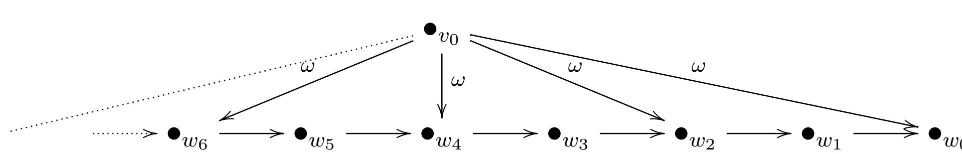
If $m > 0$ and $n = 0$, k is necessarily finite. If $m = n = 0$, the set of i-values such that w_i receives an exit may or many not have the largest element. If there is no largest element, we say that the spine of E is of *infinite length* (in which case the tail graph also has infinite spine length $k_t = \omega$). The left-infinite path passing w_i for every $i \in \omega$ is the *spine of* E. We exhibit some examples below.

For a cycle-to-sink direct-exit graph with a spine of length k, we define its c-to-d *connecting matrix* as the $m \times (k+1)$ matrix $[a_{ji}]$ where a_{ji} is the number of edges v_j emits to w_i. It is dependent on the choice of $c \in [c]$ and a different choice of an element of $[c]$ results in a matrix with rows permuted by a cyclic permutation of degree m. For example, below is an $m = 2, n = 0$ direct-exit graph with the spine of length $k = 3$ and the connecting matrix $\begin{bmatrix} 1 & 0 & 4 & 2 \\ 0 & 3 & 0 & 1 \end{bmatrix}$.

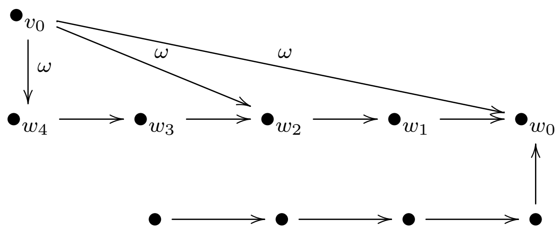


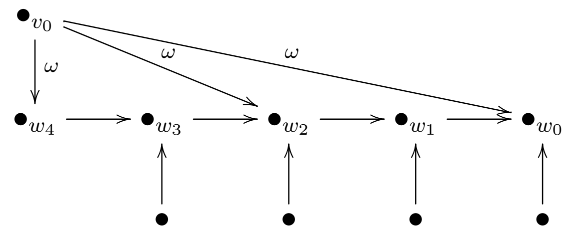
If $n = m = 0$ and if k is the spine length (possibly ω), the connecting matrix is a $1 \times (k+1)$ matrix with the cardinality of the exits ending at w_i at the i-th spot. The first graph below is a direct-exit graph with a finite spine of length two and connecting matrix $[\omega\ 0\ \omega]$. The second graph below is a direct-exit graph with the spine length infinite and the connecting matrix $[\omega\ 0\ \omega\ 0\ \omega\ 0\ \ldots]$.



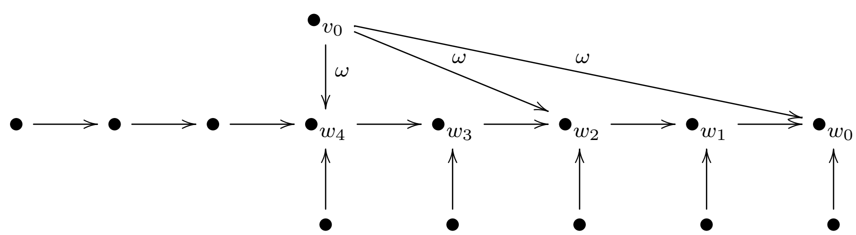


If E is the first graph below, its c-to-d part is "direct" but the path originating at a source in H and ending in w_0 is not "canonical". The second graph is E_{dir}.





If the length of the path with all its vertices not in $T(v_0)$ which ends at w_0 were 7 instead of 4, the following graph would be E_{dir}. This graph is also an example of a graph with $k = 4 < k_t = 7$.



4.4. **Connecting polynomial.** Let $E = E_{\mathrm{dir}}$ be a direct-exit 2-S-NE graph with $m > 0$ and let k be the spine length in the case that $n = 0$. Let

$$a_E(t) = \sum_{j \in m, i \in n} a_{ji}\, t^{j+1+n-_n i} \text{ if } n > 0 \quad \text{and} \quad a_E(t) = \sum_{j \in m, i \leq k} a_{ji}\, t^{j+1+i} \text{ if } n = 0.$$

We refer to a_E as a *connecting polynomial*. It depends on $c \in [c]$ and $d \in [d]$ but if it is clear which cycles we use, we will shorten the notation to a_E. The following lemma focuses on this polynomial.

Lemma 4.1. *Let E be a direct-exit 2-S-NE graph with $m > 0$ and let a_E be its connecting polynomial computed using c and d. If $v_0 = \mathbf{s}(c)$ and $l \in \mathbb{Z}^+$, then*

$$[v_0] = t^{lm}[v_0] + \sum_{j=0}^{l-1} t^{jm} a_E[w_0].$$

Proof. We have that $v_0 \to t^m v_0 + a_E w_0 \to t^{2m} v_0 + t^m a_E w_0 + a_E w_0 \to \ldots \to t^{lm} v_0 + \sum_{j=0}^{l-1} t^{jm} a_E w_0$ holds in $\mathcal{F}_E^\Gamma$ for any nonegative integer l, so $[v_0] = t^{lm}[v_0] + \sum_{j=0}^{l-1} t^{jm} a_E[w_0]$ holds in M_E^Γ. □

We consider the $m = 0$ case next and prove a lemma analogous to Lemma 4.1. If $m = 0$, the concept of a single connecting polynomial is replaced by polynomials defined for each nonempty and finite $Z \subseteq \mathbf{s}^{-1}(v_0)$ where v_0 is the infinite emitter. For such Z, let P_Z be the set of paths in $P^{w_0}_{\not d}$ which have the first edge in Z and let $a_Z \in \mathbb{Z}^+[t]$ be the polynomial $\sum_{p \in P_Z} t^{|p|}$. We let $q_\emptyset$ denote v_0 to unify the treatment. If we define the relation $\leq$ on the set of polynomials such that $a \leq b$ if $a + c = b$ for some polynomial c, we have that $Z \subseteq W$ if and only if $a_Z \leq a_W$.

Lemma 4.2. *Let $E = E_{\mathrm{dir}}$ be a 2-S-NE graph with $m = 0$ and let v_0 be the infinite emitter of E. For any finite $Z \subseteq \mathbf{s}^{-1}(v_0)$, $[v_0] = [q_Z] + a_Z[w_0]$.*

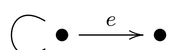
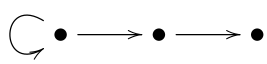
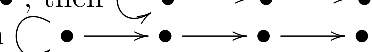
Let $b \in \mathbb{Z}^+[t]$, $[x] \in M_E^\Gamma$, and let Z be finite subset of $\mathbf{s}^{-1}(v_0)$. If $[q_Z] = [x] + b[w_0]$ holds, then there is finite $W \subseteq \mathbf{s}^{-1}(v_0)$ disjoint from Z such that $[x] = [q_{Z \cup W}]$ and $b = a_W[w_0]$.

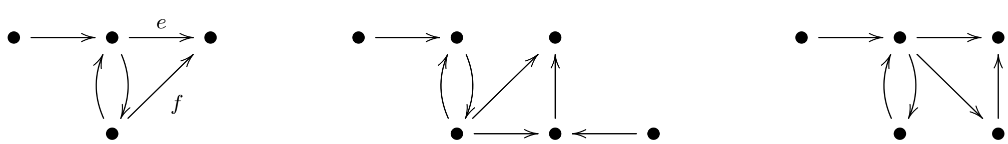
Proof. The first part holds since $v_0 \to q_Z + a_Z w_0$ holds in $\mathcal{F}_E^\Gamma$. To show the second part, assume that $[q_Z] = [x] + b[w_0]$ holds for some $b, [x]$, and Z as in the statement of the lemma. This implies that $q_Z \to y$ and $x + bw_0 \to y$ for some $y \in \mathcal{F}_E^\Gamma$ by the Confluence Lemma (Lemma 2.4). The first relation implies that such y can be chosen to have the form $y = q_{Z\cup W'} + a_{W'}t^{kn}w_0$ for some finite W' disjoint from Z and some nonnegative integer k. Since $a_{W'}t^{kn}[w_0] = a_{W'}[w_0]$ and w_0 is a terminal vertex, the relation $x + bw_0 \to y$ implies that such W' can be chosen so that $b[w_0]$ is a summand of $a_{W'}[w_0]$. As any summand of $a_{W'}[w_0]$ is of the form $a_W[w_0]$ for some finite subset W of W', we have that $b[w_0] = a_W[w_0]$ for some such W. Thus, we have that

$$[x] + a_W[w_0] = [x] + b[w_0] = [q_Z] = [q_{Z\cup W}] + a_W[w_0].$$

By canceling $a_W[w_0]$, we obtain that $[x] = [q_{Z\cup W}]$. □

4.5. **The exit moves.** Let E be a direct-exit graph with $m > 0$ and let $c \in [c]$ be such that $v_j \in c$ emits an exit e. We consider the out-split with respect to $\{e\}$ and $\mathbf{s}^{-1}(v_j) - \{e\}$ followed by the total out-split of the new graph. We continue to use the label v_j for the second new vertex in the out-split because this vertex is between v_{j-_m1} and v_{j+_m1} in the cycle with exits of the new graph which we continue to call c. With these labels, the parts E/H and d are unchanged in the out-split graph. If v_e is the second new vertex, it receives one new edge g_e from v_{j-_m1} and a copy of E^{v_j} ending at $\mathbf{s}(g_e)$ instead of v_j. The vertex v_e emits the edge which we continue to call e since it ends in $\mathbf{r}(e)$. We refer to the resulting graph as the *e-blow-up* of E.

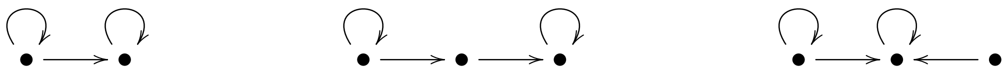
For example, if E is the Toeplitz graph  , then  is the e-blow-up of E. If f is the newly added exit from the cycle, then  is the f-blow-up of the previous graph. As another example, let E be the first graph below. The second graph is the e-blow-up and the third graph is the f-blow-up.



If $m = 0$ and e is an edge v_0 emits, the *e-blow-up* is the graph obtained by the out-split with respect to $\{e\}$ and $\mathbf{s}^{-1}(v_0) - \{e\}$ and then considering the total out-split of the obtained graph. For example, if E is the first graph below and e is the first edge of the c-to-d paths of length two, then the second graph is the e-blow-up of E.

Performing a blow-up with respect to an exit e results in a graph which may not be direct-exit any more. Let E_e be the graph obtained by transforming the resulting graph to a direct-exit graph. We call such an operation an *exit move* and write $E \to_1 E_e$.

For example, if E is the first graph below, the second graph is the blow-up with respect to the only exit and the third graph is the direct form of the second, hence, the exit-move of E.



If $E \to_1 E_e$ and $E_e \to_1 F = (E_e)_g$ for some exit g, we write $E \to_2 F$. For example, if both exits of the first graph are moved, the second graph is obtained.

We consider the impact of an exit move to the number of tails and the connecting matrix. If $n > 0$, we let l_i denote the cardinality of the set of i-tails for $i \in n$. If $n = 0$ and k_t is the spine of the tail-graph H_t, we let l_i denote the cardinality of the set of i-tails for $i \leq k_t$.

An exit with source $v_j, j \in m$ and range w_i is an ji-*exit* where $i \in n$ if $n > 0$ and $i \leq k$ if $n = 0$. If $m > 0, n > 0$ and an ji-exit e is moved, the only change in the connecting matrix is that the a_{ji}-value decreases by one and the $a_{(j-_m 1)(i-_n 1))}$-value increases by one. The cardinality l_i increases by at least one (corresponding to e). To describe the increase in the number of i'-tails, we extend the definition of $\mathcal{P}_k^{v_j}$ to include the value $k = 0$ and let this set be the moved exit e. If $\mathcal{P}_k^{v_j}$ is nonempty and $i' \in n$ is such that $k + i' = i (\operatorname{mod} n)$, then $w_{i'}$ gets $|\mathcal{P}_k^{v_j}|$ new tails. Thus, the number of i'-tails in the resulting graph is

$$l_{i'} + \sum_{k \in \{k | k = i - i' (\operatorname{mod} n)\}} |\mathcal{P}_k^{v_j}| \tag{1}$$

where the standard cardinal arithmetic holds if either $l_{i'}$ or $|\mathcal{P}_k^{v_j}|$ is infinite.

If $m > 0$, $n = 0$, k_t is the spine length of the tail graph H_t, k_j is the spine length of E^j, and a ji-exit is moved for some $j \in m, i \leq k$, then a_{ji} decreases by one and $a_{(j-_m 1)(i+1)}$ increases by one. If $i = k$, the length of the spine of the new graph increases by one. If $k_j > k_t$, the length of the spine graph increases. The number of i'-tails in the new graphs is equal to

$$l_{i'} + |\mathcal{P}_{k'}^{v_j}| \quad \text{for } k' \text{ such that } k' = i' - i. \tag{2}$$

Formula (1) has $i - i'$ and formula (2) has $i' - i$. This difference is present since the distance from w_i to w_0 is $n - i$ if $n > 0$ and it is i if $n = 0$.

If $m = 0$ and $n > 0$ and if a $0i$-exit is moved, the $0i$-value of the connecting matrix of the resulting graph is $a_{0i} - 1$ (here we use cardinal arithmetic to have that $\omega - 1 = \omega$ if $a_{0i} = \omega$). No other entries of the matrix are changed. The number of i'-tails in the new graph is given by the same formula as in the $n > 0$ and $m > 0$ case except that the only possible value of j is 0.

If $m = n = 0$, k is the spine length, and a $0i$-exit is moved, the $0i$-value of its connecting matrix is $a_{0i} - 1$. The formula for the number of the i'-tails in the new graph is the same as in the $m > 0$ and $n = 0$ case except that the only possible j-value is zero.

Because $\mathcal{P}_0^{v_j}$ is nonempty, if l_i is finite, then a move of any ji-exit increases the number of i-tails in the resulting graph. If l_i is infinite, the move of an exit may produce a graph isomorphic to the original graph. Example 4.3 below exhibits some such graphs.

Example 4.3. Moving any of the exits of the two graphs below produces a graph isomorphic to the initial graph.

We prove a short lemma we use for the cycle-to-cycle graphs.

Lemma 4.4. *Let $E=E_{\text{dir}}$ be a cycle-to-cycle graph with $|c|=m, |d|=n$, and let G be the greatest common divisor of m and n. A ji-exit can be moved to become a lk-exit of the resulting graph if and only if $i-j=l-k(\operatorname{mod} G)$.*

Proof. Let $m=Gm'$ and $n=Gn'$ for m' and n' which are mutually prime.

By the definition of an exit move, a ji-exit can be moved to become a lk-exit of the resulting graph if and only if there is $k'\in\mathbb{Z}$ such that $j-k'=k(\operatorname{mod} m)$ and $i-k'=l(\operatorname{mod} n)$. Assuming this holds, let $k'=j-k+m_0m=i-l+n_0n$ for some $m_0,n_0\in\mathbb{Z}$. We have that $j-k=i-l+G(n_0n'-m_0m')$ which shows that $j-k=i-l(\operatorname{mod} G)$ and so $i-j=l-k(\operatorname{mod} G)$.

Conversely, suppose that $i-j=l-k(\operatorname{mod} G)$ so that $j-k=i-l(\operatorname{mod} G)$ and let $j-k=i-l+k''G$ for some $k''\in\mathbb{Z}$. Since m' and n' are mutually prime, let $m_0,n_0\in\mathbb{Z}$ be such that $1=m'm_0+n'n_0$. Hence, $k''G=k''Gm'm_0+k''Gn'n_0=k''mm_0+k''nn_0$ so that $j-k=i-l+k''mm_0+k''nn_0$. Let $k'=i-l+k''nn_0$ so that $j-k=k'(\operatorname{mod} m)$ and that $i-l=k'(\operatorname{mod} n)$. Thus, $j-k'=k(\operatorname{mod} m)$ and $i-k'=l'(\operatorname{mod} n)$ showing that a ji-exit can be moved to become a lk-exit. □

4.6. **Reduction and reducibility.** We would like to introduce the inverse of an exit-move which we refer to as *reduction*. Let us start with such consideration for cycle-to-cycle graphs.

Assume that E is a cycle-to-cycle graph for which there are $j\in m$ and $i\in n$ such that $a_{(j-_m1)(i-_n1)}\neq 0$ and such that

$$l_{i'}\geq \sum_{k\in\{k|k=i-i'(\operatorname{mod} n)\}}|\mathcal{P}_k^{v_j}| \tag{3}$$

holds for every $i'\in n$. In this case, there is an operation inverse to a move of a ji-exit and we say that E is *ji-reducible*. Note that the relation (3) trivially holds for i' such that $l_{i'}=\omega$. We say that E is *reducible* if there are $i\in n$ and $j\in m$ such that E is ji-reducible.

If E is ji-reducible and $[a_{ji}]$ is the connecting matrix of E corresponding to $c\in[c]$ and $d\in[d]$, we define *a single ji-reduction* $E_{\text{red},1,ji}$ of E as follows. The quotient of $E_{\text{red},1,ji}$ is the same as the quotient E. The c-to-d part of $E_{\text{red},1,ji}$ is determined by the connecting matrix $a'_{j'i'}$ given by

$$a'_{ji}=a_{ji}+1,\quad a'_{(j-_m1)(i-_n1)}=a_{(j-_m1)(i-_n1)}-1 \text{ and}$$
$$a'_{j'i'}=a_{j'i'} \text{ if } (j'\neq j \text{ or } i'\neq i) \text{ and } (j'\neq j-_m1 \text{ or } i'\neq i-_n1).$$

The graph $E_{\text{red},1,ji}$ has the number of i'-tails equal to

$$l_{i'}-\sum_{k\in\{k|k=i-i'(\operatorname{mod} n)\}}|\mathcal{P}_k^{v_j}|$$

if $l_{i'}$ is finite and it has ω i'-tails otherwise. If $l_{i'}$ is finite, the cardinal subtraction is subtraction of finite numbers because if $l_{i'}$ is finite and relation (3) holds, then $|\mathcal{P}_k^{v_j}|$ is finite for any k such that $k=i-i'(\operatorname{mod} n)$. The scenario when both sides of relation (3) are ω is considered in section 4.7.

By the definition of $E_{\text{red},1,ji}$, we have that $E_{\text{red},1,ji}\to_1 E$ holds. Hence, the algebras of E and $E_{\text{red},1,ji}$ are in the same graded $*$-isomorphism class. Thus, if E is ji-reducible, there is a graph F such that $F\to_1 E$ is the move of a ji-exit. The converse also holds: if there is such a graph F, then $a_{(j-_m1)(i-_n1)}\neq 0$ and relation (3) holds. We use this condition to define reducibility for S-NE graphs with composition length larger than two in section 5.

If $E_{\text{red},1,ji}$ is $j'i'$-reducible, we let $E_{\text{red},2,ji,j'i'}$ be the graph $(E_{\text{red},1,ji})_{\text{red},1,j'i'}$. Continuing this argument, we define $E_{\text{red},l,j_1i_1,\ldots,j_li_l}$ as $(E_{\text{red},l-1,j_1i_1,\ldots,j_{l-1}i_{l-1}})_{\text{red},1,j_li_l}$. We write $E_{\text{red},l,ji}$ for the graph

$E_{\text{red},l,(j-_m(l-1))(i-_n(l-1))\ldots,(j-_m1)(i-_n1),ji}$ where $E_{\text{red},l,(j-_m(l-1))(i-_n(l-1))\ldots,(j-_m1)(i-_n1),ji} \to_l E$ is the move of the same ji-exit l times.

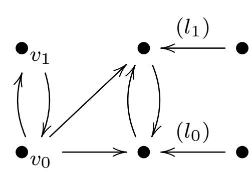
Let E be a graph in which l_i is finite for at least one i. If E is not reducible, we say that E is *reduced* and write $E = E_{\text{red}}$. If E is reducible, the process of reduction terminates after finitely many steps producing a *reduction* E_{red} of E. If E is a graph with $l_i = \omega$ for all $i \in n$, we say that E is ji-reduced for every $j \in m$ and $i \in n$. We also say that E is *reduced* and write $E = E_{\text{red}}$. Thus, the case $l_i = \omega$ for all $i \in n$ is the only case when a graph is *both reduced and reducible.*

We are interested in one particular type of a reduced graph – a graph reduced only with respect to full revolutions of exits. If $m > 0$, $n > 0$, and if L is the least common multiple of m and n, let $N_{jii'}$ be the number of i'-tails added to $w_{i'}$ when a ji-exit is moved L-times. An explicit formula for $N_{jii'}$ can be obtained using formula (1) but we do not display it since we do not use it.

We repeat the process of defining a reducible and a reduced graph, but by considering reduction of exits only for a full L-revolution instead of a single exit move. In particular, if E is such that there are $j \in m$ and $i \in n$ such that $a_{ji} \neq 0$ and such that $l_{i'} \geq N_{jii'}$ holds for every $i' \in n$, then E is L, ji*-reducible.* We say that E is *L-reducible* if it is L, ji-reducible for some $i \in n, j \in m$.

Let E be a graph in which l_i is finite for at least one i. If E is not L-reducible, then E is *L-reduced.* If E is L-reducible, the process of reduction terminates after finitely many steps and it produces an L-reduced graph which we refer to as an *L-reduction* of E. If E is a graph with $l_i = \omega$ for all i, we say that E is *L-reduced.* We can unify the two cases by stating that E is L-reduced if and only if for every $j \in m$ and $i \in n$, if $E_{\text{red},L,ji}$ is defined, then $E \cong E_{\text{red},L,ji}$.

Let E be the graph below where l_0 and l_1 countable cardinalities. This graph is reduced if and only if both l_0 and l_1 are in $\{0, \omega\}$. This is because if $0 < l_0 < \omega$, then E can be 01-reduced and, if $0 < l_1 < \omega$ then E can be 00-reduced. Two consecutive moves of any of the exits create one 0-tail and one 1-tail, so E is 2-reduced if and only if at least one of l_0 and l_1 is in $\{0, \omega\}$.



Next, let E be a graph with an infinite-emitter. If there is F such that $F \to_1 E$ and E is obtained by a move of a $0i$-exit, we say that E is $0i$-reducible. We define a *$0i$-reduction, $0i$-reducibility,* and *being reduced* analogously as for $m > 0$. Instead of L-reductions, we consider reductions with respect to $0i$-exits such that $a_{0i} = \omega$ and we call such reductions *ω-reductions.* Thus, E is ω-reducible if there is a graph F such that $F \to_1 E$ holds and it is a move of a $0i$-exit with $a_{0i} = \omega$. We say that E is *ω-reduced* if E is either not ω-reducible or if it is isomorphic to any of its ω-reductions. For example, let E be an infinite-emitter-to-sink graph with quotient consisting of a single vertex, with the infinite spine length, and such that $a_{0i} = a'_{0i} = \omega$ for all $0 \neq i \in \omega$ and with $a_{00} < \omega$. Then E is reduced if any only if $l_i \in \{0, \omega\}$ for $i \geq 0$ and E is ω-reduced if and only if $l_i \in \{0, \omega\}$ for $i > 0$.

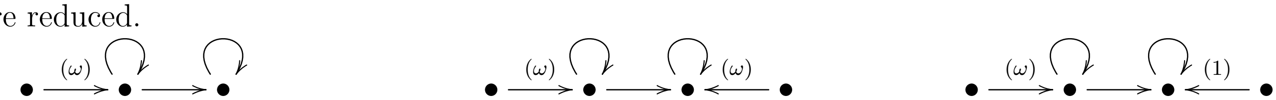
A cycle-to-sink graph E is *ji-reducible* if $F \to_1 E$ for a graph F such that E is obtained by moving a ji-exit of F where $j \in m$ and i is less than or equal to the spine length of F. We define the concepts of a *ji-reduction,* of *ji-reducibility* and of *being reduced* analogously as when $n > 0$.

If k is the spine length of a cycle-to-sink graph E, and if all exits except those with range w_k are moved until they end in w_k, then the spine length of the resulting graph is still k. We are interested in graphs reduced up to the exit moves which would increase the spine length. In particular, we say that E is *spine-reducible* if there is F and l such that $F \to_l E$ and F has shorter spine length. If E is not spine-reducible, we say that E is *spine-reduced*.

For example, let us consider the class of graphs with their quotients being a single loop and with H_t having the spine of infinite length with infinitely many i-tails for all $i \in \omega$. Let E_k be one such graph with the connecting matrix $[a_{0i}]$, $i = 0, \ldots, k$ and with $k > 0$. Let E_{k-1} be another such graph with the same quotient and the tail graph as E_k and with the first $k-2$ entries of the connecting matrix the same, with the spine length $k-1$ and with $a_{0(k-1)} + a_{0k}$ at the $k-1$-spot of the connecting matrix. The graph E_k is the result of moving a_{0k}-many of $0(k-1)$-exits of E_{k-1}. We can continue this process to obtain E_{k-2} so that $E_{k-2} \to_{a_{k-1}} E_{k-1}$ and so on until E_0 is obtained. The spine length of E_0 is zero, its connecting matrix is $\left[\sum_{i=0}^{k} a_{0i}\right]_{1\times 1}$, and E_0 is spine-reduced.

4.7. **Tails cutting.** Reducing a graph limits the number of tails if the number of tails obtained by an exit move is not infinite. To be able to control this case also, we consider another type of tail minimization. The following example illustrates this.

Example 4.5. Let E, E' and F be the three graphs below. The graph E' is obtained by moving the only exit of E. The graph E' is *also* obtained by moving the (only) exit of F. All three graphs are reduced.



Let v_0 be the exit-emitter in all three graphs and let w_0 be the vertex of the terminal cycle. Let $\phi_E : E \to E'$ and $\phi_F : F \to E'$ be the exit moves. They both induce $\mathbf{POM}^D$-isomorphisms given by $[w_0] \mapsto [w_0]$ and $[v_0] \mapsto [v_0] + t[w_0] = [v_0] + [w_0]$ with inverses such that $[v_0] \mapsto t[v_0] + t[w_0] = t[v_0] + [w_0]$. So, the $\mathbf{POM}^D$-isomorphism $f = \overline{\phi}_E^{-1}\overline{\phi}_F : M_F^\Gamma \to M_E^\Gamma$ has $[v_0] \mapsto [v_0]$ and, hence, it is the identity map on the monoid level but not on the graph or the algebra levels. If g is the exit, $\{g_n \mid n \in \omega\}$ is the set of 0-tails of v_0, $u_n = \mathbf{s}(g_n)$, f_1 is the 0-tail of F, and $w_1 = \mathbf{s}(f_1)$, a graded $*$-isomorphism which realizes f can be obtained by

$$w_1 \mapsto g_1 g g^* g_1^*,\ f_1 \mapsto g_1 g d^*,\ u_n \mapsto u_n - g_n g g^* g_n^* + g_{n+1} g g^* g_{n+1}^* \text{ and } g_n \mapsto g_n - g_n g g^* + g_{n+1} g g^*$$

and by mapping all other vertices and edges of F identically onto themselves. It is direct to check that such a correspondence extends to a graded $*$-monomorphism with the inverse such that $u_1 \mapsto u_1 - g_1 g g^* g_1^* + w_1$, $g_1 \mapsto g_1 - g_1 g g^* + f_1 d g^*$, and

$$u_n \mapsto u_n - g_n g g^* g_n^* + g_{n-1} g g^* g_{n-1}^* \text{ for } n > 1, \text{ and } g_n \mapsto g_n - g_n g g^* + g_{n-1} g g^* \text{ for } n > 1.$$

In examples like these, if E' is obtained from an exit move which increased the number of i-tails by ω for some i, we would like to highlight the fact that E is obtained from E' by the inverse of such exit-move which is decreasing the number of tails by the *maximal* possible number. In contrast, the inverse of the exit move producing F from E' is not such. Based on this, we define a cut graph as follows. Note that the graphs E and F have isomorphic quotients and equal connecting matrices so only the number of tails differentiates them.

If graphs have a sink, the tail cutting process requires some additional consideration so we consider this case first. We let $E_{\text{cut}} = E$ if E/H is finite. Let k_t be the length of the tail graph and let $C(i)$ stands for the following statement for $i > 0$.

$C(i)$ There are $j, j' \in m, i' \in k_t$, and $k', l \in \omega$ such that $|\mathcal{P}_l^{v_j}| = \omega, a_{j'i'} \neq 0$, and $l + k'|c| + d(j, j') + i' = i$.

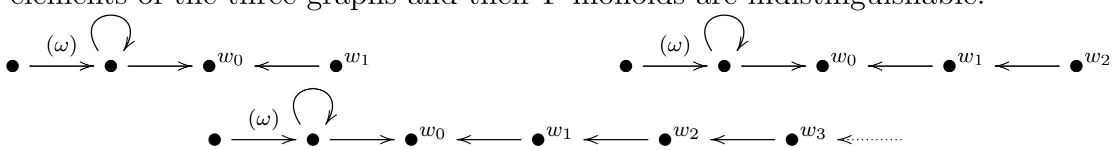
If $C(i)$ holds, we say that i-tails are *cuttable*. Note that here we are treating the edge of the spine from w_{i+1} to w_i as an i-tail. This will enable us to decrease the length of the spine of the tail graph if $(k_t - 1)$-tails are cuttable.

If $m = 0$, the condition $C(i)$ simplifies to:

$$\text{There are } i' \in k_t, \text{ and } l \in \omega \text{ such that } |\mathcal{P}_l^{v_0}| = \omega,\ \ a_{0i'} \neq 0,\ \text{ and }\ \ l + i' = i.$$

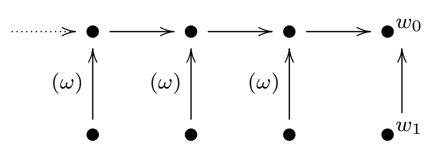
The following example illustrates our approach to the tail cutting in the $n = 0$ case.

Example 4.6. Let E_1, E_2 and E_ω be the three graphs below. The tail graph is the only distinguishing elements of the three graphs and their Γ-monoids are indistinguishable.



The quotient graph has spine length one. We consider one edge of the quotient graph g_t to be the spine of the quotient graph and we let $u_t = \mathbf{s}(g_t)$. Let u_n be the other sources of the quotient graph and g_n be the edges for $n \in \omega$. Let e be the loop and g the exit in each graph and let f_n be the edge which w_n emits in each graph for $n = 1$ in E_1, and $n = 1, 2$ in E_2, and $n > 0$ in E_ω.

In this case, all i-tails for $i > 0$ are cuttable in E_2 and E_ω. These graphs have no tails for $i > 0$ but we can shorten the tail graph spine. The porcupine graphs of all three graphs are 1-S-NE equivalent to the graph below and the existence of a bijection of the sets P^{w_0} of these graphs enables us to define the tail cutting maps.



We obtain one such bijection by matching f_2f_1 of E_2 with g_0g of E_1, and g_0g of E_2 with g_1g of E_1. Continuing this consideration, we match g_ng of E_2 with $g_{n+1}g$ of E_1 for $n > 0$. This bijection gives rise to the tail-cutting operations $\iota_{\text{cut}} : E_2 \to_{\text{cut}} E_1$ and tail-creating operation $\iota_{\text{cut}}^{-1} : E_1 \to_{\text{cut},-1} E_2$ defined similarly as in Example 4.5. The vertices v_0, w_0, w_1, and u_t and the edges e, g, f_1, and g_t are mapped identically on themselves by both maps. The map ι_{cut} is such that

$$w_2 \mapsto g_0gg^*g_0^*,\ \ u_n \mapsto u_n - g_ngg^*g_n^* + g_{n+1}gg^*g_{n+1}^*,\ \ f_2 \mapsto g_0gf_1^*,\ \ g_n \mapsto g_n - g_ngg^* + g_{n+1}gg^*$$

which implies that $\iota_{\text{cut}}(g_ngg^*g_n^*) = g_{n+1}gg^*g_{n+1}^*$ for $n > 0$. The map ι_{cut}^{-1} is such that

$$u_1 \mapsto u_1 - g_0gg^*g_0^* + w_2,\ \ u_n \mapsto u_n - g_ngg^*g_n^* + g_{n-1}gg^*g_{n-1}^*,\ \text{for } n > 0$$
$$g_1 \mapsto g_1 - g_0gg^* + f_2f_1g^*,\quad g_n \mapsto g_n - g_ngg^* + g_{n-1}gg^* \text{ for } n > 0$$

so that $\iota_{\text{cut}}^{-1}(g_0gg^*g_0^*) = w_2$, $\iota_{\text{cut}}^{-1}(g_0gf_1^*) = f_2$, and that $\iota_{\text{cut}}^{-1}(g_ngg^*g_n^*) = g_{n-1}gg^*g_{n-1}^*$ for $n > 0$.

It is direct to check that the maps extend to mutually inverse graded $*$-homomorphisms. The maps $\overline{\iota}_{\text{cut}}$ and $\overline{\iota_{\text{cut}}^{-1}}$ are identities since $[g_ngg^*g_n^*] = t^2[w_0] = [g_lgg^*g_l^*]$ for any $n > 0$ and $l > 0$.

Let us consider E_ω and E_1 and the bijection of paths ending at w_0 in the two graphs given by matching f_2f_1 with g_0g, $f_3f_2f_1$ with g_1eg, $f_4f_3f_2f_1$ with g_2e^2g, and so on. This bijection gives rise to the tail-cutting operations $\iota_{\text{cut}} : E_\omega \to_{\text{cut}} E_1$ and tail-creating operation $\iota_{\text{cut}}^{-1} : E_1 \to_{\text{cut},-1} E_\omega$ which map the vertices v_0, w_0, w_1 and the edges e, g, f_1 identically on themselves and which are given on vertices by formulas below where e^0 stands for v_0 and e^{-n} stands for $(e^*)^n$ for $n>0$. We also shorten the notation of $e^n gg^* e^{-n}$ to e_n for $n>0$ and we let $e_0 = gg^*$ so that

$$w_n \mapsto g_{n-2}e_{n-2}g^*_{n-2}, \quad \text{for } n\geq 2 \quad f_n \mapsto g_{n-2}ee_{n-3}g^*_{n-3} \quad \text{for } n>2, \quad f_2 \mapsto g_0 g f_1^*,$$

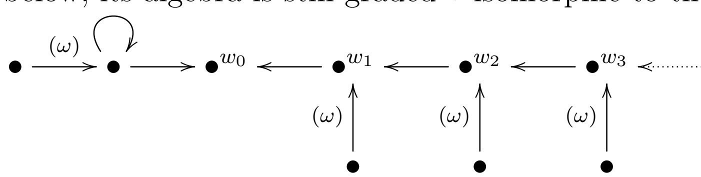
$$u_n \mapsto u_n - \sum_{j=0}^{n-1} g_n e_j g_n^* + \sum_{j=0}^{n-1} g_{n+1}e_j g^*_{n+1}, \qquad g_n \mapsto g_n - \sum_{j=0}^{n-1} g_n e_j + \sum_{j=0}^{n-1} g_{n+1}e_j \text{ for } n\geq 0.$$

With these definitions, we have that $\iota_{\text{cut}}(g_n e_l g_n^*) = g_{n+1}e_l g^*_{n+1}$ for $n \geq 0$ and $l = 0, \ldots, n-1$ and $\iota_{\text{cut}}(g_n e_l g_n^*) = g_n e_l g_n^*$ for $l \geq n$. The map ι_{cut}^{-1} is such that

$$g_0 \mapsto g_0 - g_0 e_0 + f_2 f_1 g^*, \qquad g_n \mapsto g_n - \sum_{j=0}^{n-1} g_n e_j + \sum_{j=0}^{n-2} g_{n-1}e_j + f_{n+1}f_2\ldots f_1 g^* e^{-(n-2)} \text{ for } n>0.$$

It is direct to check that the maps extend to mutually inverse graded $*$-homomorphisms.

If $E_{\omega,\omega}$ is the graph below, its algebra is still graded $*$-isomorphic to that of E_1.



We can choose to map f_2f_1 to g_0g, $f_3f_2f_1$ to g_2eg, $f_4f_3f_2f_1$ to g_4e^2g and, continuing this trend, $f_l \ldots f_1$ to $g_{2(l-2)}e^{l-2}g = g_{2l-4}e^{l-2}g$ for $l\geq 2$. If $w_{(l,n)}$ are the sources of tails $f_{(l,n)}$ ending at w_l for $l \geq 0$, we can choose to map $f_{(0,n)}f_1$ to $g_{2n}gf_1^*$ and

$$f_{(l,n)}f_l\ldots f_1 \quad \text{to} \quad g_{2(l-2)+2n+2}e^{l-1}g = g_{2l+2n-2}e^{l-1}g \text{ for } l\geq 2 \text{ and } n \geq 0.$$

Let ι_{cut} map $v_0, w_0, w_1, u_t, e, g, f_1$, and g_t identically onto themselves and let

$$w_l \mapsto g_{2l-4}e_{l-2}g^*_{2l-4}, \quad \text{for } l\geq 2 \quad f_l \mapsto g_{2l-4}ee_{l-3}g^*_{2l-6} \quad \text{for } l>2, \quad f_2 \mapsto g_0 g f_1^*,$$

$$w_{(l,n)} \mapsto g_{2l+2n-2}e_{l-1}g^*_{2l+2n-2}, \qquad f_{(l,n)} \mapsto g_{2l+2n-2}ee_{l-2}g^*_{2l-4} \quad \text{for } l\geq 2, n\geq 0$$

$$u_l \mapsto u_l - \sum_{j=0}^{\lfloor \frac{l}{2}\rfloor} g_l e_j g_l^* + \sum_{j=0}^{\lfloor \frac{l}{2}\rfloor} g_{2l+1}e_j g^*_{2l+1}, \qquad g_l \mapsto g_l - \sum_{j=0}^{\lfloor \frac{l}{2}\rfloor} g_l e_j + \sum_{j=0}^{\lfloor \frac{l}{2}\rfloor} g_{2l+1}e_j \text{ for } l\geq 0$$

where $\lfloor \frac{l}{2}\rfloor$ is the floor function returning the value $\frac{l}{2}$ if l is even and $\frac{l-1}{2}$ if l is odd. It is direct to check that ι_{cut} extends to a graded $*$-homomorphism. The inverse ι_{cut}^{-1} can be obtained by considerations as in the proof of Proposition 4.8.

The edge f_1 cannot be cut from any of the graphs since 0-tails are not cuttable for these graphs.

The approach to obtaining the cut maps and their inverses in the previous example generalizes to arbitrary graphs. We continue to consider graphs with sinks first.

Definition 4.7. Let l_i be the number of i-tails of a graph E with a sink for $i<k_t$. We define *the cut form* E_{cut} of E so that its quotient, d and the c-to-d part are the same as for E and the number of tails is specified as follows. For $i<k$, E_{cut} has l_i i-tails if i-tails are not cuttable and it has zero i-tails otherwise. For the rest of the tails, if any, we consider the following cases.

(1) There is no $k_0 \in k_t, k_0 \geq k$ such that i-tails are cuttable for all $i\in k_t, i\geq k_0$. Then, the number of i-tails is l_i if i-tails are not cuttable and zero otherwise.

(2) There is $k_0 \in k_t, k_0 \geq k$ such that i-tails are cuttable for all $i \in k_t, i \geq k_0$. The length of the spine graph of E_{cut} is k_0 in this case. For $i < k_0$, the number of i tails is l_i if i-tails are cuttable and zero otherwise.

We say that a graph E is *cut* or that it is in its *cut form* if $E = E_{\text{cut}}$.

For example, for E_2, E_ω, and $E_{\omega,\omega}$ of the previous example, $k_0 = 1$ and the tail spine length of E_2, E_ω, and $E_{\omega,\omega}$ can be reduced to one.

Proposition 4.8. *If E is a graph with a sink, the map $\iota_{\text{cut}} : E \to_{\text{cut}} E_{\text{cut}}$ extends to a graded $*$-isomorphism of the corresponding algebras.*

Proof. We fix some notation first. Since E has a sink, $|d| = 0$, so we can use "n" for other values throughout the proof.

For $i \in k_t$, let l_i be the cardinality of i-tails, for $i \leq k_t$, let w_i be the vertices on the spine of H_t emitting the edge f_i on the spine and let $f_{(i,n)}$ be the i-tails for $n \in l_i$ and $w_{(i,n)}$ be their sources. Let $C \subseteq k_t$ be the set of all i such that i-tails are cuttable.

For $j \in m$ if $m > 0$ and $j = 0$ if $m = 0$, let k_j be the spine length of E^{v_j}. For $l \in k_j, l > 0$, there is a unique path $p_{j,l-1}$ of length $l-1$ ending at v_j by the definition of 1-S-NE canonical form of a 1-S-NE graph with a sink. Let $g_{j,l,n}, n \in |\mathcal{P}_l^{v_j}| - 1$ be the edge of E^{v_j} ending in $\mathbf{s}(p_{j,l-1})$ and originating at a source $u_{j,l,n} = \mathbf{s}(g_{j,l,n})$. So, $g_{j,l,n}, n \in |\mathcal{P}_l^{v_j}| - 1$ are the tails to $\mathbf{s}(p_{j,l-1})$ and the remaining element of the set of the first edges of paths in $\mathcal{P}_l^{v_j}$ is on the spine of E^{v_j}.

Let c_j be the element of $[c]$ starting at v_j, let $c_{jj'}$ be the shortest path from v_j to $v_{j'}$, let $g_{j'i'}$ be any $j'i'$ exit if $a_{j'i'} \neq 0$, and let d_i be the part of the tail graph spine from w_i to w_0.

For $i \in C$, there are j_i, l_i, k_i, j_i', and i_i' such that $|\mathcal{P}_{l_i}^{v_{j_i}}| = \omega, a_{j_i'i_i'} \neq 0$, and $l_i + k_i|c| + d(j_i, j_i') + i_i' = i$. This enables us to map $f_{i+1}d_i$ if $i \leq n_0$ in case (2) and $f_{(i,n)}d_i$, in any case, to

$$g_{j_i,l_i,\sigma(n)} p_{l_i-1} c_{j_i}^{k_i} c_{j_i j_i'} g_{j_i' i_i'} d_{i_i'}$$

where $\sigma(n)$ is appropriately chosen value we specify in the rest of the proof and which depends on cases (1) and (2) and on whether l_i is finite or not. No matter our choice of $\sigma(n)$, the lengths of the corresponding paths match because $l_i + k_i|c| + d(j_i, j_i') + 1 + i_i' = i + 1$.

Let us consider the case $m > 0$ first. Let $JL = \{(j,l) \in m \times k_j \mid j = j_i, l = l_i \text{ for some } i \in C\}$ and, for $(j,l) \in JL$, let

$$C_{j,l} = \{i \in C \ \mid \ j = j_i, \ l = l_i\}.$$

Note that any of C, JL, and $C_{j,l}$ can be infinite. For every $(j,l) \in JL$, let us index the elements of $C_{j,l}$ as $i_{j,l,0}, i_{j,l,1}, \ldots$ where the list ends after finitely many steps if the cardinality of $C_{j,l}$ is finite and it is infinite otherwise. The indexing of $C_{j,l}$ enables us to assign a unique triple (j,l,n) such that $(j,l) \in JL$ and $n \in |C_{j,l}|$ to any $i \in C$. We denote this correspondence by $i \mapsto (j_i, l_i, n_i)$ and its inverse by $(j,l,n) \mapsto i_{(j,l,n)}$. Note that even when $C_{j,l}$ is infinite, for any $n \in \omega$, the set

$$C_{j,l,\leq n} = \{i \in C_{j,l} \mid n_i \leq n\}$$

is finite. If $C_{j,l}$ is finite and $n \geq |C_{j,l}|$, then $C_{j,l,\leq n} = \{i_{j,l,0}, \ldots, i_{j,l,|C_{j,l}|-1}\}$. If $C_{j,l}$ is infinite, $C_{j,l,\leq n} = \{i_{j,l,0}, \ldots, i_{j,l,n}\}$ for any $n \in \omega$.

We start to create a map $\iota_{\text{cut}} : E \to_{\text{cut}} E_{\text{cut}}$. Let us shorten

$$p_{l_i-1} c_{j_i}^{k_i} c_{j_i j_i'} g_{j_i' i_i'} d_{i_i'} \text{ to } q_i \quad \text{and} \quad q_i q_i^* \text{ to } e_i.$$

It is directly to check that the elements e_i are orthogonal to each other, that $e_iq_i = q_i$, that $q_i^*e_i = q_i^*$, and that the quintuple $(l_i, j_i, k_i, j_i', i_i')$ is unique for $i \in C$.

With the introduced abbreviations, for $i \in C$ and $n' \in l_i$, we let

$$f_{(i,n')}d_i \mapsto \begin{cases} g_{j_i,l_i,2n_i+2n'} \quad q_i & \text{if (1) holds or if (2) holds and } i < k_0 \\ g_{j_i,l_i,2n_i+2n'+2}q_i & \text{if (2) holds and } i \geq k_0 \end{cases}$$

If (1) holds, then we map d_i to d_i for any $i \in k_t$. If (2) holds, then we map d_i to d_i for $i < k_0$ and, for $i = k_0 + n, n \geq 0$, we treat $f_{k_0+n+1} = f_{i+1}$ as a $k_0 + n$ tail and let

$$d_{i+1} = f_{i+1}d_i \mapsto g_{j_i,l_i,2n_i}q_i,$$

so this agrees with the formula for the image of $f_{(i,n')}d_i$ in the sense that it explains the lag of "+2" in the formula for the image of $f_{(i,n')d_i}$ the case that (2) and $i \geq k_0$ hold.

We extend ι_{cut} to $E^0 \cup E^1$ by the formulas which ensure that the axioms hold as follows. We let ι_{cut} be the identity on the vertices and edges of c, d, any c-to-d path, any non-cuttable i-tails for $i \in k_t$, the spines of E^{v_j} for $j \in m$ and on the tails of $\mathbf{s}(p_{j,l-1})$ and their sources for all j and l such that $(j, l) \notin JL$. For the remaining tails, we let

$$f_{(i,n')} = f_{(i,n')}d_id_i^* \mapsto \iota_{\text{cut}}(f_{(i,n')}d_i)\iota_{\text{cut}}(d_i)^*, \quad w_{(i,n')} = f_{(i,n')}f_{(i,n')}^* \mapsto \iota_{\text{cut}}(f_{(i,n')})\iota_{\text{cut}}(f_{(i,n')})^* \quad \text{for } n \in l_i$$

If (2) holds and $i \geq k_0$, we let $f_{i+1} = d_{i+1}d_i^* \mapsto \iota_{\text{cut}}(d_{i+1})\iota_{\text{cut}}(d_i)^*$ and

$$w_{i+1} \mapsto \iota_{\text{cut}}(f_{i+1})\iota_{\text{cut}}(f_{i+1})^* = \iota_{\text{cut}}(d_{i+1})\iota_{\text{cut}}(d_i)^*\iota_{\text{cut}}(d_i)\iota_{\text{cut}}(d_{i+1})^* = \iota_{\text{cut}}(d_{i+1})\iota_{\text{cut}}(d_{i+1})^*.$$

With these definitions, for $n \in \omega$,

$$\iota_{\text{cut}}(f_{i+1})^*\iota_{\text{cut}}(f_{i+1}) = \iota_{\text{cut}}(d_i)\iota_{\text{cut}}(d_{i+1})^*\iota_{\text{cut}}(d_{i+1})\iota_{\text{cut}}(d_i)^* = \iota_{\text{cut}}(d_i)\iota_{\text{cut}}(d_i)^* = \iota_{\text{cut}}(w_i)$$

and one checks that

$$\iota_{\text{cut}}(f_{(i,n')})^*\iota_{\text{cut}}(f_{(i,n')}) = \iota_{\text{cut}}(d_i)\iota_{\text{cut}}(f_{(i,n')}d_i)^*\iota_{\text{cut}}(f_{(i,n')}d_i)\iota_{\text{cut}}(d_i)^* = \iota_{\text{cut}}(d_i)\iota_{\text{cut}}(d_i)^* = \iota_{\text{cut}}(w_i).$$

We define the images on the rest of the vertices and edges of E^{v_j} next. For $n' \in \omega$ we consider the floor function $\lfloor \frac{n'}{2} \rfloor$, returning the value $\frac{n'}{2}$ if n' is even and $\frac{n'-1}{2}$ if n' is odd (same as in Example 4.6). For $(j, l) \in JL$ and $n' > 0$, we let

$$g_{j,l,n'} \mapsto g_{j,l,n'} - \sum_{i \in C_{j,l,\leq \lfloor \frac{n'}{2} \rfloor}} g_{j,l,n'}e_i + \sum_{i \in C_{j,l,\leq \lfloor \frac{n'}{2} \rfloor}} g_{j,l,2n'+1}e_i,$$

and $u_{j,l,n'} = g_{j,l,n'}g_{j,l,n'}^* \mapsto \iota_{\text{cut}}(g_{j,l,n'})\iota_{\text{cut}}(g_{j,l,n'})^*$. Defining $\iota_{\text{cut}}(g^*)$ to be $\iota_{\text{cut}}(g)^*$ ensures that the extension is a $*$-map. We have that $\iota_{\text{cut}}(e_i) = e_i$ for every $i \in C$. Thus, for $(j, l) \in JL$ and $i \in C$

$$\iota_{\text{cut}}(g_{j,l,n'}e_i) \text{ is } \begin{array}{ll} 0 & \text{if } i \notin C_{j,l} \text{ (and } g_{j,l,n'}e_i = 0 \text{ in this case also)} \\ g_{j,l,n'}e_i & \text{if } i \in C_{j,l} - C_{j,l,\leq \lfloor \frac{n'}{2} \rfloor} \\ g_{j,l,2n'+1}e_i & \text{if } i \in C_{j,l,\leq \lfloor \frac{n'}{2} \rfloor}. \end{array}$$

This ensures that $\iota_{\text{cut}}(g_{j,l,n'}e_ig_{j,l,n'}^*) = g_{j,l,2n'+1}e_ig_{j,l,2n'+1}^*$ for all $i \in C_{j,l,\leq \lfloor \frac{n'}{2} \rfloor}$.

It is direct to check that the axioms hold so the above map extends to a graded $*$-homomorphism by the Universal Property. The extension is injective by the Graded Uniqueness Theorem.

We aim to define the inverse of ι_{cut} next. First we consider the values of such inverse on elements of the form $g_{j,l,n'}e_i$ for $(j, l) \in JL$ and $i \in C_{j,l}$. For $i \in C_{j,l,\leq \lfloor \frac{2n'}{2} \rfloor} = C_{j,l,\leq n'}$, $i = i_{(j,k,n_i)}$ for some $n_i \leq n'$. Hence, $n' = n_i + n$ for some $n \geq 0$. This enables us to define $\iota_{\text{cut}}^{-1}(g_{j,l,n'}e_i)$ as

$$\begin{array}{ll} g_{j,l,n'}e_i & \text{if } i\in C_{j,l}-C_{j,l,\leq\lfloor\frac{n'}{2}\rfloor},\\ g_{j,l,n}\ e_i & \text{if } n'=2n+1 \text{ and } i\in C_{j,l,\leq\lfloor\frac{n'}{2}\rfloor},\\ f_{(i,n)}d_iq_i^* & \text{if } i\in C_{j,l,\leq\lfloor\frac{n'}{2}\rfloor},\ 2n'=2n_i+2n \text{ for } n\geq 0 \text{ if (1) holds or (2) holds and } i<k_0,\\ f_{(i,n)}d_iq_i^* & \text{if } i\in C_{j,l,\leq\lfloor\frac{n'}{2}\rfloor} \text{ and } n'=2n_i+2n+2 \text{ for } n\geq 0 \text{ if (2) holds and } i\geq k_0,\\ d_{i+1}q_i^* & \text{if } i\in C_{j,l,\leq\lfloor\frac{n'}{2}\rfloor} \text{ and } 2n'=2n_i \text{ if (2) holds and } i\geq k_0. \end{array}$$

With $\iota_{\rm cut}^{-1}$ defined on $g_{(j,l,n')}e_i$, we automatically have that $g_{(j,l,n')}=\iota_{\rm cut}^{-1}(\iota_{\rm cut}(g_{(j,l,n'))})$ if we let

$$\iota_{\rm cut}^{-1}(g_{(j,l,n')})=g_{j,l,n'}+\sum_{i\in C_{j,l,\leq\lfloor\frac{n'}{2}\rfloor}}\iota_{\rm cut}^{-1}(g_{j,l,n'}e_i)-\sum_{i\in C_{j,l,\leq\lfloor\frac{n'}{2}\rfloor}}\iota_{\rm cut}^{-1}(g_{j,l,2n'+1}e_i)$$

and one checks that $g_{(j,l,n')}=\iota_{\rm cut}(\iota_{\rm cut}^{-1}(g_{(j,l,n'))}))$. Finally, we let $u_{(j,l,n')}=g_{(j,l,n')}g^*_{(j,ln')}$ be mapped to $\iota_{\rm cut}(g_{(j,l,n')})\iota_{\rm cut}(g_{(j,ln')})^*$ for $(j,l)\in JL$ and $n'\in\omega$ which ensures that (CK2) holds for these images. It is directly to check that $\iota_{\rm cut}^{-1}$ and $\iota_{\rm cut}$ are inverse to each other.

If $m=0$, the only possible value of j is zero, so some definitions simplify and some formulas are shorter. For example, the elements q_i become $p_{l_i-1}g_{0i'_i}d_{i'_i}$, the set JL becomes the set $L=\{l\in k_0\mid l=l_i \text{ for } i\in C\}$ and the sets $C_{j,l}$ become $C_l=\{i\in C\mid l_i=l\}$ for $l\in L$. With these modifications, the definitions of $\iota_{\rm cut}^{-1}$ and $\iota_{\rm cut}$ are analogous to those in the case $m>0$. Note that if the spine length k is infinite, the case (2) cannot happen (because $k=k_t$ in that case). □

Let us move on to the case when E has a proper terminal cycle in which case we continue to use n only for $|d|>0$. Let $E_{\rm cut}$ be the graph with the same quotient and c-to-d part as E. If E/H is finite, $E_{\rm cut}=E$. If E/H is infinite, let $d(j,j')$ stands for the length of the shortest path in c from v_j to v_j' and let $C(i)$ be the following statement.

$C(i)$ There are $j,j'\in m, i'\in n$, and $k',l\in\omega$ such that $|\mathcal{P}_l^{v_j}|=\omega,\ \ a_{j'i'}\neq 0,$ and $l+k'|c|+d(j,j')+n-i'=n-i(\operatorname{mod} n)$.

If $m=0$, we have that $j=j'=|c|=0$ in condition $C(i)$ so it simplifies as follows.

There are $i'\in n$, and $l\in\omega$ such that $|\mathcal{P}_l^{v_0}|=\omega,\ a_{0i'}\neq 0,$ and $l+n-i'=n-i(\operatorname{mod} n)$.

We say that the i-tails are *cuttable* if $C(i)$ holds or if there is $j\in m$ such that $\mathcal{P}_l^{v_j}$ is nonempty for infinitely many l (equivalently, the spine of E^{v_j} is infinite). In this last case, we also say that i-tails are cuttable for every $i\in n$. Using the introduced terminology, if l_i is the number of i-tails of E, then $E_{\rm cut}$ has l_i tails if i-tails are not cuttable and it has zero i-tails otherwise.

Condition $C(i)$ for $n>0$ is completely analogous to condition $C(i)$ for $n=0$ except that $n-i$ and $n-i'$ in the last equation for $n>0$ are replaced by i and i' for $n=0$. This is due to the fact that the distance from w_i to w_0 is $n-i$ if $n>0$ and it is i if $n=0$. Since mod 0 is trivial relation, the presence of $\operatorname{mod} n$ in the $n>0$ case matches the absence of mod 0 in the $n=0$ case.

If $n>0$, then n plays the role of both k and k_t and condition (2) never holds. If E is a graph with $n>0$ and such that E^{v_j} has finite spine length for all $j\in m$, then the proof of Proposition 4.8 carries when using n instead of k_t. The proof is simplified since condition (1) is always in effect. This enables us to have the tail cutting operation $\iota_{\rm cut}: E\to_{\rm cut}E_{\rm cut}$ which cuts all cuttable tails of E and produces $E_{\rm cut}$ for graph with proper terminal cycle.

Assume that E is a graph with $n>0$ and such that E^{v_j} has infinite spine length for some $j\in m$. If $m>0$ and L is the least common multiple of m and n, let E' be the graph obtained by moving any exit of E L times. If $m=0$, let E' be the graph obtained by moving any $0i$-exit of E with

$a_{0i} = \omega$. In each case, E' has infinitely many i-tails for every $i \in n$ and E_{cut} is defined as the graph with zero i-tails for every $i \in n$. Let ϕ_E be the move operation which transforms E to E' and let ϕ_{cut} be the same move but applied to E_{cut}. It produces E' also since infinitely many tails are created by such operation. Thus, we can let $\iota_{\text{cut}} : E \to_{\text{cut}} E_{\text{cut}}$ be $\phi_{\text{cut}}^{-1}\phi_E$ so that $\iota_{\text{cut}}^{-1} = \phi_E^{-1}\phi_{\text{cut}}$.

4.8. **Canonical quotients.** Next we turn to the quotient E/H. We introduce some terminology needed for the $m > 0$ case. We say that E is a *single exit-emitter* graph if only one vertex of c emits exits. If $m = 0$, this trivially holds.

Let $v_0, \ldots, v_{m-1}$ be the vertices of c. Recall the definitions of the sets $\mathcal{P}_k^{v_j}$, $j \in m$ and $\mathcal{P}^{v_j} = \bigcup_{0<k\in\omega} \mathcal{P}_k^{v_j}$ and graphs $E^{v_j}, j \in m$, from section 3.4. We say that E^{v_j} is the *j-quotient graph.*

First, we ensure that the j-quotient graph is in the 1-S-NE canonical form for all $j \in m$. Let ϕ_j be a graded $*$-isomorphism of E^{v_j} and its canonical form $E_{\text{can}}^{v_j}$ and let E' be the graph obtained from E by replacing E^{v_j} with $E_{\text{can}}^{v_j}$. Let ϕ be the identity on the subgraph generated by $H \cup c^0$ and let it be ϕ_j on E^{v_j}. Thus, ϕ is defined on $E^0 \cup E^1$ and we let $\phi(e^*) = \phi(e)^*$. It is direct to check that the axioms (V) and (CK1) hold. The axiom (E1) trivially holds for edges with sources in $H \cup c^0$ and for edges of E/H with ranges not in c, it is sufficient to consider an edge $g \in \mathcal{P}_1^{v_j}$ for some $j \in m$. By the definition of the canonical form of an acyclic graph with a sink, $\phi(g)$ is another edge in $\mathcal{P}_1^{v_j}$. So, $\phi(g)\phi(\mathbf{r}(g)) = \phi(g)v_j = \phi(g)$. The axiom (CK2) holds for vertices in $H \cup c^0$ since it holds in E and it holds for vertices of $E - H$ since it holds for the images of ϕ_j. By the Universal Property and definition of ϕ, ϕ extends to a graded $*$-monomorphism. By considering the inverse of ϕ_j, we obtain the inverse of ϕ, so ϕ is a graded $*$-isomorphism.

If $m = 0$ or $m = 1$ this is all that is needed: we let $E_{\text{can quot}}$ be the graph obtained by replacing E/H with $(E/H)_{\text{can}}$ in E. From here to the end of the section, we assume that $m > 1$.

When each j-quotient graph is in a 1-S-NE canonical form, we let k_j be the spine length of E^{v_j} for any $j \in m$ and we consider the feasibility of an operation which would "move" $E^{v_{j-_m 1}}$ to E^{v_j}. Such an operation is possible under the following conditions.

- The vertex $v_{j-_m 1}$ does not emit exits.
- The graph E^{v_j} has at least one edge, i.e. $k_j > 0$.

If there are no edges other than those in c in E/H, then we say that any $j \in m$ is *feasible* and we let $\text{Fea} = m$. If E/H has edges other than those in c (equivalently $E^0 - H - c^0 \neq \emptyset$), we determine the set of feasible vertices by the following process consisting of "moving" as many of j-quotient graphs in the direction of c and then considering the length of their spines.

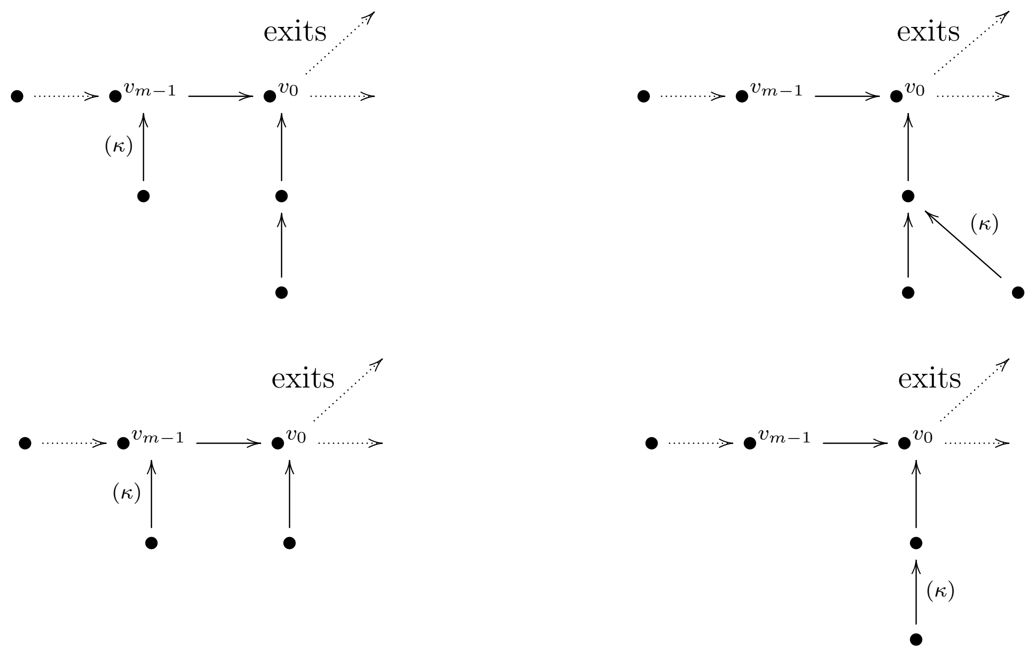
If the two conditions are met, we define an operation which produces a new graph with the $(j -_m 1)$-quotient graph consisting only of a sink and j-quotient graph consisting of new edges and vertices obtained by "moving" $E^{v_{j-_m 1}}$ to E^{v_j}. For simplicity, let us assume that $j = 0$ and let E_0 be the new graph we aim to define.

The graph E_0 has the vertices and edges of c, c-to-d, d and the tail graph the same as E and we use same labels for these elements. Every j-quotient of E_0 is isomorphic to that of E if $j \neq 0$ and $j \neq m - 1$. So, it remains to define the zero- and $(m - 1)$-quotient of E_0.

For $j \in m$, let $p_l^j = e_l^j e_{l-1}^j \ldots e_1^j$ be the path on the spine of E^{v_j} of length $l \leq k_j$ and g_{li}^j be the edges of the tails ending in $\mathbf{s}(p_{l-1}^j)$. Thus, $\{e_l^j p_{l-1}^j\} \cup \{g_{li}^j p_{l-1}^j \mid i \in |\mathcal{P}_l^{v_j}| - 1\}$ is the set $\mathcal{P}_l^{v_j}$.

We consider whether $k_{m-1} + 1 \leq k_0$ or $k_{m-1} + 1 > k_0$. If κ is any countable cardinal and E is the first graph below, it has $k_{m-1} + 1 = 1 + 1 \leq k_0 = 2$. The second graph below is E_0 in this case. If E

is the third graph below, it has $k_{m-1}+1=1+1>k_0=1$ and the fourth graph is E_0 in this case.



If $k_{m-1}+1\leq k_0$ we replace g_{li}^{m-1} for $i\in|\mathcal{P}_l^{v_{m-1}}|, i\neq 0$ and e_l^{m-1} by new tails $f^0_{(l+1)i}$, where i now ranges in $i\in|\mathcal{P}_l^{v_{m-1}}|$, ending at the source of p_l^0 for all $l\leq k_{m-1}$. If $k_{m-1}+1>k_0$, we do the same but only for $l\leq k_0$. For $l>k_0$ and $l\leq k_{m-1}$, we move the portion of $E^{v_{m-1}}$ below the root of $\mathbf{s}(p^{m-1}_{k_0-1})$ to E^{v_0} so that its spine now becomes $k_0+(k_{m-1}-k_0+1)=k_{m-1}+1$ and we continue to use p_l^0 for the path of length l on the spine where l now ranges from 1 to $k_{m-1}+1$. The added l-tails to this new portion are f^0_{li} for $i\in|\mathcal{P}_l^{v_{m-1}}|-1$ and $l=k_0+1,\ldots,k_{m-1}+1$.

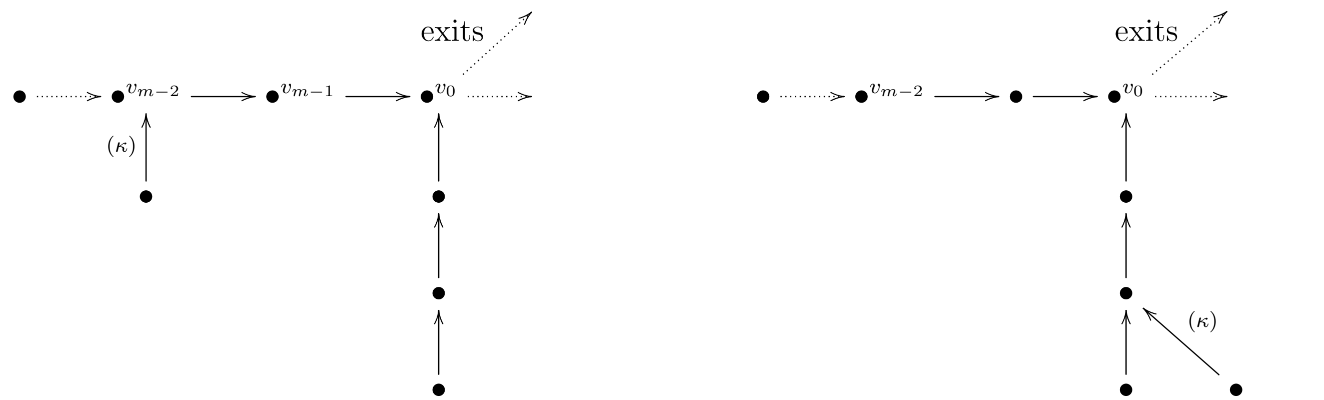
Let e_{m-1} be the edge of c from v_{m-1} to v_0. If $k_{m-1}+1\leq k_0$, we define a function ϕ mapping $g_{li}p^{m-1}_{l-1}e_{m-1}$ to $f^0_{(l+1)i}p^0_l, i>0, i\in|\mathcal{P}_l^{v_{m-1}}|$ and $e_l^{m-1}p^{m-1}_{l-1}e_{m-1}$ to $f^0_{(l+1)0}p^0_l$ for $l\leq k_{m-1}$. If $k_{m-1}+1>k_0$, the images of ϕ are the same for $l<k_0$. For $l=k_0,\ldots,k_{m-1}$, we map $p_l^{m-1}e_{m-1}$ to p^0_{l+1} and we let $\phi(g_{li}p^{m-1}_{l-1}e_{m-1})=f^0_{(l+1)i}p^0_l$ where now i takes value zero also and $i\in|\mathcal{P}_l^{v_{m-1}}|$. Since every edge in E^{v_j} is equal to pq^* for some l and $p\in\mathcal{P}_l^{v_{m-1}}$ and $q\in\mathcal{P}_{l-1}^{v_{m-1}}$, this enables us to define ϕ on the edges of $E^{v_{m-1}}$ by $\phi(p)\phi(q)^*$. Every source of such an edge is mapped to $\phi(p)\phi(p)^*$. Defining ϕ on the ghost edges by $\phi(g^*)=\phi(g)^*$ assures that the extension is a $*$-map. We map the rest of the elements of E and E_0 identically onto themselves. It is direct to check that the axioms hold for these images, so ϕ extends to a graded $*$-monomorphism. Checking (CK2) critically involves the fact that v_{m-1} does not emit any exits in E. There is an inverse of ϕ defined by moving the new paths of E_0 back to their originals in E. Hence, ϕ extends to a graded $*$-isomorphism.

We refer to such move of the $(j-_m 1)$-quotient graph to the j-quotient graph as a *one-step move*.

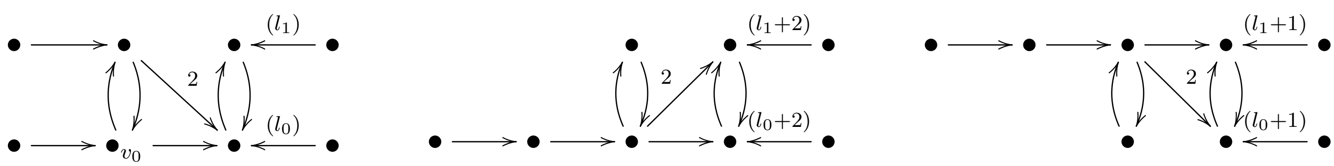
If $m>2$, we consider the scenario when $E^{v_{j-_m 1}}$ consists only of a sink and we intend to move $E^{v_{j-_m 2}}$ to E^{v_j}. This is possible under the following two conditions.

- The vertices $v_{j-_m 1}$ and $v_{j-_m 2}$ do not emit exits.
- The graph E^{v_j} has a path of length two, i.e $k_j>1$.

The operation transforming E to a graph with both $(j-_m 2)$- and $(j-_m 1)$-quotient graphs consisting only of sinks is defined analogously as for a one-step move and we refer to it as a *two-step* *move*. The figure below illustrates one such move for a graph with $m > 2$.



For any $c \in [c]$, one can move the exits to start at $\mathbf{s}(c)$ and then make a maximal possible number of all step moves towards $\mathbf{s}(c)$. If any of such step moves is possible, we let $E_{\text{can quot},c}$ be the graph resulting from all such possible step moves. Otherwise, we let $E_{\text{can quot},c}$ be the graph resulting only from moving all exits to $\mathbf{s}(c)$. We say that $E_{\text{can quot},c}$ is the *canonical quotient graph with the single exit-emitter* $\mathbf{s}(c)$. If $m > 1$, the different canonical quotient graphs do not have to be isomorphic as the following example shows. If E is the first graph below, c is the cycle with $\mathbf{s}(c) = v_0$ and c_1 is the other element of $[c]$, then the second graph is $E_{\text{can quot},c}$ and the third is $E_{\text{can quot},c_1}$.



The consideration of $E_{\text{can quot},c}$ has the following advantage. For any E, we have that

$$|P_1^{v_0}| = |\mathcal{P}_1^{v_0}| + 1, \quad |P_2^{v_0}| = |\mathcal{P}_2^{v_0}| \cup |\mathcal{P}_1^{v_{m-_m 1}}| + 1, \quad |P_3^{v_0}| = |\mathcal{P}_3^{v_0}| \cup |\mathcal{P}_2^{v_{m-_m 1}}| \cup |\mathcal{P}_1^{v_{m-_m 2}}| + 1 \ldots$$

If E is a canonical quotient with v_0 as its single-exit emitter, then

$$|P_1^{v_0}| = |\mathcal{P}_1^{v_0}| + 1, \quad |P_2^{v_0}| = |\mathcal{P}_2^{v_0}| + 1 \text{ or } |P_2^{v_0}| = |\mathcal{P}_1^{v_{m-_m 1}}| + 1,$$
$$|P_3^{v_0}| = |\mathcal{P}_3^{v_0}| + 1 \text{ or } |P_3^{v_0}| = |\mathcal{P}_2^{v_{m-_m 1}}| + 1 \text{ or } |P_3^{v_0}| = |\mathcal{P}_1^{v_{m-_m 2}}| + 1 \ldots$$

So, only one of the $\mathcal{P}$-sets impacts the cardinality of a P-set.

In particular, for every $j \in m$, the graph generated by the vertices of $R(v_j)$ has a simple form: if the spine length of E^{v_0} is longer than m then $E_{\text{can quot},c}$ is such that only E^{v_0} has edges. If the spine length k_0 of E^{v_0} is shorter than m, then $E_{\text{can quot},c}$, where c is such that $\mathbf{s}(c) = v_0$, is such that $E^{v_j} = \{v_j\}$ for all $j = m - k_0, \ldots m - 1$. More generally, if the j-quotient graph contains edges and k_j is its spine length, then the l-quotient graph for $l = j -_m 1, \ldots, j -_m (k_j + 1)$ is $\{v_l\}$. This enables us to choose preferred locations for moving all exits to: we say that v_j is *0-feasible* if the spine length k_j of E^{v_j} is the maximum of the set $\{k_j \mid j \in m\}$. Let Fea_0 be the set of 0-feasible vertices. In the previous example, $\text{Fea}_0 = c^0$.

We say that $v_j \in \text{Fea}_0$ is *1-feasible* and write $v_j \in \text{Fea}_1$ if $|\mathcal{P}_1^{v_j}|$ is the maximum of the set $\{|\mathcal{P}_1^{v_j}| \mid v_j \in \text{Fea}_0\}$. We continue this consideration and define Fea_l for $l > 1$. If at any point the set Fea_l becomes a single element set, we have a preferred location for moving all exits to. If the process does not finish after finitely many steps, then the graphs E^{v_j} are isomorphic for all $j \in \text{Fea}_l$ where l is such that $\text{Fea}_l = \text{Fea}_{l'}$ for all $l' > l$. In this case, we stop at step l and start considering the spacing between the Fea_l values: consider the lengths l_j of the path on c_j between $v_j \in \text{Fea}_l$ and the first next $v_{j'} \in \text{Fea}_l$. Let Fea_{l+1} be the set of all $v_j \in \text{Fea}_l$ such that the length l_j is maximal. Then, let Fea_{l+2} be the set obtained by considering $v_j \in \text{Fea}_{l+1}$ such that the first j'-quotient graph which has edges for $v_{j'}$ between v_j and the next vertex in Fea_{l+1}, has the same distinguishing features as those considered when Fea_l was being determined: longest spine first, then most 1-tails, then most 2-tails etc. If $\text{Fea}_{l'}$ is the resulting set, we move on to considering the second j'-quotient graph which has edges for $v_{j'}$ between v_j and the next vertex in $\text{Fea}_{l'}$. We terminate the process whenever any set $\text{Fea}_{l'}$ becomes a single-element set. If this does not happen, we have that there is $l_0 \geq l$ such that $\text{Fea}_{l_0} = \text{Fea}'_l$ for all $l' > l_0$. We let $\text{Fea} = \text{Fea}_{l_0}$ and say that a vertex in Fea is *feasible*.

If Fea has more than one element, then the quotient graph has a complete *symmetry* in the following sense: each two consecutive vertices in Fea are equally spaced by a period k and if v_j and v_{j+k} are two consecutive feasible vertices and $v_{j'}$ and $v_{j'+k}$ are another two consecutive feasible vertices, then $E^{v_{j+i}}$ is isomorphic to $E^{v_{j'+i}}$ for every $i \in k$. This symmetry enables us to define a *canonical quotient form* $E_{\text{can quot}}$ as any graph $E_{\text{can quot},c}$ with $\mathbf{s}(c) = v_j$ and $j \in \text{Fea}$. If we consider more than one graph at a time, we write Fea_E for Fea.

For instance, in the last example above, $\text{Fea} = \text{Fea}_0 = c^0$.

Note that $E_{\text{can quot}}$ may not be reduced because some exit moves were involved in its definition. Reducing this graph does not change its quotient. We prove some further properties of $E_{\text{can quot}}$.

Lemma 4.9. *Let $E = E_{\text{dir}}$ be with $m > 0$. If $v_0 = \mathbf{s}(c)$ and $v_j = \mathbf{s}(c_j)$ are feasible, then $E_{\text{can quot},c}$ and $E_{\text{can quot},c_j}$ have isomorphic quotients and their algebras are graded $*$-isomorphic to the algebra of E. Moreover, the following conditions are equivalent and any of them implies that* $\text{Fea} = \{v_0\}$.

(1) *The spine length of E^{v_0} in $E_{\text{can quot},c}$ is at least m.*
(2) *The 1-S-NE canonical form of E/H is such that every vertex of c has tails.*

Proof. Since $E_{\text{can quot},c}$ is obtained from E by moving exits and considering step operations, corresponding algebras are graded $*$-isomorphic. If (1) holds, then $(E/H)_{\text{can}}$ has at least one j-tail for each $j \in m$ by the definition of a 1-S-NE canonical form. If (2) holds, then conditions for making m-step moves are met in E and we can move every E^{v_j} to E^{v_0}. Thus, (1) holds. This also implies that $j = 0$, that $\text{Fea} = \{v_0\}$ and that the spine of E^{v_0} in $E_{\text{can quot}}$ is at least m. □

4.9. **Canonical forms.** We define a canonical form by first considering the canonical quotient in order to ensure that the quotient is canonical. Then, considering a reduced and cut form of $E_{\text{can quot}}$ does not impact its quotient but it ensures that the number of tails is as small as possible. After this, we move exits to certain form depending on m and n values which are specified in the definition below. Finally, we L-, spine- or ω-reduce (whatever is applicable based on the m and n values) and cut the tails to regain the minimality of the tails. We shorten $(E_{\text{red}})_{\text{cut}}$ to $E_{\text{red,cut}}$.

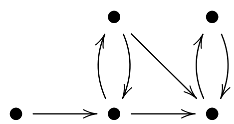
Definition 4.10. Let E be a direct-exit 2-S-NE graph and let E_0 be $((E_{\text{red,cut}})_{\text{can quot}})_{\text{red,cut}}$. Move the exits, if needed, as specified below and then, let E_{can} be the L-, spine- or ω-reduced and cut form, whatever is applicable, of the resulting graph.

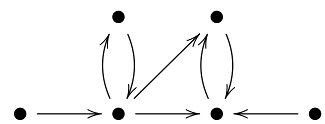
- If $m > 0$ and $n > 0$, then the required form is such that a feasible vertex is a single exit-emitter and the ranges of the exits are among $G =\text{GCD}(m, n)$ consecutive vertices of d. In particular, for $c \in [c]$ and $d \in [d]$ such that $v_0 = \mathbf{s}(c)$ is feasible, v_0 is a single exit-emitter and the vertices from w_{n-G+1} to w_0 are receiving exits. This is possible by Lemma 4.4. Thus, E_{can} depends on the choice of c and d.

- If $m > 0$ and $n = 0$, the required form is such that c emits exits only to the spine source.
- If $m = 0$ and $n > 0$ or $m = n = 0$ and $k < \omega$, the required form is a graph in which w_i receives either zero or infinitely many exits for $i \in n$ if $n > 0$ and for $i \leq k$ if $n = 0$.
- If $m = n = 0$ and $k = \omega$, we impose no requirements.

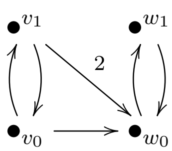
If E is a 2-S-NE graph, we let $E_{\text{can}} = (E_{\text{dir}})_{\text{can}}$. We say that E is *canonical* or that it is *in a canonical form* if $E \cong F_{\text{can}}$ for some graph F.

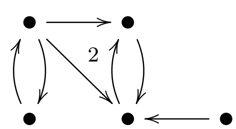
For example, let E be the first graph below. As E has only one feasible vertex, the canonical form is unique up to a graph isomorphism. The second graph is $E_{\text{can quot}}$ and E_{can}.

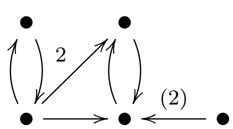




Let E_0, E, and F be three graphs below in that order. Both v_0 and v_1 are feasible vertices of E_0 and both E and F are canonical forms of E_0.



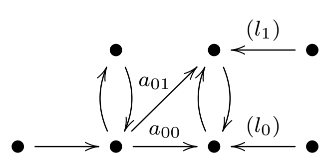




Note that E is obtained by moving the 00-exit and F by moving both 10-exits. A graded $*$-isomorphism of the algebras of E and F can be obtained by composing the inverse of the moves $E_0 \to E$ with the moves $E_0 \to F$. We generalize this observation in the following lemma whose proof is contained in its statement.

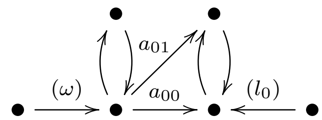
Lemma 4.11. *Let E_0 be a direct-exit graph and such that $E_0 = ((E_0)_{\text{can quot}})_{\text{red,cut}}$ and let E and F be canonical forms of E_0. Let $\phi_E : E_0 \to E$ and $\phi_F : E_0 \to F$ be the move maps (composed possibly with spine-reduction maps if $m > 0$ and $n = 0$) followed by the cut maps and let ϕ_E^{-1} and ϕ_F^{-1} be their inverses composed by the cut maps. Then, $\phi_F \phi_E^{-1}$ maps E to F and $\phi_E \phi_F^{-1}$ maps F to E.*

We consider another example illustrating the impact of the number of tails on the graded $*$-isomorphism class of the corresponding algebra. Let us consider graphs of with $m = n = 2$ with the quotient consisting of one edge ending in the cycle (so the range of such edge is the only feasible vertex). A canonical form of such a graph has the form as in the figure below.



If both a_{00} and a_{01} are nonzero, then $l_0, l_1 \in \{0, 1, 2, \omega\}$ and the possible values for the pair (l_0, l_1) include any with at least one entry zero and $(1,1), (1,\omega), (\omega, 1)$, and (ω, ω). If $a_{00} \neq a_{01}$, then all the different possibilities for the l_0 and l_1 values correspond to different graded $*$-isomorphism classes of the corresponding algebras. If $a_{00} = a_{10}$, then the map interchanging w_0 and w_1, their tails and the exits they receive and which is the identity on other graph elements is a graph isomorphism. This makes the algebra of a graph with (l_0, l_1) graded $*$-isomorphic to the algebra with (l_1, l_0), so the number of graded $*$-isomorphism classes is smaller than when $a_{00} \neq a_{01}$.

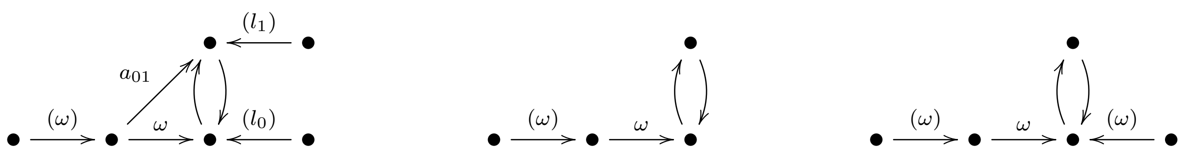
If ω tails are added to the feasible vertex of the quotient and if $a_{00} \neq 0$, then the 1-tails are cuttable, so $l_1 = 0$ in any canonical form. The 0-tails are cuttable if and only if $a_{01} \neq 0$.



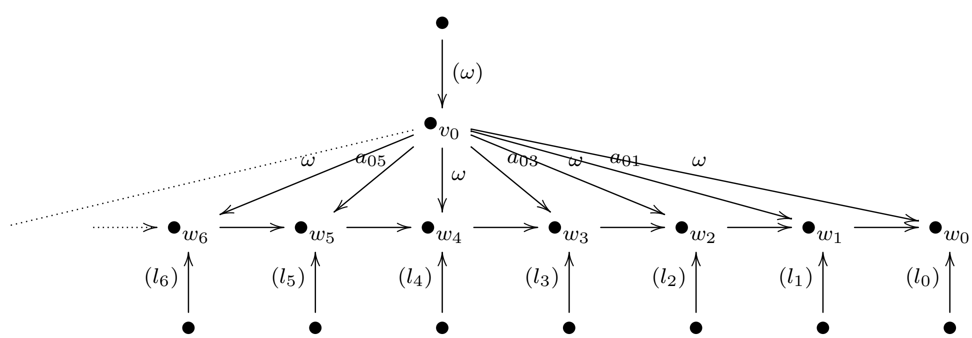
If $a_{01} \neq 0$, a canonical form has $l_0 = l_1 = 0$. If $a_{01} = 0$, then $l_0 \in \{0, 1, \omega\}$. So, there are exactly three graded $*$-isomorphism classes of algebras of such graphs.

The Toeplitz graph is a canonical form of any graph with quotient consisting of a single loop, the spine of the tail graph being the spine of the graph and the tail graph having no tails.

Let us look at an example with $m = 0$ and $n > 0$. Let E_0 be the first graph below where $0 \leq a_{01} < \omega$. In this graph, the 1-tails are cuttable and 0-tails are cuttable if and only if $a_{01} \neq 0$. If $a_{01} \neq 0$, then a cut form has neither 0- nor 1-tails and a canonical form, the second graph below, is obtained by moving all a_{01} exits and then cutting the resulting graph. If $a_{01} = 0$, then only 1-tails are cuttable but then 0-tails are reducible to zero if their number is finite. In this case, the second and the third graphs are possibilities for a canonical form based on whether l_0 is finite or not.



For an example with $m = n = 0$ and $k = \omega$, let E be the graph below so that its connecting matrix is $[\omega\ a_{01}\ \omega\ a_{03}\ \omega\ a_{05} \ldots])$ where $a_{0(2k+1)}$ is finite for all $k \in \omega$. Let $0 < l_i \leq \omega$ for $i \geq 0$.



The $2k+1$-tails are cuttable for all $k \geq 0$ and $2k+2$-tails are cuttable if $a_{0(2k+1)} \neq 0$. In addition, $2k$-tails are reducible to zero if their number is finite. So, the number of $2k$-tails determines the graded $*$-isomorphism class of the algebra of such a graph.

4.10. **2-S-NE equivalence.** For 2-S-NE graphs E and F, we define the following relation.

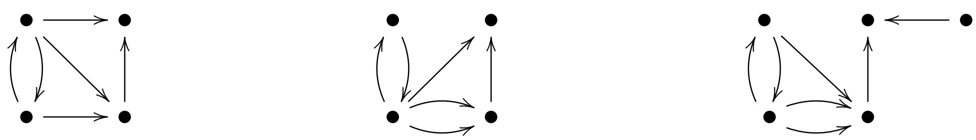
$$E \approx F \quad \text{if there are canonical forms } E_{\text{can}} \text{ and } F_{\text{can}} \text{ such that } E_{\text{can}} \cong F_{\text{can}}$$

and we say that E and F are *2-S-NE equivalent* if $E \approx F$ holds. It is clear that the relation $\approx$ is reflexive and symmetric. We show that transitivity follows from Lemma 4.11.

Lemma 4.12. *The relation $\approx$ is transitive on 2-S-NE-graphs.*

Proof. If E, F, and G are graphs such that $E \approx F$ and $F \approx G$, then there are canonical forms E_1 of E, F_1 and F_2 of F, and G_2 of G such that $E_1 \cong F_1$ and $F_2 \cong G_2$. Let $\phi_{F_1F_2} : F_1 \to F_2$ be the operation from Lemma 4.11. Applying the same operation to G_2 results in another canonical form G_1 of G and $F_2 \cong G_2$ implies $F_1 \cong G_1$. Thus, $E_1 \cong F_1$ and $F_1 \cong G_1$ hold, so $E_1 \cong G_1$ holds. □

There are graphs E and F such that $E \approx F$ and $E \not\cong F$. For example, let E be the first and F the second graph. The last graph is a canonical form of both E and F so we have that $E \approx F$.



4.11. **Theorem 4.13 and its proof.** In this section, we formulate and prove that GCC holds for countable 2-S-NE graphs and we realize every $\mathbf{POM}^D$-isomorphism by a graph operation.

Theorem 4.13. The GCC holds for the 2-S-NE graphs. *Let E and F be two countable 2-S-NE graphs. The following conditions are equivalent.*

(1) *There is a $\mathbf{POM}^D$-isomorphism $f : M_E^\Gamma \to M_F^\Gamma$.*
(2) *The relation $E \approx F$ holds.*
(3) *There is a graded $*$-isomorphism $\phi : L_K(E) \to L_K(F)$.*

If (1) holds, there are canonical forms E_{can} and F_{can} and operations $\phi_E : E \to E_{\text{can}}$, $\iota : E_{\text{can}} \cong F_{\text{can}}$, and $\phi_F : F \to F_{\text{can}}$, such that $\overline{\phi_F^{-1}\iota\phi_E} = f$.

The implications (2) ⇒ (3) and (3) ⇒ (1) are direct. In the rest of this section, we show that (1) ⇒ (2) and that the last sentence of the theorem holds.

Before going any further, we fix some notation and establish a general set-up for two 2-S-NE graphs E and F and a $\mathbf{POM}^D$-isomorphism f between their Γ-monoids. Let $\phi_E : E \to E_{\text{can}}$ and $\phi_F : F \to F_{\text{can}}$ be operations transforming the graphs to their canonical forms. We can consider $\overline{\phi}_F f \overline{\phi_E^{-1}}$ instead of f and so we can assume that $E = E_{\text{can}}$ and $F = F_{\text{can}}$.

Let E have composition factors P_H and E/H and let $\bar\iota_H$ be the map induced by the inclusion $\iota_H : I(H) \to L_K(E)$ and $\overline{\pi}_{E/H}$ be the map induced by the natural map $\pi_{E/H} : L_K(E) \to L_K(E/H)$. Let G be a hereditary and saturated subset of F^0 such that $f\bar\iota_H(J^\Gamma(H)) = J^\Gamma(G)$ and let ι_G and $\pi_{F/G}$ be analogous to ι_H and $\pi_{E/H}$. By the choice of G, $f\bar\iota_H : J^\Gamma(H) \to J^\Gamma(G)$ is a $\mathbf{POM}^D$-isomorphism and f induces a $\mathbf{POM}^D$-isomorphism $f_{E/H} : M_{E/H}^\Gamma \to M_{F/G}^\Gamma$ such that the diagram below commutes.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & J^\Gamma(H) & \xrightarrow{\bar\iota_H} & M_E^\Gamma & \xrightarrow{\overline{\pi}_{E/H}} & M_{E/H}^\Gamma & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f\bar\iota_H} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_{E/H}} & & \\
0 & \longrightarrow & J^\Gamma(G) & \xrightarrow{\bar\iota_G} & M_F^\Gamma & \xrightarrow{\overline{\pi}_{F/G}} & M_{F/G}^\Gamma & \longrightarrow & 0
\end{array}$$

The two-cluster case. In the easy case when E has two connected components, the first row of the above diagram splits and the existence of f implies that there is a splitting of the bottom row. Thus, any two generators of M_F^Γ are $\mathbb{Z}^+[t, t^{-1}]$-independent. This implies that that there are no paths from any vertex of the cluster of F/G to G (because there would be two $\mathbb{Z}^+[t, t^{-1}]$-dependent generators of M_F^Γ otherwise). As $F = F_{\text{tot}}$, F has two connected components, one 1-S-NE-equivalent to H and the other to E/H. Since E and F are canonical, their 1-S-NE components are canonical also and, hence, isomorphic by Propositions 3.4 and 3.5. Thus, we have that $E \cong F$. In addition, there are $\chi : P_H \to P_G$ and $\psi : E/H \to F/G$ such that $\overline{\chi} = f\bar\iota_H$ and $\overline{\psi} = f_{E/H}$ by Propositions

3.4 and 3.5. This implies the existence of a graph operation ϕ which is defined on P_H as χ and on E/H as ψ and which is such that $\overline{\phi} = f$.

The one-cluster case. Having the easy case out of the way, let us assume that E has only one terminal cluster. As the previous paragraph shows, F also has only one terminal cluster.

Let c, c', d, and d' be the terminal cycles of E/H, F/G, P_H, and P_G, respectively and let $|c| = m$, $|c'| = m'$, $|d| = n$, and $|d'| = n'$. We let c^0 consists of $v_0, \ldots, v_{m-_m1}$ where these vertices are listed in the same order they appear in c and we let $v'_0, \ldots, v'_{m'-_{m'}1}$ be analogously ordered vertices of c', $w_0, \ldots, w_{n-_n1}$ of d, and $w'_0, \ldots, w'_{n'-_{n'}1}$ of d'.

By Propositions 3.4 and 3.5, the existence of $f\bar{\iota}_H$ implies that $P_H \approx P_G$, so $n = |d| = |d'| = n'$. By the same propositions, the existence of $f_{E/H}$ implies that $E/H \approx F/G$, so $m = |c| = |c'| = m'$.

If $n = 0$, we have that f maps $[w_0]$ to $[w'_0]$. If $n > 0$, for any $d \in [d]$ and $d' \in [d']$, if $f([\mathbf{s}(d)]) = t^i[\mathbf{s}(d')]$ for some $i \in n$, we replace d' with the element $d'' \in [d']$ which originates at w_{n-_ni}. If $d'' \neq d'$, relabel the vertices so that $d'' = d'$. Thus, we have that $f([w_0]) = [w'_0]$ holds.

<u>**The m > 0 case.**</u> If $n > 0$, then E and F are single exit-emitters. We can choose $c \in [c]$ is such that v_0 is the single exit-emitter in E. We can choose $c' \in [c']$ so that $f_{E/H}([v_0]) = [v'_0]$ but we do not have a guarantee that v'_0 is the single exit-emitter in F – we will have this just after we show Proposition 4.15. Regardless of whether $n > 0$ or $n = 0$, we have that $f_{E/H}([v_0]) = [v'_0]$, so

$$f([v_0]) = t^{lm}[v'_0] + b(t)[w'_0]$$

for some $l \in \mathbb{Z}$ and $b(t) \in \mathbb{Z}^+[t, t^{-1}]$.

If a_E and a_F are the connecting polynomials of E and F, we have that $f([v_0]) = t^{lm}[v'_0] + b[w'_0] = t^{(l+1)m}[v'_0] + t^{lm}a_F[w'_0] + b[w'_0]$ by Lemma 4.1. Thus, while l and b are not uniquely determined, changing l changes b in a specific way: increasing l by one changes b to $t^{lm}a_F + b$. This shows that we can assume that $l \geq 0$. We claim that we can choose b to be in $\mathbb{Z}^+[t]$. If $n > 0$, then $[w'_0] = t^n[w'_0]$, so any $t^k[w'_0]$ with $k < 0$ is equal to $t^{k+k'n}[w'_0]$ for $k' \in \mathbb{Z}^+$ large enough so that $k + k'n \geq 0$. If $n = 0$, $[w'_0]$ is incomparable. The elements of the form $t^k[w'_0]$ for $k \in \mathbb{Z}$ are minimal elements and D_F does not contain an element of the form $t^k[w'_0]$ for $k < 0$ (see [14, Proposition 3.4]). So, $b \in \mathbb{Z}^+[t]$.

By Lemma 4.1, we have that $f([v_0]) = f(t^m[v_0] + a_E[w_0]) = t^{(l+1)m}[v'_0] + t^m b[w'_0] + a_E[w'_0]$. As we also have that $f([v_0]) = t^{(l+1)m}[v'_0] + t^{lm}a_F[w'_0] + b[w'_0]$ and, as M_F^Γ is cancellative, we have that

$$t^{lm}a_F[w'_0] + b[w'_0] = t^m b[w'_0] + a_E[w'_0]. \tag{4}$$

If $f^{-1}([v'_0]) = t^{l'm}[v_0] + b'[w_0]$, the requirement that $f^{-1}(f([v_0])) = [v_0]$ produces

$$t^{(l+l')m}[v_0] + \sum_{j=0}^{l+l'-1} t^{jm}a_E[v_0] = [v_0] = t^{(l+l')m}[v_0] + t^{lm}b'[w_0] + b[w_0].$$

Canceling the term with $[v_0]$ produces the first relation below and using $f(f^{-1}([v'_0])) = [v'_0]$ similarly produces the second relation below.

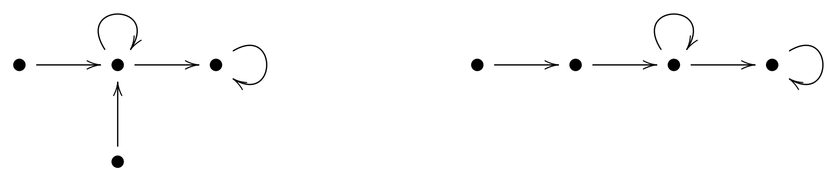
$$(t^{lm}b' + b)[w_0] = \sum_{j=0}^{l+l'-1} t^{jm}a_E[w_0] \qquad\qquad (t^{l'm}b + b')[w'_0] = \sum_{j=0}^{l+l'-1} t^{jm}a_F[w'_0] \tag{5}$$

Let $|b|$ denote the number of terms of the form t^i for $i \in \mathbb{Z}^+$ of b (so the $\mathbb{Z}^+[t]$ monomial of the form kt^i for $k \in \mathbb{Z}^+$ is considered to be the sum of k-terms of the form t^i) and let the same notation be used for any polynomial of $\mathbb{Z}^+[t]$. Equation (4) implies that $|t^{lm}a_F| + |b| = |t^m b| + |a_E|$. Since

$|b| = |t^m b|$ and $|t^{lm} a_F| = |a_F|$, we have that $|a_F| + |b| = |b| + |a_E|$. After canceling $|b|$, this produces the relation $|a_E| = |a_F|$. So, c and c' have the same number of exits.

Next, we show that the quotients of two canonical quotient graphs with **POM**-isomorphic Γ-monoids are isomorphic and the following example illustrates this.

Example 4.14. Let E and F be the two cycle-to-cycle graphs below.



Both graphs have canonical quotients. The quotients are 1-S-NE equivalent but not isomorphic. Since the number of vertices is small, one can check directly that there is no $\mathbf{POM}^D$-isomorphism between the graph Γ-monoids by considering the order-units. However, we cannot use the order-units if the number of vertices is infinite, so we present an alternative argument: M_E^Γ contains one element of the form $t^2[\mathbf{s}(c)]$ and M_F^Γ contains two elements of the form $t^2[\mathbf{s}(c')]$. Also, there are three elements of the form $t[\mathbf{s}(c)]$ in M_E^Γ and only two elements of the form $t[\mathbf{s}(c')]$ in M_F^Γ.

Proposition 4.15. *Let $m > 0$ and E, F, H, G, v_0, v_0' and f be as in the rest of this section. Then, v_0' is feasible and there is $\iota : E/H \cong F/G$ such that $\iota(v_0) = v_0'$.*

Proof. Recall that we have chosen $c \in [c]$ so that $v_0 \in \mathrm{Fea}_E$. If j_0 is such that v'_{j_0} is a single exit emitter in F, we have that $v'_{j_0} \in \mathrm{Fea}_F$. If v_0' is also in Fea_F, we have that $E^{v'_{j_0}} \cong E^{v'_0}$ by the definition of feasible vertices. By composing f with the $\mathbf{POM}^D$-isomorphism induced by the operation of transforming the canonical quotient in which v'_{j_0} emits exits to the canonical quotient in which v_0' emits exits, we can assume that f is such that $f([v_0]) = [v_0']$ and that v_0' emits exits.

This shows that it is sufficient to establish that $v_0' \in \mathrm{Fea}_F$. If $\mathrm{Fea}_F = (c')^0$, this trivially holds, so it is sufficient to consider the case when Fea_F is strictly smaller than $(c')^0$. In this case, v'_{j_0} receives some edges outside of c'. By considering the map $f_{E/H}$, we have that E/H also has more edges than only those in c, so v_0 also receives edges outside of c. Hence, the spine k_0 of E^{v_0} is positive.

If $k_0 \geq m$, then v_0 is the only vertex in E/H which receives edges outside of c (see section 4.8). By Lemma 4.9, the 1-S-NE canonical form of E/H has tails for every $j \in m$. By considering $f_{E/H}$, we have that the canonical form of F/G also has tails for every j. By Lemma 4.9, $\mathrm{Fea}_E = \{v_0\}$ and $\mathrm{Fea}_F = \{v'_{j_0}\}$. We claim that $j_0 = 0$.

The generating interval D_E contains $|P_1^{v_0}| = |\mathcal{P}_1^{v_0}|+1$ elements of the form $t[v_0]$ (if the cardinality $|\mathcal{P}_1^{v_0}|+1$ is infinite, this means that D_E contains finite sums of the form $lt[v_0]$ for every finite cardinal in $|\mathcal{P}_1^{v_0}|+1$). Since $f(D_E) = D_F$, the number of elements of the form $t[v_0']$ is also $|\mathcal{P}_1^{v_0}|+1$ on one hand, and, on the other, it is $|P_1^{v_0'}| = |\mathcal{P}_1^{v_0'}| + 1$. So, $|\mathcal{P}_1^{v_0}| = |\mathcal{P}_1^{v_0'}|$. Since $|\mathcal{P}_1^{v_0}| \geq 1$, we have that $|\mathcal{P}_1^{v_0'}| \geq 1$ which implies that $j_0 = 0$ as $\mathrm{Fea}_F = \{v'_{j_0}\}$.

Hence, it remains to consider the case $k_0 < m$. In this case, we consider the multiplicity of the elements $t^i[v_0]$ in D_E, for $i \geq 0$. Since $f(D_E) = D_F$, this multiplicity is the same for $t^i[v_0']$ and $|P_i^{v_0}| = |P_i^{v_0'}|$. If $i = 1, \ldots, k_0$, $|\mathcal{P}_i^{v_0}| > 0$, so $|P_i^{v_0'}| = |P_i^{v_0}| = |\mathcal{P}_i^{v_0}| + 1 > 1$. Since $|P_i^{v_0'}| > 1$ holds for every $i = 1, \ldots, k_0$, the spine length k_0' of $E^{v_0'}$ is at least k_0. The existence of $f_{E/H}$ implies that $(E/H)_{\mathrm{can}} \cong (F/G)_{\mathrm{can}}$, and v_{m-k_0} has no tails in $(E/H)_{\mathrm{can}}$. Hence, v'_{m-k_0} has no tails in $(F/G)_{\mathrm{can}}$

and so $E^{v'_{m-k_0}}$ has no edges in F/G. Thus, $k'_0 \leq k_0$. Since we also have $k'_0 \geq k_0$, $k_0 = k'_0$. For every $i \leq k_0$, $|\mathcal{P}_i^{v_0}| + 1 = |P_i^{v_0}| = |P_i^{v'_0}| = |\mathcal{P}_i^{v'_0}| + 1$ and so $|\mathcal{P}_i^{v_0}| = |\mathcal{P}_i^{v'_0}|$. Thus, $E^{v_0} \cong E^{v'_0}$ holds.

Let v_{m-i_0} be the first vertex before v_0 such that $E^{v_{m-i_0}}$ has edges. In this case, $m - i_0 < m - k_0$, so $i_0 > k_0$ and $i_0 + 1$ is the first integer larger than k_0 such that $|P_{i_0+1}^{v'_0}| = |P_{i_0+1}^{v_0}| = |\mathcal{P}_1^{v_{m-i_0}}| + 1 > 1$. This shows that $i_0 + 1$ is the first integer larger than k_0 such that $E^{v'_{m-i_0}}$ has edges. Since $|P_{i+i_0}^{v'_0}| = |P_{i+i_0}^{v_0}| = |\mathcal{P}_i^{v_{m-i_0}}| + 1 > 1$ for every $i = 1, \ldots, k_{m-i_0}$, the spine length k'_{m-i_0} of $E^{v'_{m-i_0}}$ is at least k_{m-i_0}. As $(E/H)_{\text{can}} \cong (F/G)_{\text{can}}$, and $v_{m-k_0-k_{m-i_0}}$ has no tails in $(E/H)_{\text{can}}$, $v'_{m-k_0-k_{m-i_0}}$ has no tails in $(F/G)_{\text{can}}$. Thus, $E^{v'_{m-k_0-k_{m-i_0}}}$ has no edges in F/G and so $k'_{m-i_0} \leq k_{m-i_0}$. This shows that $k_{m-i_0} = k'_{m-i_0}$. For every $i = 1, \ldots, k_{m-i_0}$, $|\mathcal{P}_i^{v_{m-i_0}}| + 1 = |P_{i+i_0}^{v_0}| = |P_{i+i_0}^{v'_0}| = |\mathcal{P}_i^{v'_{m-i_0}}| + 1$ and so $|\mathcal{P}_i^{v_{m-i_0}}| = |\mathcal{P}_i^{v'_{m-i_0}}|$. Thus, $E^{v_{m-i_0}} \cong E^{v_{m-i_0}}$ holds.

We continue the argument by considering i_1 such that $v_{m-i_0-i_1}$ is the first vertex before v_{m-i_0} such that $E^{v_{m-i_0-i_1}}$ has edges and show that $v'_{m-i_0-i_1}$ is the first vertex before v'_{m-i_0} such that $E^{v'_{m-i_0-i_1}}$ has edges and that $E^{v_{m-i_0-i_1}} \cong E^{v'_{m-i_0-i_1}}$ holds. After finitely many steps, the process terminates and $\iota : E/H \cong F/G$ for some ι such that $\iota(v_0) = v'_0$. As v_0 is feasible, v'_0 is feasible. □

This proposition enables us to assume that $E/H \cong F/G$ in the rest of the proof. If $n > 0$, we claim that we can also assume that v'_0 is the single exit-emitter of F. Indeed, if v'_0 is not emitting exits, then we can use Lemma 4.11 to consider another canonical form F' of $F_{\text{red,cut}}$ which has v'_0 emitting exits. Let $\overline{\psi} : F \to F'$ be the operation from Lemma 4.11. By considering F' instead of F, $\overline{\psi}f$ instead of f, and c'_{j_0} instead of c', we can assume that $f_{E/H}([v_0]) = [v'_0]$ where both v_0 and v'_0 are single exit-emitters.

At this point, we prove another lemma. It critically uses the fact that E and F are in their cut forms and it does not hold without that assumption, as some of our earlier examples showed.

Lemma 4.16. *Let $E = E_{\text{cut}}$ and $F = F_{\text{cut}}$ be such that the connecting matrices computed by considering some c, c', d, and d' are the same. If there is a $\mathbf{POM}^D$-isomorphism f of the Γ-monoids such that $f([v_0]) = [v'_0]$ and $f([w_0]) = [w'_0]$, then there is a graph isomorphism $\iota : E \cong F$ such that $f = \bar{\iota}$, that $\iota(v_j) = v'_j$ for $j \in m$, and that $\iota(w_i) = w'_i$ for $i \in n$.*

Proof. Let $\iota : E/H \cong F/G$ be such that $\iota(v_0) = v'_0$. Thus, $\iota(v_j) = v'_j$ for $j \in m$. If $m > 0$, such ι exits by Proposition 4.15. If $m = 0$, such ι exits since both quotient graphs are in their 1-S-NE canonical forms by Proposition 3.4 (also see section 4.8). To unify the cases, if $m = 0$, writing $j \in m$ means $j = 0$.

As the connecting matrices are the same, we can extend ι to $T(v_0)$ by mapping d onto d' where $w_0 = \mathbf{s}(d)$ and $w'_0 = \mathbf{s}(d')$, for those $d \in [d]$ and $d' \in [d']$ which are used to compute the connecting matrices. Thus, $\iota(w_i) = w'_i$ and the ji-exits of E are mapped bijectively onto the ji-exits of F for all $j \in m$ and $i \in n$ if $n > 0$ and $i \leq k$ if $n = 0$. Thus, the extension, which we still call ι, is an isomorphism of the subgraph $E_{T\cup R}$ of E generated by $T(v_0) \cup R(v_0)$ and the subgraph $F_{T\cup R}$ of F generated by $T(v'_0) \cup R(v'_0)$. We show that ι can be extended to the tails.

Condition $C(i)$, considered on E, depends only on the subgraph $E_{T\cup R}$ and $C(i)$, considered on F, depends only on the subgraph $F_{T\cup R}$. Thus, for every $i \in n$ if $n > 0$ and every $i \in \omega$ if $n = 0$, we have that i-tails of E are cuttable if and only if i-tails of F are cuttable.

Because $n - i$ in condition $C(i)$ for $n > 0$ corresponds to i in condition $C(i)$ for $n = 0$, we consider the cases $n > 0$ and $n = 0$ separately. The proof follows the same arguments in both cases.

If $n>0$ and $i\in n$, let $P^{w_0}_{\not d,i}$ be the set of paths of $P^{w_0}_{\not d}$ of length $n-i+1$ modulo n. Let us partition $P^{w_0}_{\not d,i}$ into three disjoint sets, some possibly empty, depending on the location of their sources:

$$P^{w_0}_{\text{exits},i}=\{p\in P^{w_0}_{\not d,i}\mid \mathbf{s}(p)\in T(v_0)\},\ P^{w_0}_{\text{quot},i}=\{p\in P^{w_0}_{\not d,i}\mid \mathbf{s}(p)\in E^0-(H\cup T(v_0))\}\text{ and}$$
$$P^{w_0}_{\text{tails},i}=\{p\in P^{w_0}_{\not d,i}\mid \mathbf{s}(p)\in H-T(v_0)\}.$$

Thus, $P^{w_0}_{\text{tails},i}$ is the set of i-tails. The length of a path $p\in P^{w_0}_{\text{quot},i}$ is $l+k'|c|+d(j,j')+1+n-i'$ for some $i'\in n$, $l,k'\in\omega$, and $j,j'\in m$ such that $a_{j'i'}\neq 0$ and $|p|=n-i+1(\operatorname{mod} n)$. Thus, $C(i)$ holds for such values l,k',j,j' and i' if and only if $|\mathcal{P}^{v_j}_l|=\omega$.

Let $P^{w_0'}_{\not d',i}$ be analogously defined for F and let $P^{w_0'}_{\text{exits},i}, P^{w_0'}_{\text{quot},i}$, and $P^{w_0'}_{\text{tails},i}$ be analogously defined for $P^{w_0'}_{\not d',i}$. The existence of f ensures that there is a bijection $\sigma_i: P^{w_0}_{\not d,i}\to P^{w_0'}_{\not d',i}$ and the existence of ι ensures that such σ_i can be found so that it maps $P^{w_0}_{\text{exits},i}$ onto $P^{w_0'}_{\text{exits},i}$.

If condition $C(i)$ holds for $i\in n$ (in E and, equivalently, in F), then i-tails are cuttable both in E and in F and we have that $P^{w_0}_{\text{tails},i}=P^{w_0'}_{\text{tails},i}=\emptyset$. So, the existence of ι enables us to find σ_i so it maps the elements of $P^{w_0}_{\text{quot},i}$ with the same l,k',j,j' and i' values onto the elements of $P^{w_0'}_{\text{quot},i}$ with the same l,k',j,j' and i' values. If $|\mathcal{P}^{v_j}_l|<\omega$, then the existence of ι also ensures the existence of such σ_i. Thus, we necessarily have that such σ_i maps $P^{w_0}_{\text{tails},i}$ onto $P^{w_0'}_{\text{tails},i}$. Since this holds for every $i\in n$, E and F have the same number of i-tails for any $i\in n$. If ι_i is a bijection of i-tails for $i\in n$, let us extend ι to the tails by mapping an i-tail e to $\iota_i(e)$. Such extension, let us keep calling it ι, is an isomorphism $E\cong F$ such that $\bar\iota=f$.

If $n=0$, then $P^{w_0}_{\not d}=P^{w_0}$. Let $P^{w_0}_i$ be the set of paths of length $i+1\in\omega$ ending at w_0 and let $P^{w_0}_{\text{exits},i}$, $P^{w_0}_{\text{quot},i}$, and $P^{w_0}_{\text{tails},i}$ be defined analogously as when $n>0$. The length of $p\in P^{w_0}_{\text{quot},i}$ is $l+k'|c|+d(j,j')+1+i'=i+1$ for some $i'\leq k$, $l,k'\in\omega$, and $j,j'\in m$ such that $a_{j'i'}\neq 0$. Thus, just as in the $n>0$ case, $C(i)$ holds for such values l,k',j,j' and i' if and only if $|\mathcal{P}^{v_j}_l|=\omega$. The rest of the arguments is the same as when $n>0$: there is a bijection σ_i which maps $P^{w_0}_{\text{tails},i}$ onto $P^{w_0'}_{\text{tails},i}$. This implies that both tail graphs have the same spine length k_t and that the number of i-tails is equal for all $i\leq k_t$. So, there is a bijective correspondence ι_i of the i-tails for every $i\leq k_t$ enabling us to extend ι to the tails and bringing us to an isomorphism $\iota: E\cong F$ such that $\bar\iota=f$. □

The $\mathbf{m>0,n>0}$ case. Let L be the least common multiple of m and n and G be the greatest common divisor of m and n. Let m' be such that $m=Gm'$ and n' be such that $n=Gn'$.

Let us choose $d\in[d]$ so that the exits receivers in E are among w_{n-G+1} to w_0. Let $i_0\in n$ be such that $w'_{i_0-_n(G-1))}$ to w'_{i_0} receive exits in F. By moving all exits of F by a multiple of m and then L-reducing and cutting, if needed, we can assume that $i_0\in\{n-G+1,\ldots,n-1,0\}$. Hence $n-_n i_0\in\{0,\ldots,G-1\}$.

With this set up, we have that $a_E=a_{00}t+a_{0(0-_n1)}t^2+\ldots+a_{0(0-_n(G-1))}t^G$ holds, so that $a_E[w_0]=\sum_{j=1}^G a_{0(0-_n(j-1))}t^j[w_0]$ and $a_F=a_{0i_0}t^{n-_ni_0+1}+a_{0(i_0-_n1)}t^{n-_ni_0+2}+\ldots+a_{0(i_0-_n(G-1))}t^{n-_ni_0+G}$. Thus, we have that $a_F[w_0']=\sum_{j=1}^G a'_{0(i_0-(j-1))}t^{n-_ni_0+j}[w_0']$.

Let $A_1=a_{0(0-_n(G-1))}$, $A_0=a_E-A_1$, and let us consider a_E as the sum of n' blocks $A_0+A_1t^G+0t^{2G}+\ldots+0t^{(n'-1)G}$. We consider these blocks because we have that $t^Ga_E[w_0]=(0t^0+A_0t^G+A_1t^{(1+_{n'}1)G}+0+\ldots+0t^{(n'-_{n'}1)G})[w_0]$. Using these blocks, the multiplication of a_E by a term of the form t^{lm} for $l\in\mathbb{Z}^+$ is well-represented. Let us represent a_F using similar blocks. If G_0 is such that $n-_n i_0+G_0=G$, let $A_0'=a'_{0i_0}t^{n-_ni_0+1}+\ldots+a'_{0(i_0-_n(G_0-2))}t^{n-_n(i_0-G_0+2)+1}=a'_{0i_0}t^{n-_ni_0+1}+\ldots+$

$a'_{0(n-G-2)}t^{G-1}$ and $A'_1 = a'_{0(i_0-_n(G_0-1))} + \ldots + a'_{0(i_0-_n(G-1))}t^{n-_n i_0} = a'_{0(n-G-1))} + \ldots + a'_{0(i_0-_n(G-1))}t^{n-_n i_0}$ so that we have that $a_F[w'_0] = (A'_0 + A'_1 t^G)[w'_0]$ and use the same rules for computing $t^m a_F[w'_0]$.

We represent b using similar blocks. Let $b = \sum_{i\in n} b_i t^i$. We can assume that the degree of b is at most $n-1$ since b is always considered either applied to $[w_0]$ or $[w'_0]$. So, we have that

$$b = \sum_{k'\in n'}\sum_{j\in G} b_{j+k'G}t^{j+k'G} = B_0 + B_1 t^G + \ldots + B_{n'-_{n'}1}t^{(n'-_{n'}1)G} \text{ for } B_{k'} = \sum_{j\in G} b_{j+k'G}t^j.$$

Let $m_0 < n'$ be such that $m' = m_0 (\operatorname{mod} n')$ (thus $m_0 G < n'G = n$ and $m = m_0 G (\operatorname{mod} n)$) and $n_0 < n'$ be such that $n_0 = lm' (\operatorname{mod} n')$ (thus $n_0 G < n'G = n$ and $lm = n_0 G (\operatorname{mod} n)$). With these values, $t^{lm} a_F[w'_0] = t^{n_0 G} a_F[w'_0] = (A_0 t^{n_0 G} + A_1 t^{(n_0 +_{n'} 1)G})[w'_0]$ and $t^m b[w'_0] = t^{m_0 G} b[w'_0] = (B_{n'-_{n'}m_0} + B_{(n'-m_0)+_{n'}1}t^G + \ldots + B_{(n'-m_0)-_{n'}1}t^{(n'-_{n'}1)G})[w'_0]$. So, the coefficients with $[w'_0]$ on both sides of the equation (4) are

$$(0 + \ldots + 0 + A'_0 t^{n_0 G} + A'_1 t^{(n_0+_{n'}1)G} + 0 + \ldots + 0) + (B_0 + B_1 t^G + \ldots + B_{n'-_{n'}1}t^{(n'-_{n'}1)G}) \text{ and}$$

$$(B_{n'-_{n'}m_0} + B_{(n'-m_0)+_{n'}1}t^G + \ldots + B_{(n'-m_0)-_{n'}1}t^{(n'-_{n'}1)G}) + (A_0 + A_1 t^G + 0t^{(1+_{n'}1)G} + \ldots + 0t^{(n'-_{n'}1)G}).$$

Equating the terms with jG for $j \in n'$ produces n' equations. Adding them all up produces an equation which has $\sum_{j\in n'} B_j$ on both sides. Canceling this sum produces $A'_0 + A'_1 = A_0 + A_1$. This implies that $a_{0(n-_n j)} = a'_{0(n-_n j)}$ for $j = i_0, \ldots, G-1$ and $a_{0(n-_n j)} = a'_{0(n-_n j-_n G)}$ for $j = 0, \ldots, n-i_0+1$. By Lemma 4.4, we can move $0(n-_n j -_n G)$-exits of F to $0(n-_n j)$-exits for every $j = 0, \ldots, n-i_0+1$. If F' is an L-reduced and cut form of the resulting graph and if $\psi : F \to F'$ is the exit-move operation followed by an L-reduction and the cut map, we have that the connecting matrices of E and F' are the same and that $\overline{\psi} f([w_0]) = [w'_0]$. We can consider F' instead of F and $\overline{\psi} f$ instead of f and we will continue to refer to these new elements as F and f. Thus, we have that $a_E = a_F$ (i.e $A_0 = A'_0$ and $A_1 = A'_1$).

Returning to the n' equations, we note that if $n_0 > 1$, then $n' > 2$ and the $(n_0 +_{n'} 1)$-th equation is $A'_0 + B_{n_0} = B_{n'-_{n'}m_0+n_0}$, so A'_0 is a summand of $B_{n'-_{n'}m_0+n_0}$. The $(n_0 +_{n'} 2)$-th equation is $A'_1 + B_{n_0+_{n'}1} = B_{n'-_{n'}m_0+n_0+1}$ so A'_1 is a summand of $B_{n'-_{n'}m_0+n_0+1}$. Thus, $t^{(l-1)m} a_F[w'_0] = (A'_0 + A'_1 t^G) t^{lm-m}[w'_0] = (A'_0 + A'_1 t^G) t^{(n_0-m_0)G}[w'_0]$ is a summand of $b[w'_0]$. Thus, we have that

$$f([v_0]) = t^{lm}[v'_0] + b[w'_0] = t^{(l-1)m}(t^m[v'_0] + a_F[w'_0]) + b_{l-1}[w'_0] = t^{(l-1)m}[v'_0] + b_{l-1}[w'_0]$$

for a summand b_{l-1} of b. Since all the subscripts of the n' equations are considered modulo n', one shows that the same simplification is possible whenever $n_0 > 0$: if $n_0 = 1$, we can simplify $f([v_0])$ in the same way. This simplification enables us to consider $l-1$ instead of l. By renaming the new l-value to be l and the new b-value to be b, we can continue this process until $l = 0$ or $n_0 = 0$.

Since $l = 0$ implies $n_0 = 0$, it is sufficient to consider the case $n_0 = 0$. Returning the values $n_0 = 0, A_0 = A'_0$, and $A_1 = A'_1$ to the n' equations coming out of the equation (4), we have that $B_j = B_{j+_{n'}(n'-m_0)}$ for every $j \in n'$. If n does not divide m, then $m_0 \neq 0$ so all blocks of b are mutually equal to each other: $B_j = B_k$ for all $j, k \in n'$. If n divides m, there is only one block B_0, so this trivially holds. In any case, since $lm' = 0 (\operatorname{mod} n')$ and m' and n' are mutually prime, we necessarily have that $l = 0 (\operatorname{mod} n')$.

If l' and b' are analogous to l and b for f^{-1}, and n'_0 is analogous to n_0, the same argument shows that $n'_0 = 0$ so that n' divides l'. So, there are nonnegative integers k and k' such that $l = kn'$ and $l' = k'n'$. Hence, n divides both lm and $l'm$ and we have that $t^{lm}b = b$, $t^{l'm}b' = b'$, $t^{l'm}b = b$, and $t^{lm}b' = b'$ hold in $\mathbb{Z}[t]/(t^n-1)$. The equations (5) become $(b'+b)[w_0] = \sum_{j=0}^{l+l'-1} t^{jm}a_E[w_0]$ and $(b'+b)[w_0'] = \sum_{j=0}^{l+l'-1} t^{jm}a_E[w_0']$ so we have that

$$(B_0+B_0')\sum_{j\in n'} t^{jG} = b+b' = \sum_{j=0}^{l+l'-1} t^{jm}a_E$$

holds in $\mathbb{Z}[t]/(t^n-1)$ and this implies that $B_0+B_0' = (k+k')(A_0+A_1)$ holds. This shows that B_0 is equal to $k_0a_{0(0-_n(G-1))}+d_0+\sum_{j\in G, j\neq 0}(k_ja_{0(0-_n(j-1))}+d_j)t^j$ for some nonnegative integers k_j and $d_j < a_{0(0-_n(j-1))}$ for $j\in G, j\neq 0$ and $d_0 < a_{0(0-_n(G-1))}$.

Let E' be the cut form of the graph obtained from E by moving each $0(n-_n(j-1))$-exit Lk_j times and moving d_j-many of the $0(n-j+1)$-exits L times. Let $\phi_{+b}: E\to E'$ be this move operation followed by the cut map. We denote this operation by ϕ_{+b} since $\overline{\phi}_{+b}([v_0]) = [v_0]+b[w_0]$. Thus, the composition $\overline{\phi}_{+b}f^{-1}$ maps $[w_0']$ to $[w_0]$ and $[v_0']$ to

$$t^{l'm}[v_0]+(b'+b)[w_0] = t^{l'm}[v_0]+\sum_{j=0}^{l+l'-1} t^{jm}a_E[w_0] = t^{l'm}[v_0]+\sum_{j=0}^{l'-1}t^{jm}a_E[w_0]+\sum_{j=l'}^{l+l'-1}t^{jm}a_E[w_0] =$$

$$[v_0]+\sum_{j=l'}^{l+l'-1}t^{jm}a_E[w_0] = [v_0]+t^{l'm}\sum_{j=0}^{l-1}t^{jm}a_E[w_0] = [v_0]+\sum_{j=0}^{l-1}t^{jm}a_E[w_0].$$

The L-reduction of E' followed by the cut map is ϕ_{+b}^{-1} and its image is E since E is L-reduced and cut. We have that $\overline{\phi}_{+b}^{-1}([v_0]) = t^{(l+l')m}[v_0]+(\sum_{j=0}^{l+l'} t^{jm}a_E - b)[w_0'] = t^{(l+l')m}[v_0]+b'[w_0']$ holds. Thus, $f\overline{\phi}_{+b}^{-1}([v_0]) = t^{(2l+l')m}[v_0']+(b+b')[w_0'] = t^{lm}[v_0']$.

Let F' be the graph obtained by moving all exits of F kL times and then cutting the resulting graph (recall that $k=\frac{n}{l}$). Let $\phi_{+kL}: F\to F'$ be this operation. The composition $\overline{\phi}_{+kL}f\overline{\phi}_{+b}^{-1}$ is defined and it maps $M_{E'}^{\Gamma}$ onto $M_{F'}^{\Gamma}$. We have that $\overline{\phi}_{+kL}f\overline{\phi}_{+b}^{-1}([v_0]) = \overline{\phi}_{+kL}(t^{lm}[v_0']) = [v_0']$. Both E' and F' are in their cut forms so we can use Lemma 4.16. By this lemma, there is a graph isomorphism $\iota': E'\cong F'$ such that $\iota'(v_j)=v_j'$ and $\iota'(w_i)=w_i'$ for every $j\in m$ and $i\in n$ and that $\overline{\phi}_{+kL}f\overline{\phi}_{+b}^{-1} = \overline{\iota'}$ so that

$$f = \overline{\phi_{+kL}^{-1}\iota'\phi_{+b}}$$

holds. This shows that we can realize f. So, it remains to show that $E\cong F$.

Since all four graphs E, F, E' and F' have isomorphic quotients and the same c-to-d part, only the number of tails possibly distinguishes them. If t_i is the number of tails added to l_i in the process $\phi_{+b}: E\to E'$ and t_i' in the process $\phi_{+kL}: F\to F'$, we have that $l_i+t_i = l_i'+t_i'$ since $E'\cong F'$. For i such that $t_i=\omega$, we have that $t_i'=\omega$ also since the quotients are isomorphic and the connecting matrices the same. The converse $t_i'=\omega\Rightarrow t_i=\omega$ also holds, so the i-tails are cuttable in E if and only if they are cuttable in F and $l_i=l_i'=0$ for such i. For i such that $t_i<\omega$ (thus $t_i'<\omega$), we have that $l_i=\omega$ if and only if $l_i=\omega$, so $l_i=l_i'$ again in this case. Thus, if $l_i=\omega$ for all $i\in n$, then $E\cong E'\cong F'\cong F$, so E' and F' are also canonical forms.

Hence, it remains to consider i such that all four values in the formula $l_i+t_i=l_i'+t_i'$ are finite. If $k_ja_{0(0-_n(j-1))}+d_j < ka_{0(0-_n(j-1))}$ for some $j\in G, j\neq 0$ or $k_0a_{0(0-n(G-1))}+d_0 < ka_{0(0-_n(G-1))}$, then a $0(0-_n(j-1))$-exit of F for $j\in\{1,\ldots,G\}$ is moved more times by ϕ_{+kL} than any $0(0-_n(j-1))$-exit of E by ϕ_{+b}. As F is L-reduced, this cannot happen if there is $i\in n$ such that $l_i'<\omega$, so $k_ja_{0(0-_n(j-1))}+d_j \geq ka_{0(0-_n(j-1))}$ and $k_0a_{0(0-n(G-1))}+d_0\geq ka_{0(0-_n(G-1))}$. If $k_ja_{0(0-_n(j-1))}+d_j >$

$ka_{0(0-_n(j-1))}$ for some $j \in G, j \neq 0$ or $k_0 a_{0(0-n(G-1))} + d_0 > ka_{0(0-_n(G-1))}$, then a $0(0-_n(j-1))$-exit of E for $j \in \{1, \ldots, G\}$ is moved more times by ϕ_{+b} than any $0(0-_n(j-1))$-exit of F by ϕ_{+kL}. This also cannot happen if there is $i \in n$ such that $l_i < \omega$, so we have that $k_j a_{0(n-_n(j-1))} + d_j = ka_{0(n-_n(j-1))}$ for all $j \in G, j \neq 0$ and $k_0 a_{0(0-n(G-1))} + d_0 = ka_{0(0-_n(G-1))}$. By the definition of d_j for $j \in G$, this implies that $d_j = 0$ for all $j \in G$ and so $k_j = k$ for all $j \in G$. This shows that ϕ_{+b} and ϕ_{+kL} are the same moves and that, in particular, $b[w_0'] = \sum_{j=0}^{l-1} t^{jm} a_E[w_0']$ so that $f([v_0]) = [v_0']$. Thus, starting with isomorphic graphs $E' \cong F'$, the maps ϕ_{+b}^{-1} and ϕ_{+kL}^{-1} produce isomorphic graphs. The map $\phi_{+kL}^{-1}\iota'\phi_{+b}$ is a graph isomorphism $\iota : E \cong F$ and its existence also follows from Lemma 4.16 at this point. So, we have that $f = \bar{\iota}$ in this case. This shows that we have $E \cong F$ in any case.

Before continuing with the proof of the next case, let us present some nonidentity $\mathbf{POM}^D$-automorphisms of the graphs from Example 4.3. For both graphs of that example, the map given by $f([v_0]) = [v_0] + t[w_0]$ and $f([w_0]) = [w_0]$ corresponds to moving one of the exits. For the first graph, f has the inverse such that $f^{-1}([v_0]) = t[v_0] + t[w_0]$ and for the second graph, $f^{-1}([v_0]) = t^2[v_0]$.

The $\mathbf{m > 0, n = 0}$ case. Since $n = 0$, we have that $f([w_0]) = [w_0']$. We can pick $c \in [c]$ so that v_0 is feasible. By Proposition 4.15, we can choose a canonical form of F_0 such that $f_{E/H}([v_0]) = [v_0']$ and that v_0' is feasible. Let k and k' be the lengths of the spines of E and F. Assume that $k \leq k'$. The case $k \geq k'$ is analogous by considering f^{-1} instead of f.

Since $n = 0$ and a sink is incomparable, we can write equation (4) as $t^{lm} a_F + b = t^m b + a_E$ which holds in $\mathbb{Z}[t]$. By the definition of a canonical form, a_E contains no monomial of degree smaller than $k+1$ and of degree larger than $k+m$ and $t^{lm} a_F$ contains no monomial of degree smaller than $k'+1+lm$ and of degree larger than $k'+m+lm$. Thus, b does not contain a monomial of degree smaller than $k+1$ which implies that $t^m b$ does not contain a monomial of degree smaller than $k+m+1$. Since $k \leq k'$, $k+m+1 \leq k'+m+1$ and, if $l > 0$, then $k'+m+1 \leq k'+lm+1$, so $t^m b$ contains $t^{lm} a_F$ as a summand. Consequently, b contains $t^{(l-1)m} a_F$ as a summand so that

$$f([v_0]) = t^{lm}[v_0'] + t^{(l-1)m} a_F [w_0'] + b_{l-1}[w_0] = t^{(l-1)m}[v_0'] + b_{l-1}[w_0']$$

holds for a summand b_{l-1} of b. Thus, we can decrease the value of l if $l > 0$ and $b \neq 0$. By renaming the new l-value to l and the new b-value to b, we can continue this process until we eventually get that $l = 0$ or $b = 0$. If $b = 0$, then $t^{lm} a_F = a_E$. Since $k' \geq k$, this would be possible only if $l = 0$ in which case $a_E = a_F$ and there is a graph isomorphism $\iota : E \cong F$ such that $f = \bar{\iota}$ by Lemma 4.16. Thus, it remains to consider the case $l = 0$ and $b \neq 0$.

Returning $l = 0$ to the equation (4) produces $a_F + b = t^m b + a_E$ which implies that the degree of $t^m b$ is at most $m + k'$. Hence, the degree of b is at most k'. The equation (4) for f^{-1} produces $t^{l'm} a_E + b' = t^m b' + a_F$ and it implies that b' does not contain a monomial with degree smaller than $k'+1$. Hence, all monomials in b' have degrees strictly larger than those in b. The first equation of (5) implies that $b + b' = \sum_{j=0}^{l'-1} t^{jm} a_E$, so there is $j_0 \in l'$ such that $b = \sum_{j=0}^{j_0-1} t^{jm} a_E + b_0$ and $b' = \sum_{j=j_0+1}^{l'-1} t^{jm} a_E + b_0'$ where b_0 and b_0' are such that their sum is $t^{j_0 m} a_E$.

Let E' be the cut form of the graph obtained by moving all exits of E $j_0 m$ times and then moving those which correspond to the monomials present in b_0 m more times. Let $\phi_{+b} : E \to E'$. Since all exits of E end in w_k, $b \neq 0$ holds if and only if the spine of E' is longer than k. If ϕ_{+b}^{-1} is the inverse operation defined on E'. As

$$t^{l'm}[v_0] + b'[w_0] + b[w_0] = t^{l'm}[v_0] + \sum_{j=0}^{l'-1} t^{jm} a_E [w_0] = [v_0] = \phi_{+b}^{-1}\phi_{+b}([v_0]) = \phi_{+b}^{-1}([v_0]) + b[w_0],$$

we can cancel $b[w_0]$ in the first and the last expression and have that $\phi_{+b}^{-1}([v_0]) = t^{l'm}[v_0] + b'[w_0]$.

The composition $f\overline{\phi_{+b}}^{-1} : E' \to F$ maps $[v_0]$ to $t^{l'm}[v_0'] + (t^{l'm}b + b')[w_0'] = t^{l'm}[v_0'] + \sum_{j=0}^{l'-1} t^{jm} a_F[w_0'] = [v_0']$. Equation (4) for this composition, implies that $a_F = a_{E'}$. Both F and E' are cut, so there is a graph isomorphism $\iota : E' \cong F$ such that $f\overline{\phi_{+b}}^{-1} = \overline{\iota}$ by Lemma 4.16. Thus, $f = \overline{\iota\phi_{+b}}$.

The relation $E' \cong F$ implies that the spine length of E' is also k'. If $b \neq 0$, then $k' > k$ and E' can be spine-reduced to E, so E' is not spine-reduced. However, since $F \cong E'$ and F *is* spine-reduced, this would be a contradiction. Hence, $b = 0$ and so $k' = k$, $E' = E$, and $\iota : E \cong F$ is such that

$$f = \overline{\iota}.$$

<u>The m = 0 case.</u> Since $m = 0$, v_0 and v_0' are infinite emitters. The relation $f_{E/H}([v_0]) = [v_0']$ implies that $f([v_0])$ has exactly one of the elements $[v_0']$ and $[q_{Z'}]$ for some nonempty and finite $Z' \subseteq \mathbf{s}^{-1}(v_0')$ present in its representation via the generators. Letting $q_\emptyset = v_0'$, we unify the treatment of these two cases so that for any finite $Z \subseteq \mathbf{s}^{-1}(v_0)$, we write

$$f([q_Z]) = [q_{Z_F}] + b_Z[w_0']$$

for some finite $Z_F \subseteq \mathbf{s}^{-1}(v_0')$ and some $b_Z \in \mathbb{Z}^+[t, t^{-1}]$. Using the same argument as in the $m > 0$ case, one shows that b_Z can be taken to be in $\mathbb{Z}^+[t]$. The map f^{-1} necessarily has the same form as f so, let $f^{-1}([q_{Z'}]) = [q_{Z'_E}] + b'_{Z'}[w_0]$ for some $b'_{Z'} \in \mathbb{Z}^+[t]$ and some finite $Z'_E \subseteq \mathbf{s}^{-1}(v_0)$. As $[q_{(\emptyset_E)_F}] + a_{(\emptyset_E)_F}[w_0'] = [v_0'] = f(f^{-1}([v_0'])) = f([q_{\emptyset_E}] + b'_\emptyset[w_0]) = [q_{(\emptyset_E)_F}] + b_{\emptyset_E}[w_0'] + b'_\emptyset[w_0']$

$$(b_{\emptyset_E} + b'_\emptyset)[w_0'] = a_{(\emptyset_E)_F}[w_0'] \text{ holds and, similarly, } (b'_{\emptyset_F} + b_\emptyset)[w_0] = a_{(\emptyset_F)_E}[w_0] \text{ holds.}$$

Thus, $b_\emptyset[w_0] = a_W[w_0]$ and $b'_\emptyset[w_0'] = a_{W'}[w_0']$ for some finite sets $W \subseteq \mathbf{s}^{-1}(v_0)$ and $W' \subseteq \mathbf{s}^{-1}(v_0')$.

With this notation, we show that $a_{0i} = \omega$ implies $a'_{0i} = \omega$ for all $i \in n$ if $n > 0$ and $i \leq k$ if $n = 0$. If $a_{0i} = \omega$ and $k \in \mathbb{Z}^+$ is arbitrarily large, then $kt^{n-_n i+1}[w_0]$ is a summand of $[v_0]$ if $n > 0$ and $kt^{i+1}[w_0]$ is a summand of $[v_0]$ if $n = 0$. Since the power of t is the only difference between the two cases, we prove the claim for $n > 0$ and the proof for $n = 0$ is analogous. So, $n > 0$ and we have that $kt^{n-_n i+1}[w_0'] = f(kt^{n-_n i+1}[w_0])$ is a summand of $f([v_0]) = [q_{\emptyset_F}] + b_\emptyset[w_0']$. By taking $k > |b_\emptyset|$, we have that $(k - |b_\emptyset|)t^{n-_n i+1}[w_0']$ is a summand of $[q_{\emptyset_F}]$. Since $\emptyset_F$ is finite and k can be taken arbitrarily large, this implies that v_0' emits at least $k - |b_\emptyset| - |\emptyset_F|$ paths of length $n -_n i + 1$ to $[w_0']$. Since this holds for any k, a'_{0i} is infinite. Repeating the same argument for f^{-1} ensures that the converse holds, so for any $i \in n$, $a_{0i} = \omega$ if and only if $a'_{0i} = \omega$.

The $\mathbf{m = 0, n > 0}$ case. Let E' be the cut form of the graph obtained by moving exits in $\emptyset_E$ and $\phi_E : E \to E'$ be the corresponding map. We have that $\overline{\phi}_E([v_0]) = [v_0] + a_{\emptyset_E}[w_0]$ and that $\overline{\phi}_E^{-1}([v_0]) = [q_{\emptyset_E}]$. Let F' be the cut form of the graph obtained by moving the exits in W' and ϕ_F the corresponding map so that $\overline{\phi}_F([v_0']) = [v_0'] + a_{W'}[w_0'] = [v_0'] + b'_\emptyset[w_0']$. Thus, $\overline{\phi}_F f\overline{\phi}_E^{-1}([v_0]) = \overline{\phi}_F f([q_{\emptyset_E}]) = \overline{\phi}_F([q_{(\emptyset_E)_F}] + b_{\emptyset_E}[w_0']) = [q_{(\emptyset_E)_F}] + (b_{\emptyset_E} + b'_\emptyset)[w_0'] = [q_{(\emptyset_E)_F}] + a_{(\emptyset_E)_F}[w_0'] = [v_0']$. By Lemma 4.16, there is an isomorphism $\iota' : E' \cong F'$ such that $\overline{\phi}_F f\overline{\phi}_E^{-1} = \overline{\iota}'$ so that

$$f = \overline{\phi_F^{-1}\iota'\phi_E}.$$

This shows that we can realize f. It remains to show that $E \cong F$.

Since $a_{0i} = \omega$ if and only if $a'_{0i} = \omega$ for all $i \in n$, E and F have their connecting matrices equal by the definition of a canonical form. Thus, there is a bijection $\sigma : \mathbf{s}^{-1}(v_0) \to \mathbf{s}^{-1}(v_0')$ such that $a_Z[w_0] = a_{\sigma(Z)}[w_0]$ and $a_{Z'}[w_0'] = a_{\sigma^{-1}(Z')}[w_0']$ for $Z \subseteq \mathbf{s}^{-1}(v_0)$ and $Z' \subseteq \mathbf{s}^{-1}(v_0')$. The formula for

$f([v_0])$ can be simplified if $\sigma(W) \cap \emptyset_F \neq \emptyset$, so we can assume that $\sigma(W) \cap \emptyset_F = \emptyset$ and, similarly, that $\sigma^{-1}(W') \cap \emptyset_E = \emptyset$.

Since $\sigma^{-1}(W')$ and $\emptyset_E$ are disjoint and both E and F are ω-reduced, we have that either $\emptyset_E$ and W' are empty sets (so that $E = E'$ and $F = F'$) or that they consists of exits whose move does not impact the cardinality of any of the tails. Since all the graphs are in their cut form, the second case implies that E/H is finite and that $l_i = l'_i = \omega$ for all i such that i-tails are impacted by a move of any of the exits from $\emptyset_E$ and W'. Thus, the existence of ι' implies that $l_i = l'_i$ for all $i \in n$ in any case. Hence, there is a graph isomorphism $\iota : E \cong F$.

The $\mathbf{m = 0, n = 0}$ case. Let k be the spine length of E and k' the spine length of F. If $k < \omega$, we claim that $k \leq k'$. Indeed, if $k < \omega$, then v_0 emits either zero or ω many exits to w_i for every $i \leq k$ by the definition of a canonical form and we have that $a_{0i} = \omega$ implies $a'_{0i} = \omega$. Since $a_{0k} = \omega$ by the definition of a canonical form, we have that $a'_{0k} = \omega$ so $k' \geq k$. Repeating the same argument for f^{-1}, we have that $k' < \omega$ implies $k' \leq k$.

If $k = \omega$, let l be larger than i such that an exit ending in w_i is in W. Since $b_\emptyset[w_0] = a_W[w_0]$, the degree of $b_\emptyset$ is smaller than l. Since $k = \omega$, for every j arbitrarily larger than l, there is $i \geq j$ such that v_0 emits an exit to w_i. Then $t^{i+1}[w'_0] = f(t^{i+1}[w_0])$ is a summand of $f([v_0]) = [q_{\emptyset_F}] + b_\emptyset[w'_0]$. Since $t^{i+1}[w'_0]$ is not a summand of $b_\emptyset[w'_0]$, it is a summand of $[q_{\emptyset_F}]$. As we can do this for arbitrarily large i-values, we necessarily have that for some i, w'_i is different that the range of edges in $\emptyset_F$. This shows that v'_0 emits an exit to w_i. Since we can take j to be arbitrarily large, $k' = \omega$. By considering f^{-1} and repeating this argument, if $k' = \omega$, then $k = \omega$. This shows that $k = \omega$ if and only if $k' = \omega$.

Thus, if $k < \omega$, we have that $k' < \omega$ and that both $k \leq k'$ and $k' \leq k$ hold, so $k = k'$. If $k = \omega$, then $k' = \omega$, so we we have that $k = k'$ in any case.

If $k = k' < \omega$ the proof of the case $m = 0, n > 0$ applies: E and F have the same connecting matrix by the definition of a canonical form and we can define a bijection σ same as in the previous case and use it to show that there are E', F', ι', ϕ_E and ϕ_F such that $E \cong F$ and that $f = \overline{\phi_F^{-1}\iota'\phi_E}$.

If $k = k' = \omega$, let l be larger than i such that an exit ending in w_i is in $W \cup \emptyset_E$ and that an exit ending in w'_i is in $W' \cup \emptyset_F$. We claim that $a_{0i} = a'_{0i}$ for all $i > l$. If $a_{0i} \neq 0$ for some $i > l$ and if $j \leq a_{0i}$, then $[v_0]$ contains a summand $jt^{i+1}[w_0]$. Hence, $jt^{i+1}[w'_0] = f(jt^{i+1}[w_0])$ is a summand of $f([v_0]) = [q_{\emptyset_F}] + b_\emptyset[w'_0]$. Since i is larger than the degree of $b_\emptyset$, we have that $jt^{i+1}[w'_0]$ is a summand of $[q_{\emptyset_F}]$. By the choice of l, we have that v'_0 emits at least j $0i$-exits, and so $a_{0i} \leq a'_{0i}$. By repeating this argument for f^{-1}, we have that $a'_{0i} \leq a_{0i}$, so $a_{0i} = a'_{0i}$. This shows that the entries of the connecting matrices of E and F can differ only for $i \leq l$ such that $a_{0i} < \omega$ (thus $a'_{0i} < \omega$ also).

Let E' be the cut form of the graph obtained by moving all exits ending in w_i for $i \leq l$ such that $a_{0i} < \omega$ and let $\psi_E : E \to E'$ be the corresponding operation. Let F' be analogously defined graph and $\psi_F : F \to F'$ be analogously defined operation. The graphs E' and F' are also canonical forms of E_0 and F_0 and the composition $\overline{\psi}_F f \overline{\psi}_E^{-1}$ is defined.

Since E' and F' are canonical, we can consider E' and F' instead of E and F and $\overline{\psi}_F f \overline{\psi}_E^{-1}$ instead of f. Our new E and F have the same connecting matrix. Repeating the earlier arguments of the $m = 0$ and $n > 0$ case, we can show that there are graphs E'' and F'', exit moves and cuts $\phi_E : E \to E''$ and $\phi_F : F \to F''$, and an isomorphism $\iota'' : E'' \cong F''$ such that $\overline{\phi}_F f \overline{\phi}_E^{-1} = \overline{\iota''}$ so that

$$f = \overline{\phi_F^{-1}\iota''\phi_E}.$$

Thus, we can realize f. The argument for $E \cong F$ is the same as in the $m = 0, n > 0$ case.

This completes the proof of Theorem 4.13.

5. Composition S-NE graphs

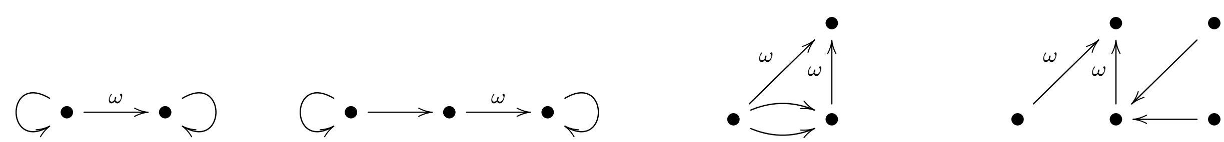
5.1. **Canonical forms of countable composition S-NE graphs.** Let $E = E_{\text{tot}}$ be a countable n-S-NE graph for any $n \geq 1$ with a composition series as below.

$$(\emptyset, \emptyset) \subsetneq (H_1, S_1) \subsetneq (H_2, S_2) \subsetneq \ldots \subsetneq (H_{n-1}, S_{n-1}) \subsetneq (H_n, \emptyset) = (E^0, \emptyset)$$

We claim that we can consider a graph with a composition series such that the sets of breaking vertices B_{H_j} are empty for all $j = 1, \ldots, n-1$ ($H_n = E^0$ so $B_{H_n} = \emptyset$). If $B_{H_1} \neq \emptyset$, and $v \in B_{H_1}$, then v emits infinitely many edges to H_1 and nonzero and finitely many edges to any of $H_{j+1} - H_j, j = 1, \ldots, n-1$. Let us consider an out-split with respect to the partition of $\mathbf{s}^{-1}(v)$ which has each of those finitely many edges in different partition sets and $\mathbf{s}^{-1}(v) \cap \mathbf{r}^{-1}(H_1)$ in the remaining partition set. Since E has finitely many infinite emitters, B_{H_1} is finite, so we can repeat this for any element of B_{H_1}. In the end, we obtain a graph with $B_{H_1} = \emptyset$. By considering E/H_1 and using induction, we can have $B_{H_j} = \emptyset$ for all j. So, we can assume that E has a composition series as below.

$$(\emptyset, \emptyset) \subsetneq (H_1, \emptyset) \subsetneq (H_2, \emptyset) \subsetneq \ldots \subsetneq (H_{n-1}, \emptyset) \subsetneq (H_n, \emptyset) = (E^0, \emptyset).$$

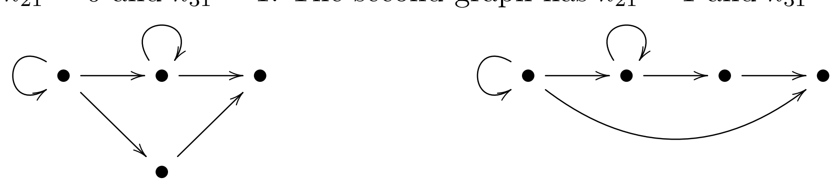
The result of this process for an n-S-NE graph E is a graph which we say is the *breaking-vertices-free form of E*. For example, the second 3-S-NE graph is the breaking-vertices-free form of the first graph and the fourth is such form of the third one.



Let c_{j+1} be a terminal cycle of H_{j+1}/H_j for $j = 0, \ldots, n-1$. If c_l emits an edge to H_j for $1 \leq j < l \leq n$ such that its range is not in $H_{j'}$ for any $j' < l$, then there is a path originating in c_l terminating at c_j and such that no edge is on c_l or c_j. We say that such path is a *c_l-to-c_j path.* The first edge of such path is an *lj-exit.*

If there are no edges emitted from E/H_j to H_j for some $j = 1, \ldots, n$, then E is a disjoint union of P_{H_j} and E/H_j. In this case, we can use induction to assume that canonical forms of P_{H_j} and E/H_j and then define a canonical form of E as a disjoint union of canonical forms of P_{H_j} and E/H_j. Because of this, it is sufficient to consider the case when E/H_j emits edges to H_j for every $j = 1, \ldots, n$ and when $E = E_{\text{tot}}$ is a connected graph in its breaking-vertices-free form. In this case, E/H_1 is a $(n-1)$-S-NE graph.

We can make c_j-to-c_1 paths to be *direct-exit* paths for $j > 1$ by considering P_{H_1} and using Proposition 3.5 with V being $P^0_{H_1} - H_1$. If c_1 is a proper cycle, this enables us to have that all c_j-to-c_1 paths for $j = 2, \ldots, n$ are of length one. If c_1 is a sink, the spine of the c_j-to-c_1 part is defined for every $j > 1$, we refer to it as the $j1$-spine and we use k_{j1} to denote its length. The spine lengths of the c_j-to-c_1 and the c_l-to-c_1 parts may be different if $j \neq l$. For example, the first graph below has $k_{21} = 0$ and $k_{31} = 1$. The second graph has $k_{21} = 1$ and $k_{31} = 0$.



Having the concept of the $j1$-spine, we can label vertices on such spine as $v_{1i}, i \leq k_{j1}$ analogously as in the $n = 2$ case. We can also let $v_{jk}, k \in m_j$ be the vertices of c_j and let an *$j1$-ki-exit* for

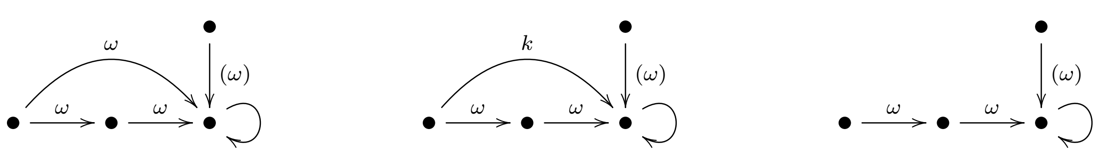
$1 < j \leq n$ be an $j1$-exit with the source $v_{jk}, k \in m_j$, and the range v_{1i} for $i \in m_1$ if c_1 is a proper cycle and $i \leq k_{j1}$ otherwise.

Having graphs in their direct forms also brings us to the concepts of the tails. If $j > 1$, the tails of the porcupine-quotients H_j/H_{j-1} correspond to the tails to c_j in E/H_{j-1}. We refer to those as the *j-tails of E*. If $i \in m_j$ for $m_j > 0$ and $i \leq k_j$ for $m_j = 0$ where k_j is the length of the tail graph of H_j, then we say that the *j-i-tails of E* are the i-tails of H_j/H_{j-1}. The tail graph $(H_1)_t$ is defined as before and we let *1-i-tails* of E be the tails of this graph. Here i is such that $i \in m_1 = |c_1|$ if c_1 is a proper cycle and $i \leq k_t$ if c_1 is a sink where k_t is the spine length of the tail graph $(H_1)_t$. If the range of a 1-i-tail is also the range of a $j1$-exit, we say that such a 1-i-tail is also an *$j1$-i-tail.*

An exit-move is defined analogously as for the 2-S-NE graphs. A *blow-up* of an exit e is an out-split with respect to the partition $\{e\}$ and $\mathbf{s}^{-1}(\mathbf{s}(e)) - \{e\}$ followed by the out-splits making the resulting graph be in its total out-split form. An *exit move* is a blow-up followed by operations making the resulting graph be a direct-exit graph. If the source v of a moved $j1$-exit does not receive any edges from H_l for any $l > j$, then only tails can be created by the move, just as in the $n = 2$ case. If v receives an edge from H_l for some $l > j$, then a c_l-to-c_j path is created in the process. In fact, $|\bigcup_{l<j} \mathbf{r}^{-1}(v) \cap \mathbf{s}^{-1}(H_l - H_j)|$ many of such paths are created by a move of such an exit. For example, the second graph is the result of a move of any the 21-exits of the first graph.

If E is obtained by a move of an $j1$-kl-exit of F for some $j > 1$ and k, l such that v_{jk} and v_{1l} are defined, we write $F \to_1 E$ and say that E is *$j1$-kl-reducible.* We define $E_{\text{red},j1,kl}$ as the graph with the same quotient and connecting matrices as F and with the number of 1-i-tails equal to that number for E if the cardinality of 1-i-tails is not impacted by the exit move $F \to_1 E$. If the number of 1-i-tails of E is strictly larger than in F and their number is finite in E, then $E_{\text{red},j1,kl}$ and F have the same number of 1-i-tails. Otherwise, $E_{\text{red},j1,kl}$ has infinitely many 1-i-tails. Since $E_{\text{red},j1,kl} \to_1 E$, the algebras of two graphs are graded $*$-isomorphic. We say that E is *$j1$-reducible* if it is $j1$-kl-reducible for some k and l and that E is *reducible* if it is $j1$-reducible for some $1 < j \leq n$.

If the cardinality of the 1-tails and $l1$-exits for $l > j$ changes by a $j1$-exit move, then E is *$j1$-reduced* if it is not $j1$-reducible. If the cardinality of the 1-tails and $l1$-exits does not change by a $j1$-exit move, then E is *$j1$-reduced.* If a graph is $j1$-reducible and $j1$-reduced and if c_j both receives and emits exits, a move of a $j1$-exit which originates in the vertex which receives a path from c_l for some $l > j$ produces a new $l1$-exit. So, in this case, c_l is an infinite emitter (otherwise the number of $l1$ exits of E and F would differ). For example, a move of any of the 21-exits of the first graph below produces a graph isomorphic to the original graph. If k is a finite and nonzero cardinality, the second graph above is not 31-reduced. The third graph is a 31-reduction of the second graph.



A graph E is *reduced* if it is $j1$-reduced for all $j = 2, \ldots, n$. In this case, we write $E = E_{\text{red}}$.

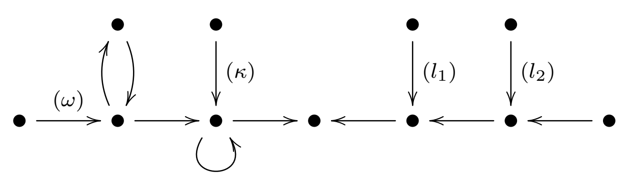
For $j > 1$, if c_j and c_1 are proper cycles and L is the least common multiple of c_j and c_1, E is *L-$j1$-kl reducible* if there is F such that $F \to_L E$ where $\to_1$ is a move of a $j1$-kl-exit and any subsequent move is the move of the resulting $j1$-exit. In this case, we define $E_{\text{red},L,j1,kl}$ analogously as when $n = 2$. We have that E is L-$j1$-*reduced* if for any k and l, when $E_{\text{red},L,j1,kl}$ is defined, then $E \cong E_{\text{red},L,j1,kl}$.

We define a *spine*-$j1$-*reduced graph* if c_1 is a sink and c_j a proper cycle and a ω-$j1$-*reduced graph* if c_j is an infinite emitter analogously as when $n = 2$. The $n = 2$ case definitions do not involve any reference to the length of a composition series, so we can generalize them directly. We unify the three cases saying that E is L_j-$j1$-*reduced* if it is L-$j1$-reduced, spine-$j1$-reduced, or ω-$j1$-reduced, depending on the values of m_j and m_1. If $\mathbf{L}$ is the $(n-1)$-tuple $(L_2, \ldots, L_n)$, we say that E is $\mathbf{L}$-*reduced* if E is L_j-$j1$-reduced for every $j > 1$. In this case, we write $E = E_{\text{red},\mathbf{L}}$.

Let $i \in m_1$ if c_1 is a proper cycle of length m_1 and let $i \in \omega$ if c_1 is a sink. For such i, we consider the following condition.

$C(1i)$ There are paths p, q, d and edges g, e such that g is a j-tail for some $j > 1$, p is a path from $\mathbf{r}(g)$ to a vertex in c_j which contains no edges of c_j, q is a path from $\mathbf{r}(p)$ to a vertex in c_l for some $1 < l \leq j$, e is an $l1$-exit and d is a path from $\mathbf{r}(e)$ to $\mathbf{s}(c_1)$ which does not contain c_1 if c_1 is a proper cycle. There are infinitely many tails with the same range as g in H_j/H_1. If c_1 is a proper cycle, then $1 + |p| + |q| + |d| = m_1 - i + 1 (\operatorname{mod} m_1)$ and, if c_1 is a sink, then $1 + |p| + |q| + |d| = i + 1$.

If this condition holds, we say that 1-i-tails are *cuttable*. For example, let E be the graph below, let κ be any countable cardinal, and let H_1 be the hereditary and saturated closure of the sink, H_2 of the source of the loop, and H_3 of the vertices of a cycle of length two.



The 1-2-tails of H_1 are cuttable because of the infinitely many tails which H_3 is receiving. The 1-1-tails of H_1 are cuttable only if $\kappa = \omega$.

If l_i is the number of 1-i-tails of a graph E, we have that E_{cut} has the same $(n-1)$-S-NE quotient, the same c_1 and the c_j-to-c_1 part as E and the number of tails is specified as follows. Let k_1 be the maximum of all spine lengths k_{j1} for $j > 1$. For $i < k_1$, E_{cut} has l_i 1-i-tails if 1-i-tails are not cuttable and it has zero 1-i-tails otherwise. Let k_t be the spine length of the tail graph of H_1. If $k_t > k_1$, the number of tails of E_{cut} is specified as follows by the consideration of two cases.

(1) There is no $k_0 \in k_t, k_0 \geq k_1$ such that 1-i-tails are cuttable for all $i \in k_t, i \geq k_0$. Then, the number of 1-i-tails is l_i if 1-i-tails are not cuttable and zero otherwise.
(2) There is $k_0 \in k_t, k_0 \geq k_1$ such that 1-i-tails are cuttable for all $i \in k_t, i \geq k_0$. The length of the spine graph of E_{cut} is k_0 in this case. For $i < k_0$, the number of 1-i-tails is l_i if 1-i-tails are cuttable and zero otherwise.

For the graph in the previous example, E_{cut} has $l_2 = 0$. If $\kappa < \omega$, the spine length of the tail graph is two and, if $\kappa = \omega$, then E_{cut} has $l_1 = 0$ and the spine length of the tail graph is one.

The definition of the cut maps of the $n = 2$ case generalizes to any n because these maps depend only on the type of the tails g from condition $C(1i)$ and the value of l_i, and they do not impact any of the exit emitters of the entire graph – it does not matter whether the paths from condition $C(1i)$ pass only two or more cycles c_j for $j \leq n$. Thus, the value of n does not matter for the definition of such maps and the arguments of section on tail-cutting for $n = 2$ are applicable to any n.

Next, we define a canonical quotient. By induction, we can assume that E/H_1 can be transformed into its $(n-1)$-S-NE canonical form. Obtaining such a canonical form does not impact the c_j-to-c_1 paths, it can only impact the tails. If c_2 emits exits to c_1, we consider the porcupine graph P_{H_2}. The concept of feasible vertices is defined for this 2-S-NE graph and we say that such vertices are *2-feasible*. If c_2 does not emit exits to c_1, then the concept of feasible vertices is not relevant for c_2-vertices since c_2 is terminal for E. If $n > 2$ and c_3 is the terminal cycle of H_3/H_2 which is not terminal for E, then it emits exits to H_1, to $H_2 - H_1$ or to both. If c_3 emits exits to $H_2 - H_1$, we say that the set of *3-feasible* vertices is the set of all vertices of c_3 such that they are 2-feasible in the $(n-1)$-S-NE graph E/H_1. If c_3 emits exits only to H_1, then P_{H_3} is a 2-S-NE graph. The concept of feasible vertices is defined for P_{H_3} and we say that such vertices are *3-feasible*. Continuing this process, we define *j-feasible* vertices for $j > 1$. A vertex is *feasible* if it is j-feasible for some $j > 1$.

To obtain a canonical quotient, move all 21-exits, if any, to start at a 2-feasible vertex of c_2, then move all 31-exits, if any, to start at a 3-feasible vertex and continue until we reach n. Then, change the $(n-1)$-S-NE graph E/H_1, if needed, so it is in its canonical form. This creates a *canonical quotient* $E_{\text{can quot}}$ of E. If $E = E_{\text{can quot}}$, we say that E has a canonical quotient form.

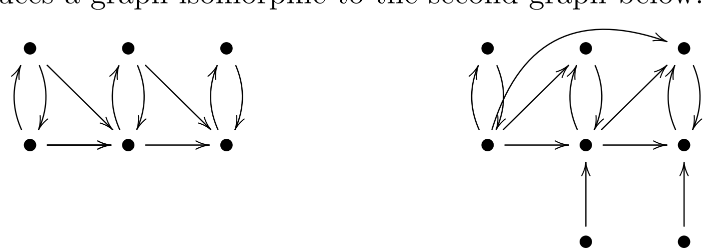
We let $E_{\text{red,cut}}$ stand for $(E_{\text{red}})_{\text{cut}}$ and we define a canonical form of E.

Definition 5.1. Let $n > 1$ and let E be a direct-exit, connected, countable n-S-NE without any breaking vertices. We write $E = E_{\text{dir}}$ for such a graph. Let E_0 be $((E_{\text{red,cut}})_{\text{can quot}})_{\text{red,cut}}$. For every $j = 2, 3, \ldots, n$, considered in this order, we move $j1$-exits, if needed, so that a form as specified below is obtained. Then, define $E_{\text{can},j}$ as a **L**-reduced and cut form of the obtained graph.

- If $m_j > 0$ and $m_1 > 0$, then the required form is a single exit-emitter graph such that the ranges of the exits are among consecutive vertices of c_1.
- If $m_j > 0$ and $m_1 = 0$, then the required form is such that m_j emits exits only to the source of the c_j-to-c_1 spine.
- If $m_j = 0$ and $m_1 > 0$ or if $m_j = m_1 = 0$ and $k_{j1} < \omega$, the required form is a graph in which v_{1i} receives either zero or infinitely many exits for every $i \in m_1$ if $m_1 > 0$ and for every $i \leq k_{j1}$ if $m_1 = 0$.
- If $m_j = m_1 = 0$ and $k_{j1} = \omega$, there are no requirements.

At the end of this process, we let $E_{\text{can}} = E_{\text{can},n}$. If E is any n-S-NE graph, we let $E_{\text{can}} = (E_{\text{dir}})_{\text{can}}$. We say that E is *canonical* or that it is *in a canonical form* if $E \cong F_{\text{can}}$ for some n-S-NE graph F.

For example, let E be the first graph below. Making E/H_1 canonical can be done in two ways since both vertices of c_3 are feasible in E/H_1. This makes P_{H_2} have only one feasible vertex in each case. Moving 21-exits to that vertex produces the second graph below. Moving the 32-exits to the other vertex of c_3 produces a graph isomorphic to the second graph below.



With the above definition of canonical form, two canonical forms obtained from the same cut and reduced graph E_0 can be transformed one to the other by the same argument as the one used in Lemma 4.11. Thus, this lemma continues to hold.

We define the relation $\approx$ for countable n-S-NE graphs by the same condition as if $n \leq 2$:

$$E \approx F \text{ if there are canonical forms } E_{\text{can}} \text{ and } F_{\text{can}} \text{ such that } E_{\text{can}} \cong F_{\text{can}}.$$

This relation is reflexive and symmetric and transitivity holds by Lemma 4.12 stated for n-S-NE graphs instead of 2-S-NE graphs. Such lemma holds by the same proof as Lemma 4.12.

5.2. **The main result and its corollaries.** In this section, we prove the main result of the paper, Theorem 5.3 and its corollaries, Corollaries 5.4, 5.5 and 5.6. First, we prove a lemma which generalizes Lemma 4.16. The arguments in the proof are analogous to the $n = 2$ case but we include them for completeness.

Lemma 5.2. *Let $E = E_{\text{cut}}$ and $F = F_{\text{cut}}$ be such that the connecting matrices computed by considering some c_j and $c'_j, j = 1, \ldots, n$ are the same. If there is a $\mathbf{POM}^D$-isomorphism f of the Γ-monoids such that $f([\mathbf{s}(c_j)]) = [\mathbf{s}(c'_j)]$ for $j = 1, \ldots, n$, then there is a graph isomorphism $\iota : E \cong F$ such that $f = \bar{\iota}$, that $\iota(\mathbf{s}(c_j)) = \mathbf{s}(c'_j)$ for $j = 1, \ldots, n$.*

Proof. Just as before in this section, we use v_{0j} for $\mathbf{s}(c_j)$ and v'_{0j} for $\mathbf{s}(c'_j)$. By the induction hypothesis, we can assume that $\iota : E/H_1 \cong F/G_1$ for some ι such that $\iota(v_{j0}) = v_{j'_0}$ for all $j > 2$. Since the connecting matrices are the same, we can extend ι to an isomorphism of the subgraph $E_{T\cup R}$ generated by $T(\bigcup_{j>1} H_j) \cup R(\bigcup_{j>1} H_j)$ and the subgraph $F_{T\cup R}$ generated by $T(\bigcup_{j>1} G_j) \cup R(\bigcup_{j>1} G_j)$.

Condition $C(1i)$, considered on E, depends only on the subgraph $E_{T\cup R}$ and $C(1i)$, considered on F, depends only on the subgraph $F_{T\cup R}$. Thus, for every $i \in m_1$ if $m_1 > 0$ and every $i \in \omega$ if $m_1 = 0$, we have that 1-i-tails of E are cuttable if and only if 1-i-tails of F are cuttable.

If $m_1 > 0$ and $i \in m_1$, let $P^{v_{10}}_{\not{c}_1,i}$ be the set of paths of $P^{v_{10}}_{\not{c}_1}$ of length $m_1 - i + 1$ modulo m_1. Let us partition $P^{v_{10}}_{\not{c}_1,i}$ into three disjoint sets analogous to those in the $n = 2$ case,

$$P^{v_{10}}_{\text{exits},i} = \{p \in P^{v_{10}}_{\not{c}_1,i} \mid \mathbf{s}(p) \in E^0_{T\cup R}\}, \quad P^{v_{10}}_{\text{quot},i} = \{p \in P^{v_{10}}_{\not{c}_1,i} \mid \mathbf{s}(p) \in E^0 - (H_1 \cup E^0_{T\cup R})\} \quad \text{and}$$
$$P^{v_{10}}_{\text{tails},i} = \{p \in P^{v_{10}}_{\not{c}_1,i} \mid \mathbf{s}(p) \in H_1 - E^0_{T\cup R}\},$$

and let such sets be analogously defined for F. The set $P^{v_{10}}_{\text{tails},i}$ is the set of 1-i-tails just as in the $n = 2$ case. Also just like for $n = 2$, we have that $C(1i)$ holds for some paths p, q, d, g, e if and only if there are infinitely many tails with the same range as g in H_j/H_1.

The existence of f ensures that there is a bijection $\sigma_i : P^{v_{10}}_{\not{c}_1,i} \to P^{v'_{10}}_{\not{c}'_1,i}$ and the existence of ι ensures that such σ_i can be found so that it maps $P^{v_{10}}_{\text{exits},i}$ onto $P^{v'_{10}}_{\text{exits},i}$. If condition $C(1i)$ holds for $i \in n$ (in E and, equivalently, in F), then i-tails are cuttable both in E and in F and we have that $P^{v_{10}}_{\text{tails},i} = P^{v'_{10}}_{\text{tails},i} = \emptyset$. So, the existence of ι enables us to find σ_i so it maps the elements of $P^{v_{10}}_{\text{quot},i}$ with the same p, q, d, e and g paths onto the elements of $P^{v'_{10}}_{\text{quot},i}$ involving $\iota(p), \iota(q), \iota(d), \iota(e)$ and $\iota(g)$. If the number of tails with the same range as g is finite, then the existence of ι also ensures the existence of such σ_i. Thus, we necessarily have that such σ_i maps $P^{v_{10}}_{\text{tails},i}$ onto $P^{v'_{10}}_{\text{tails},i}$. This shows that E and F have the same number of 1-i-tails so we can extend ι to entire graphs. Thus, we obtain an isomorphism $\iota : E \cong F$ such that $\bar{\iota} = f$. $\square$

Theorem 5.3. *Let E and F be countable composition S-NE graphs. The following conditions are equivalent.*

1. *There is a $\mathbf{POM}^D$-isomorphism $f : M^\Gamma_E \to M^\Gamma_F$.*
2. *The relation $E \approx F$ holds.*

(3) There is a graded $*$-isomorphism $\phi : L_K(E) \to L_K(F)$.

If (1) holds, there are canonical forms $E_{\rm can}$ and $F_{\rm can}$ and operations $\phi_E : E \to E_{\rm can}$, $\iota : E_{\rm can} \cong F_{\rm can}$, and $\phi_F : F \to F_{\rm can}$, such that $\overline{\phi_F^{-1}\iota\phi_E} = f$.

Proof. If E is an n-S-NE graph and if any of the four conditions holds, then F is an n-S-NE graph by Theorem 2.6. So, we can assume that E and F have composition series of the same length.

Since the implications (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (1) are direct, we prove (1) $\Rightarrow$ (2) and the last sentence of the theorem.

Let $\phi_E : E \to E_{\rm can}$ and $\phi_F : F \to F_{\rm can}$ be operations transforming the graphs to their canonical forms. We can consider $\overline{\phi}_F f \overline{\phi_E^{-1}}$ instead of f and so we can assume that $E = E_{\rm can}$ and $F = F_{\rm can}$.

Assume that (1) holds and that $H_i, i = 0, \ldots, n$ are the terms of a composition series of E. Let G_i, $i = 0, \ldots n$ be the admissible pairs of F such that $f(J^\Gamma(H_i)) = J^\Gamma(G_i)$ which implies that the sequence G_i constitutes the terms of a composition series of F. Let f_i denotes the restriction of f to $J^\Gamma(H_i)$. For $i = 1, \ldots, n-1$, $j = 2, \ldots, n$ and $i < j$, let $f_{j/i}$ be the induced $\mathbf{POM}^D$-isomorphism $M^\Gamma_{H_j/H_i} \to M^\Gamma_{G_j/G_i}$. Let m_i be the length of the terminal cycle of H_i/H_{i-1} (and G_i/G_{i-1}) and $A_{ji} = [a_{ji,lk}], i = 1\ldots, n-1, j = 2, \ldots n, i < j$ be the c_j-to-c_i connecting matrix and let $A'_{ji} = [a_{ji,lk}]$ be c'_j-to-c'_i connecting matrix of F.

If $n = 1$, the statement holds by Propositions 3.4 and 3.5. If $n = 2$, the statement holds by Theorem 4.13. Hence, let us assume that $n > 2$ and assume that the theorem holds for all S-NE graphs with composition length smaller than n.

If there are no edges emitted from E/H_j to H_j for some $j = 1, \ldots, n$, then E is a disjoint union of E/H_j and H_j and M^Γ_E is a direct sum of $J^\Gamma(H_j)$ and M^Γ_{E/H_j}. The existence of f implies that M^Γ_F splits into a direct sum of two terms each of which is $\mathbf{POM}^D$-isomorphic with $J^\Gamma(H_j)$ and M^Γ_{E/H_j}. Since the generators of $J^\Gamma(G_j)$ and M^Γ_{F/G_j} are independent, we have that there are no paths from F/G_j to G_j. So, F is a disjoint union of F/G_j and G_j. Since the composition length of P_{H_j} and E/H_j is smaller than n, we can use induction hypothesis and establish that the needed claim holds.

Thus, we can consider only connected graphs. The above consideration shows that there are edges from E/H_j to H_j if and only if there are edges from F/G_j to G_j.

By the definition of a canonical form, we have that the quotients E/H_1 and F/G_1 are canonical. By our induction hypothesis, we can assume that E and F are such that there is a graph isomorphism $\iota_{n/1} : E/H_1 = (E/H_1)_{\rm can} \cong F/G_1 = (F/G_1)_{\rm can}$ such that $f_{n/1} = \bar\iota_{n/1}$. The existence of $\iota_{n/1}$ also implies that $A_{lj} = A'_{lj}$ for every $l > j > 1$.

Let v_{ji} be the vertices of c_j for $i \in m_j = |c_j|$ and v'_{ji} are the vertices of c'_j for $i \in m_j = |c'_j|$, such that $\iota(v_{jl}) = v'_{jl}$ $j > 1$ and $l \in m_j$. Thus, we have that $f_{n/1}([v_{j0}]) = [v'_{j0}]$. By the definition of a feasible vertex and by induction hypothesis, we have that $f_{n/1}$ maps a feasible vertex on a feasible vertex. Thus, if c_j emits exits to c_l for $1 < l < j$ and in $m_l > 0$, we have that v_{j0} and v'_{j0} are j-feasible. In addition, if $m_1 > 0$ and c_j also emit exits to H_1, we can have that those exits are emitted from v_{j0} in E and from v'_{j0} in F. If c_j is terminal in E/H_1 but not terminal in E (i.e. it emits exits to H_1 only), then P_{H_j} and P_{G_j} are 2-S-NE graphs and we can have that v_{j0} and v'_{j0} are j-feasible by the definition of feasible vertices and the proof of Theorem 4.13.

For $j = 2, \ldots, n$, we consider the j-values in that order and realize f_j by a map which extends the realization of $f_{n/1}$. First, $j = 2$. If c_2 does not emit exits to H_1, then c_2 is terminal for E and we have that $f([v_{20}]) = f_{n/1}([v_{20}]) = [v'_{20}]$. If c_2 emits edges to H_1, we consider the 2-S-NE graph

P_{H_2} and the restriction f_2 of f to the Γ-monoid of P_{H_2}. By the proof of Theorem 4.13, we have that canonical forms E and F can be found so that $P_{H_2} \cong P_{G_2}$ (thus $A_{21} = A'_{21}$) and that f_2 maps the elements corresponding to 2-feasible vertices to elements corresponding to 2-feasible vertices.

If c_2 is proper and c_1 is not, then f_2 can be realized by a graph isomorphism $\iota_2 : P_{H_2} \cong P_{G_2}$ such that $\iota_2(v_{02}) = v'_{02}$ by the proof of Theorem 4.13. Thus, we have that $f([v_{20}]) = f_2([v_{20}]) = [\iota_2(v_{20})] = [v'_{20}]$. In all other cases when f_2 is not realized by a graph isomorphism, there are 2-canonical forms P'_{H_2} and P'_{G_2}, exit moves and cuts $\phi_{E,2} : P_{H_2} \to P'_{H_2}$ and $\phi_{F,2} : P_{G_2} \to P'_{G_2}$ and a graph isomorphism $\iota_2 : P'_{H_2} \cong P'_{G_2}$ such that $f_2 = \overline{\phi^{-1}_{F,2}\iota_2\phi_{E,2}}$ by the proof of Theorem 4.13.

Let E_2 be the graph obtained by the moves and cuts as in $\phi_{E,2}$ and F_2 the graph obtained by the moves and cuts as in $\phi_{F,2}$. We still use $\phi_{E,2}$ and $\phi_{F,2}$ for the operations $E \to E_2$ and $F \to F_2$. Let g be $\overline{\phi_{F,2} f \phi^{-1}_{E,2}}$. Since the moves $\phi_{E,2}$ and $\phi_{F,2}$ do not impact E/H_1 and F/G_1, $g_{n/1} = f_{n/1}$. As E and F are $\mathbf{L}$-reduced, E_2 and F_2 are still canonical forms of E_0 and F_0, so we can consider E_2 and F_2 instead of E and F and g instead of f. Hence, we can assume that $f([v_{20}]) = [v'_{20}]$.

We move on to the next j-value, $j = 3$. Let us write $f([v_{30}])$ in terms of $[v'_{20}]$, $[v'_{10}]$ and $[v'_{30}]$ if $m_3 > 0$ and $[v'_{20}]$, $[v'_{10}]$, and $[q_Z]$ for finite $Z \subseteq \mathbf{s}^{-1}(v_{30})$ if $m_3 = 0$. Just as in the proof of Theorem 4.13, we consider the cases $m_3 > 0$ and $m_3 = 0$ separately.

If $m_3 > 0$, let a_{3l} be the connecting polynomial of the c_3-to-c_l part for $l = 1, 2$, defined analogously as before, and let a'_{3l} be analogous such polynomial for F. Let $f([v_{30}]) = t^{lm_3}[v'_{30}] + q + b_1[v'_{10}]$ for some $l \geq 0$ some $b_1 \in \mathbb{Z}^+[t]$ and some element q of $M^\Gamma_{P_{G_2}}$. Since $[v'_{30}] = f_{n/1}([v_{30}]) = t^{lm_3}[v'_{30}] + q$, q is equal to $\sum_{j=0}^{l-1} t^{m_3 j} a_{32}[v'_{20}]$. Let us shorten this last term to $b_2[v'_{20}]$. The relation

$$[v_{30}] = t^{m_3}[v_{30}] + a_{32}[v_{20}] + a_{31}[v_{10}] \tag{6}$$

holds in M^Γ_E and an analogous such relation holds in M^Γ_F. Since $A_{32} = A'_{32}$, we have that $a_{32} = a'_{32}$. Using relation (6), we have that

$$t^{lm_3}[v'_{30}] + b_2[v'_{20}] + b_1[v'_{10}] = f([v_{30}]) = f(t^{m_3}[v_{30}] + a_{32}[v_{20}] + a_{31}[v_{10}]) =$$

$$t^{(l+1)m_3}[v'_{30}] + t^{m_3}b_2[v'_{20}] + t^{m_3}b_1[v'_{10}] + a_{32}[v'_{20}] + a_{31}[v'_{10}].$$

By using the analogue of relation (6) in M^Γ_F, we have that $t^{lm_3}[v'_{30}] + b_2[v'_{20}] + b_1[v'_{10}] = t^{(l+1)m_3}[v'_{30}] + t^{lm_3}a'_{32}[v'_{20}] + t^{lm_3}a'_{31}[v'_{10}] + b_2[v'_{20}] + b_1[v'_{10}]$. Thus, we have that

$$t^{(l+1)m_3}[v'_{30}] + t^{lm_3}a'_{32}[v'_{20}] + t^{lm_3}a'_{31}[v'_{10}] + b_2[v'_{20}] + b_1[v'_{10}] =$$

$$t^{(l+1)m_3}[v'_{30}] + t^{m_3}b_2[v'_{20}] + t^{m_3}b_1[v'_{10}] + a_{32}[v'_{20}] + a_{31}[v'_{10}]$$

and the terms with $[v'_{30}]$ cancel out. By using that $b_2[v'_{20}] = \sum_{j=0}^{l-1} t^{m_3 j} a_{32}[v'_{20}]$, we can cancel all terms with $[v'_{20}]$ and we arrive to

$$t^{lm_3}a'_{31}[v'_{10}] + b_1[v'_{10}] = t^{m_3}b_1[v'_{10}] + a_{31}[v'_{10}] \tag{7}$$

which generalizes (4) of the $n = 2$ case. If l', b'_1, and b'_2 are defined analogously as l, b_1, and b_2 for f^{-1}, we consider the relation $f^{-1}(f([v_{30}])) = [v_{30}]$ and have that

$$t^{(l+l')m_3}[v_{30}] + \sum_{j=0}^{l+l'-1} t^{jm_3}a_{32}[v_{20}] + \sum_{j=0}^{l+l'-1} t^{jm_3}a_{31}[v_{10}] = [v_{30}] = f^{-1}(f([v_{30}])) =$$

$$f^{-1}(t^{lm_3}[v'_{30}] + b_2[v'_{20}] + b_1[v'_{10}]) = t^{(l+l')m_3}[v_{30}] + t^{lm_3}b'_2[v_{20}] + t^{lm_3}b'_1[v_{10}] + b_2[v_{20}] + b_1[v_{10}] =$$

$$t^{(l+l')m_3}[v_{30}] + \sum_{j=0}^{l+l'-1} t^{lm_3}a_{32}[v_{20}] + t^{lm_3}b_1'[v_{10}] + b_1[v_{10}]$$

which implies that $(t^{lm_3}b_1' + b_1)[v_{10}] = \sum_{j=0}^{l+l'-1} t^{jm_3}a_{31}[v_{10}]$. Using $f(f^{-1}([v_{30}'])) = [v_{30}']$ similarly, we arrive to the analogues of relations (5) of the $n = 2$ case:

$$(t^{lm_3}b_1' + b_1)[v_{10}] = \sum_{j=0}^{l+l'-1} t^{jm_3}a_{31}[v_{10}] \quad \text{and} \quad (t^{l'm_3}b_1 + b_1')[v_{10}'] = \sum_{j=0}^{l+l'-1} t^{jm_3}a_{31}'[v_{10}']. \tag{8}$$

Thus, $f([v_{30}])$ depends only on the c_3-to-c_1 part. Having Lemma 5.2 and the formulas (7) and (8) which are analogous to (4) and (5), we have all we need to repeat the arguments from the proof of Theorem 4.13 and obtain a realization of f_3 the same way as in the proof of Theorem 4.13. Thus, there are 3-canonical forms P_{H_3}' and P_{G_3}', exit moves, possibly trivial, followed by cut maps $\phi_{E,3} : P_{H_3} \to P_{H_3}'$ and $\phi_{F,3} : P_{G_3} \to P_{G_3}'$, and a graph isomorphism $\iota_3 : P_{H_3}' \cong P_{G_3}'$ such that $f_3 = \overline{\phi_{F,3}^{-1}\iota_3\phi_{E,3}}$.

By the existence of ι_3, we have that $A_{31} = A_{31}'$. The maps $\phi_{E,3}$ and $\phi_{F,3}$ and their inverses do not impact any of c_2-to-c_1 paths, any of c_j-to-c_1 paths for $j > 3$ and any part of E/H_1 or of F/G_1. Thus, we can let E_3 be the graph obtained by the same moves and cuts as in $\phi_{E,3}$ and F_3 the graph obtained by the same operations as in $\phi_{F,3}$ and we still write $\phi_{E,3} : E \to E_3$ and $\phi_{F,3} : F \to F_3$. If g is $\overline{\phi_{F,3}}f\overline{\phi_{E,3}}^{-1}$, then $g_{n/1} = f_{n/1}$. As E and F are $\mathbf{L}$-reduced, E_3 and F_3 are still canonical forms of E_0 and F_0, so we can consider E_3 and F_3 instead of E and F and g instead of f. Hence, we can assume that $f([v_{30}]) = [v_{30}']$.

If $m_3 = 0$, then v_{30} and v_{30}' are infinite emitters. Since H_1 and G_1 do not have any breaking vertices, v_{30} and v_{30}' emit either zero or infinitely many edges to $H_2 - H_1$ and $G_2 - G_1$, respectively. Let $f([v_{30}]) = [q_{Z_1' \cup Z_2'}] + b_2[v_{02}'] + b_1[v_{10}']$ for some finite $Z_1' \subseteq \mathbf{s}^{-1}(v_{30}') \cap \mathbf{r}^{-1}(G_1)$, finite $Z_2' \subseteq \mathbf{s}^{-1}(v_{30}') \cap \mathbf{r}^{-1}(G_2 - G_1)$, and some $b_1, b_2 \in \mathbb{Z}^+[t]$. Since $f_{n/1}([v_{30}]) = [v_{30}']$, $b_2[v_{20}']$ is $a_{Z_2'}[v_{20}']$, so

$$f([v_{30}]) = [q_{Z_1' \cup Z_2'}] + a_{Z_2'}[v_{20}'] + b_1[v_{10}'] = [q_{Z_1'}] + b_1[v_{10}']$$

holds showing that $f([v_{30}])$ depends only on the c_3-to-c_1 part. The rest of the argument is the same as in the $m_3 > 0$ case: we can repeat the arguments from the proof of Theorem 4.13 to obtain 3-canonical forms P_{H_3}' and P_{G_3}', the maps, possibly trivial, $\phi_{E,3} : P_{H_3} \to P_{H_3}'$ and $\phi_{F,3} : P_{G_3} \to P_{G_3}'$, and a graph isomorphism $\iota_3 : P_{H_3}' \cong P_{G_3}'$ such that $f_3 = \overline{\phi_{F,3}^{-1}\iota_3\phi_{E,3}}$. We let E_3 be a canonical form obtained by the moves and cuts as in $\phi_{E,3}$ and F_3 be a canonical form obtained by the moves and cuts as in $\phi_{F,3}$ and g be $\overline{\phi}_{F,3}f\overline{\phi}_{E,3}^{-1}$. We consider E_3 and F_3 instead of E and F and g instead of f and have that $f([v_{30}]) = [v_{30}']$.

Continuing this argument, we have that $P_{H_j} \cong P_{G_j}$ holds (thus $A_{j1} = A_{j1}'$) for all $j > 1$ and we have graphs E_j and F_j to replace E_{j-1} and F_{j-1}, if needed, and $\overline{\phi}_{F,j}f\overline{\phi}_{E,j}^{-1}$ to replace f so that $f_j([v_{j0}]) = [v_{j0}']$ holds for all $j > 1$. For $j = n$, we have canonical forms E_n and F_n and an isomorphism $\iota_n : E_n \cong F_n$ such that $f([v_{j0}]) = [\iota_n(v_{j0})] = [v_{j0}']$, so $f = \overline{\iota}_n$. □

Theorem 5.3 implies the following corollary.

Corollary 5.4. The GCC holds for graphs with disjoint cycles, finitely many vertices and countably many edges. *If E and F are graphs with disjoint cycles, finitely many vertices and countably many edges the statement of Theorem 5.3 holds.*

Proof. If E has disjoint cycles and finitely many vertices, then there can be only finitely many cycles, sinks and infinite emitters and E is a countable n-S-NE graph for some n. □

5.3. **Graph C*-algebra version of the main result. The Graded Strong Isomorphism Conjecture.** If E is a graph, the *graph C^*-algebra $C^*(E)$ of E* is the universal C^*-algebra generated by mutually orthogonal projections $\{p_v \mid v \in E^0\}$ and partial isometries with mutually orthogonal ranges $\{s_e \mid e \in E^1\}$ satisfying the analogues of the (CK1) and (CK2) axioms and the axiom (CK3) stating that $s_e s_e^* \leq p_{\mathbf{s}(e)}$ for every $e \in E^1$ (where $\leq$ is the order on the set of projections given by $p \leq q$ if $p = pq = qp$). [1, Definition 5.2.5] has more details. By letting $s_{e_1 \ldots e_n}$ be $s_{e_1} \ldots s_{e_n}$ and $s_v = p_v$ for $e_1, \ldots, e_n \in E^1$ and $v \in E^0$, s_p is defined for every path p.

The set $\{p_v, s_e \mid v \in E^0, e \in E^1\}$ is referred to as a *Cuntz-Krieger E-family.* For such an E-family and an element z of the unit circle $\mathbb{T}$, one defines a map γ_z^E by $\gamma_z^E(p_v) = p_v$ and $\gamma_z^E(s_e) = zs_e$ and then uniquely extends this map to an automorphism of $C^*(E)$ (we assume a homomorphism of a C^*-algebra to be bounded and $*$-invariant). The *gauge action* γ^E on $\mathbb{T}$ is given by $\gamma^E(z) = \gamma_z^E$. Note that $\gamma_z^E(s_p s_q^*) = z^{|p|-|q|} s_p s_q^*$ for $z \in \mathbb{T}$ and paths p and q. The presence of the degree $|p| - |q|$ of pq^* in the previous formula explains the connection of this action and the $\mathbb{Z}$-gradings of $L_{\mathbb{C}}(E)$. In particular, the gauge action induces a $\mathbb{Z}$-grading of $C^*(E)$ (see [20, Section 3.1] for more details). If R is a C^*-algebra with an action $\beta : \mathbb{T} \to \operatorname{Aut}(R)$, we say that a homomorphism $\phi : C^*(E) \to R$ is *gauge-invariant* if $\beta_z \phi = \phi \gamma_z^E$ for every $z \in \mathbb{T}$. In particular, $\phi : C^*(E) \to C^*(F)$ is gauge-invariant if $\gamma_z^F \phi = \phi \gamma_z^E$ for every $z \in \mathbb{T}$.

In the case when the field K is the field of complex numbers $\mathbb{C}$ considered with the complex-conjugate involution, the existence of a natural injective map $L_{\mathbb{C}}(E) \to C^*(E)$ (see [1, Definition 5.2.1 and Theorem 5.2.9]) implies that if there is a $*$-isomorphism of $L_{\mathbb{C}}(E)$ and $L_{\mathbb{C}}(F)$ for graphs E and F, then there is an isomorphism of $C^*(E)$ and $C^*(F)$. In particular, if the algebra $*$-isomorphism is graded, then it induces a gauge-invariant isomorphism of the graph C^*-algebras. This holds basically because for $\phi : C^*(E) \to C^*(F)$ induced by a graded algebra map, both $\gamma_z^F \phi(s_p s_q^*)$ and $\phi \gamma_z^E(s_p s_q^*)$ are equal to $z^{|p|-|q|} \phi(s_p s_q^*)$ for any paths p, q of E.

The natural injection $L_{\mathbb{C}}(E) \to C^*(E)$ induces a natural monoid isomorphism $\mathcal{V}(L_{\mathbb{C}}(E)) \to \mathcal{V}(C^*(E))$ (see [1, Theorem 5.3.5]) and, in fact, a natural Γ-monoid isomorphism $\mathcal{V}^\Gamma(L_{\mathbb{C}}(E)) \to \mathcal{V}^\Gamma(C^*(E))$. This isomorphism maps the elements $[v]$ and $[q_Z]$ onto themselves, so we have a natural isomorphism of both $\mathcal{V}^\Gamma(L_{\mathbb{C}}(E))$ and $\mathcal{V}^\Gamma(C^*(E))$ and M_E^Γ. Because of this, the graph C^*-algebra version of the main result holds for graph C^*-algebras as the following corollary shows.

Corollary 5.5. The GCC holds for the graph C*-algebras of composition S-NE graphs. *If E and F are countable composition S-NE graphs, the conditions (1) to (3) from Theorem 5.3 are equivalent to the condition that there is a gauge-invariant isomorphism $\phi : C^*(E) \to C^*(F)$ such that $\overline{\phi} = f$ where f is a map from condition (1) of Theorem 5.3.*

Proof. If condition (1) holds and f is a map as in this condition, then there is a graded $*$-algebra isomorphism $\phi : L_{\mathbb{C}}(E) \to L_{\mathbb{C}}(F)$ such that $\overline{\phi} = f$ by Theorem 5.3. The map ϕ extends to a gauge-invariant isomorphism $C^*(E) \to C^*(F)$, which we also call ϕ, such that $\overline{\phi} = f$. □

Corollary 5.6. The Graded Strong Isomorphism Conjecture holds for composition S-NE graphs. *If E and F are countable composition S-NE graphs, the conditions below are equivalent to conditions (1) to (3) of Theorem 5.3.*

(4) *The algebras $L_K(E)$ and $L_K(F)$ are graded isomorphic as algebras.*
(5) *The algebras $L_K(E)$ and $L_K(F)$ are graded isomorphic as rings.*

(6) *The algebras* $L_K(E)$ *and* $L_K(F)$ *are graded isomorphic as* $*$*-rings.*
(7) *The algebras* $C^*(E)$ *and* $C^*(F)$ *are graded isomorphic.*

Proof. Since the implications (3) ⇒ (4) ⇒ (5), (3) ⇒ (6) ⇒ (5), and (5) ⇒ (1) are direct and (1) ⇒ (3) holds by Theorem 5.3, we have that (1) to (6) are equivalent. The equivalence of (1) and (7) follows by Corollary 5.5. □

The *diagonal* of a Leavitt path algebra of E is the K-linear span of the elements of the form pp^* where p is a path of E. By [16, Theorem 6.1], the equivalent conditions from the theorem above are also equivalent with the conditions below.

(8) There is a diagonal-preserving graded algebra isomorphism $L_K(E) \to L_K(F)$.
(9) There is a diagonal-preserving graded ring isomorphism $L_K(E) \to L_K(F)$.

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