The Graded Classification Conjecture holds for graphs with disjoint cycles

Lia Vaš

Saint Joseph's University, Philadelphia, USA

It all started with Roozbeh Hazrat...

... who got into **graded rings** some time circa 2010.

ring

graded ring

His motivation – some of the algebras we all love are naturally **graded**: graph C^*-algebras, Leavitt path algebras and their many generalizations.

The Graded Classification Conjecture (GCC)

Formed circa 2011. **Is for any two graphs E and F,**

$$L_K(E) \cong_{\text{gr}} L_K(F)$$
as graded algebras
iff
$$K_0^\Gamma(L_K(E)) \cong K_0^\Gamma(L_K(F))$$
as pointed Γ-groups?

Classification

$\Gamma = \langle t \rangle \cong \mathbb{Z}$, so K_0^Γ is a $\mathbb{Z}[\Gamma]$-module, not a $\mathbb{Z}[\mathbb{Z}]$-module.

Pointed

Order-unit = the class corresponding to sum of all vertices. **Generating interval** = summands of classes corresponding to sums of finitely many vertices.

Considered with these elements, the Grothendieck group is "**pointed**".

Recently, GCC is also referred to the statement on graded Morita equivalence (without "pointed"). We consider only the **original** formulation.

And many people jumped on the bandwagon...

... of looking at Roozbeh's conjecture
and made progress towards settling it.

The state of the GCC

Hazrat (2013) – GCC holds for finite **polycephaly** graphs (every path leads to a sink, a rose or a cycle with no exits).

Ara, Pardo (2014) – a weaker version of the GCC holds for finite graphs without sources and sinks.

Eilers, Ruiz, Sims (2020) – the GCC and its C^*-algebra version hold for countable "amplified" graphs.

Known classifications (continued)

Hazrat, Vaš (circa 2016) – the involutive version of the GCC holds for row-finite graphs in which every infinite path ends in a (finite or infinite) sink or a cycle without exits and the algebra is over a "nice enough" field (like $\mathbb{C}$, for example).

Eilers, Ruiz (2025) – the GCC holds for two subclasses of the class of graphs we consider: acyclic graphs with finitely many vertices and 2-S-NE graphs with finitely many vertices.

Vaš (2025) – the GCC holds for graphs with

- disjoint cycles,
- finitely many cycles, sinks, and infinite emitters, and
- each (right) infinite path ends in a cycle.

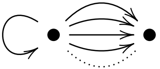
n-S-NE graphs

n = the number of cycles, sinks, and infinite emitters = the length of the **composition series** of the graphs (or algebras, or talented monoids, or K_0^{Γ}'s).

S-NE = every composition factor has either a **sink** ("S" is for a sink) or a **no-exit** cycle ("NE" is for no-exit).

Two of the four "primary colors"

in their composition series.

Example.  has one cycle, one infinite emitter and one sink $\Rightarrow$ composition length = 1+1+1= 3.

Our main prerequisite – the porcupine-quotient

$L_K(\text{quotient graph}) \cong_{\text{gr}} L_K(E)/I(H,S)$ (Tomforde 2007)

$L_K(\text{hedgehog}) \cong I(H,S) \not\cong_{\text{gr}} I(H,S)$ (Tomforde 2007)

$L_K(\text{porcupine}) \cong_{\text{gr}} I(H,S)$ (Vaš 2021)

quotient

hedgehog

porcupine

Porcupine-quotient (2023)

Given $(H, S) \leq (G, T)$ (this means $H \subseteq G$ and $S \subseteq G \cup T$)

we want to do the **quotient** construction with (H, S) but **relative** to the **porcupine** graph of (G, T) so that

$$L_K((G, T)/(H, S)) \cong_{\text{gr}} I(G, T)/I(H, S).$$

quotient

porcupine

porcupine-quotient

Example with $n = 2$

Let E be $e \circlearrowleft \bullet^v \xrightarrow{g} \bullet^w$ and $H = \{w\}$. The paths ending at w are

$$\{w, g, eg, eeg, eeeg, \ldots\}$$

and the porcupine is

$$\cdots\cdots \to \bullet \xrightarrow{e^2 g} \bullet \xrightarrow{eg} \bullet \xrightarrow{g} \bullet^w$$

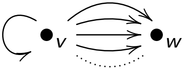
The composition series is $(\emptyset, \emptyset) \leq (\{w\}, \emptyset) \leq (E^0, \emptyset)$.

The composition factors are the above porcupine graph and the quotient $E/\{w\}$

$$\circlearrowleft \bullet^v$$

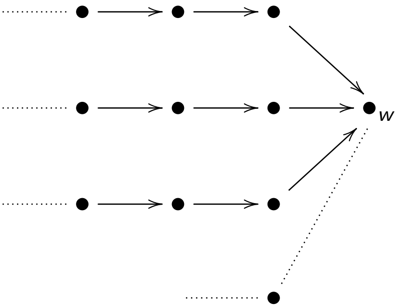
Thus, if one is to use the induction on n,
one needs to work with **infinitely many vertices**.

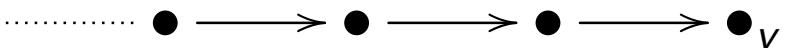
Example with $n = 3$

Let E be  . For $H = \{w\}$, the composition series is

$$(\emptyset, \emptyset) \leq (H, \emptyset) \leq (H, \{v\}) \leq (E^0, \emptyset)$$

and the three factors are: the porcupine graph of $(H, \emptyset)$



, the porcupine-quotient $(H, \{v\})/(H, \emptyset)$ 

, and the quotient $E/(H, \{v\})$ 

.

The best part

For graph monoids:

$$M_{(G,T)/(H,S)} \cong J(G,T)/J(H,S)$$

For talented monoids:

$$M^{\Gamma}_{(G,T)/(H,S)} \cong J^{\Gamma}(G,T)/J^{\Gamma}(H,S).$$

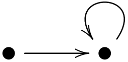
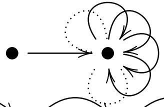
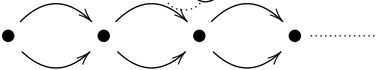
So, the requirements that a **composition series** of any of these exists are equivalent:

- admissible pairs of E,
- graded ideals of $L_K(E)$,
- order-ideals of M_E,
- Γ-order-ideals of M^{Γ}_E.

Four colors

$L_K(E)$ is graded simple iff
no nontrivial and proper admissible pairs.

The graph is the saturated closure of one of the four types of objects.

1. A **sink** 
2. A **cycle without exits** 
3. An **extreme cycle** 
4. A **terminal path**. 

S-NE and composition S-NE graphs

A graph E is an **S-NE graph** if $(G, T)/(H, S)$ has either a sink or a cycle without exits for every $(H, S) \leq (G, T)$ such that $(G, T)/(H, S)$ is cofinal.

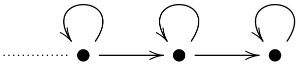
S-NE $\Rightarrow$ disjoint cycles. If E^0 is finite, disjoint cycles $\Rightarrow$ S-NE.

composition S-NE graphs $=$ S-NE $+$ has a composition series $=$

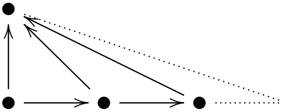
- disjoint cycles,
- each (right) infinite path ends in a cycle, and
- finitely many cycles, sinks, and infinite emitters.

Some examples

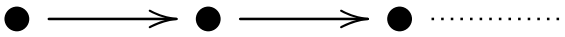
S-NE but not composition S-NE:



Acyclic but not S-NE:



and, for that matter, also



Now we can do induction!

Start with **1-S-NE graphs**.

Since 2011, it has been known that the GCC holds for finite 1-S-NE graphs (because they are polycephaly).

The proof is via **graded matrix representations**. So one would need to generalize the proof for finite-size matrices to infinite-size.

But the matrix approach does not help with $n > 1$. So,

no matrices.

The general idea – canonical forms

Want a graph which "represents" well all graphs with algebras in the same graded isomorphism class:

a canonical form.

If we define $E \approx F$ by $E_{\text{can}} \cong F_{\text{can}}$, then we aim to have:

The main result. For composition S-NE graphs E and F, TFAE.

1. K_0^Γs (equiv. talented monoids) are pointed isomorphic.
2. $E \approx F$
3. The algebras are graded $*$-isomorphic.

We **realize** any 1. isomorphism by a specific 3. isomorphism.

The timeline is also a part of this story...

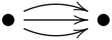
Oberwolfach, March 2024

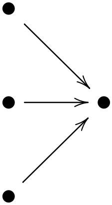
What came across...

March 2024 to January 2025 – no disjoint cycles

May 22, 2025 – v1 on arXiv

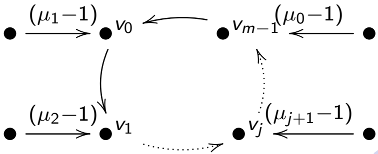
Canonical form of 1-S-NE graphs

Some notation: $\bullet \xrightarrow{3} \bullet$ stands for 

$\bullet \xrightarrow{(3)} \bullet$ stands for 

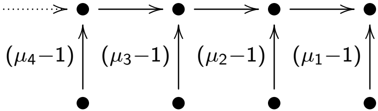
1-S-NE canonical form is a countably infinite generalization of my **2020 Beitrage paper** construction.

The NE case: $L_K(E) \cong_{\text{gr}} \mathbb{M}_\kappa[x^m, x^{-m}](\mu_0, \mu_1, \ldots, \mu_{m-1})$, then E_{can} is

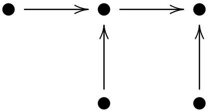


The sink case

The sink case: $L_K(E) \cong_{\text{gr}} \mathbb{M}_\kappa(1, \mu_1, \mu_2, \dots, \mu_k)$, then k=spine length and E_{can} is



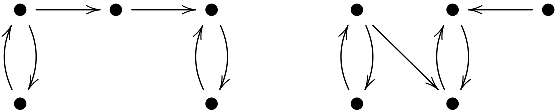
$E =$  , $\quad L_K(E) \cong_{\text{gr}} \mathbb{M}_5(K)(1,2,2), \quad k = 2$

$E_{\text{can}} =$ 

Relative construction

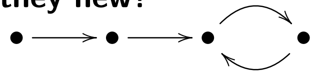
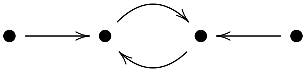
Let $V \subseteq E^0$. We repeat the E_{can} construction but without impacting the root $R(V) = \{u \in E^0 \mid u \geq v \text{ for some } v \in V\}$ and have $E_{\text{can},V}$.

The main application: E= 2-S-NE graph, H = nontrivial and proper her and sat set, and $V = R(V)$ = vertices of the porcupine of H which are not in H.



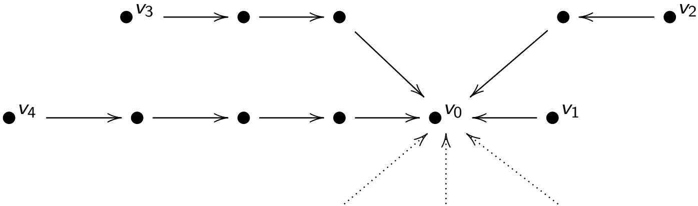
We got ourselves some moves

Are they new?

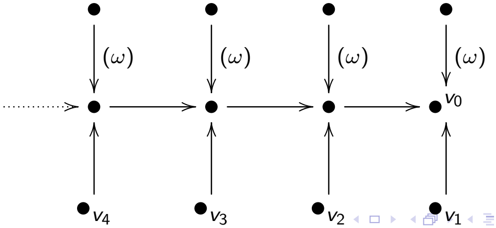
$E =$  $E_{\text{can}} =$ 

Are they moves in the Symbolic Dynamics playbook?

If $V = \{v_1, v_2, \ldots\}$ and $E =$



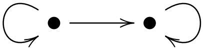
then $E_{\text{can},V} =$



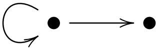
2-S-NE graphs

Two terminal clusters – easy: canonical is a disjoint union of two 1-S-NE graphs.

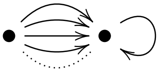
Otherwise – four types.



cycle-to-cycle



cycle-to-sink



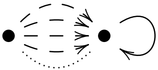
infinite-emitter-to-cycle

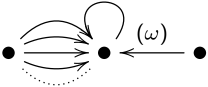
infinite-emitter-to-sink

Direct-exit forms

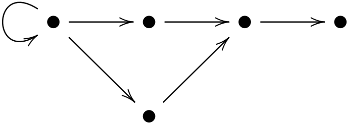
Using relative 1-S-NE canonical form,

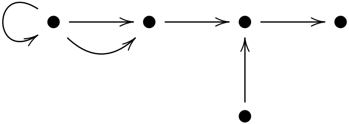
(blank)-**to-cycle: all exits end in the terminal cycle**





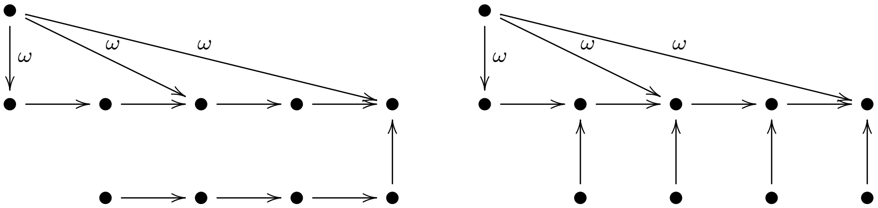
(blank)-**to-sink: all exits end in the spine**





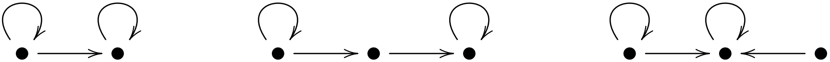
Still a lot of screws to tighten...

The tails can also be made canonical...

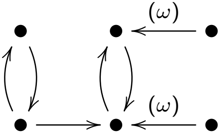


Exit moves

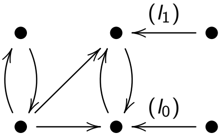
Glorified out-splits.



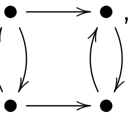
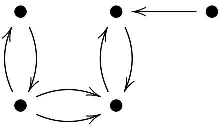
Sometimes they produce the same thing.



Reduced graphs, single-exit emitters

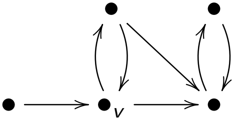


Reduced iff $l_0, l_1 \in \{0, \omega\}$.

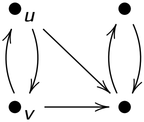
$E =$  , **single-exit emitter** for E 

Which vertex to make a single exit-emitter?

Feasible vertices

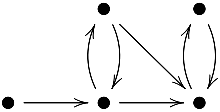


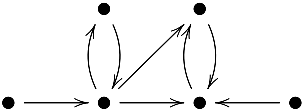
only v is feasible



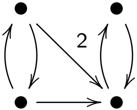
both u and v are feasible

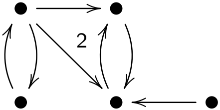
Unique feasible vertex – unique canonical form

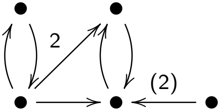




More feasible vertices – more canonical forms.

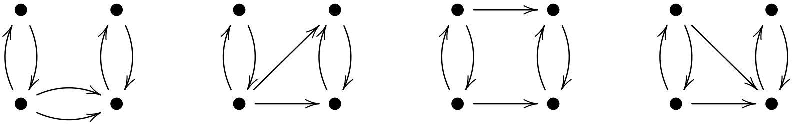






Idea of the proof

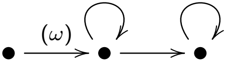
$E_1, E_2, E_3,$ and E_4 :

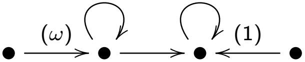


Canonical for E_3 and E_4 :

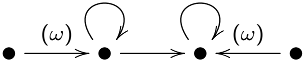
- Convert to a canonical form.
- Count the tails.

Tail cutting

$E =$ 

$F =$ 

Moving (the only) exit of both creates

$E' =$ 

Then E is the **cut form** of both E' and F.

Important because: If E and F are cut 2-S-NE and $f : M^\Gamma_E \to M^\Gamma_F$ is the "identity", then there is $\iota : E \cong F$ such that $f = \bar{\iota}$.

Realizing the talented monoid iso

The main result for 2-S-NE graphs.

TFAE

1. There is a pointed $f : M_E^\Gamma \cong M_F^\Gamma$.
2. The relation $E \approx F$ holds.
3. There is a graded $*$-isomorphism $\phi : L_K(E) \to L_K(F)$.

The "realizing" part.

If (1) holds, there are E_{can} and F_{can} and $\phi_E : E \to E_{\text{can}}$, $\iota : E_{\text{can}} \cong F_{\text{can}}$, and $\phi_F : F \to F_{\text{can}}$, such that

$$\overline{\phi_F^{-1}\iota\phi_E} = f.$$

n-S-NE – the true party starts....

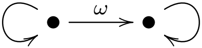
$$(\emptyset,\emptyset) \subsetneq (H_1, S_1) \subsetneq (H_2, S_2) \subsetneq \dots$$

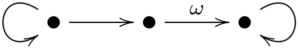
$$\subsetneq (H_{n-1}, S_{n-1}) \subsetneq (H_n, \emptyset) = (E^0, \emptyset)$$

can be made into

$$(\emptyset,\emptyset) \subsetneq (G_1,\emptyset) \subsetneq (G_2,\emptyset) \subsetneq \dots \subsetneq (G_{n-1},\emptyset) \subsetneq (G_n,\emptyset) = (E^0,\emptyset)$$

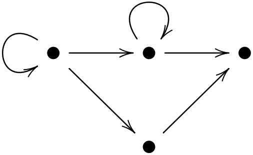
by out-splits. For example,



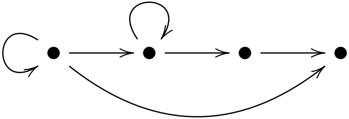


Direct-exit, spine, tails

H_j-to-H_1 parts can be made **direct-exit**. The $j1$-**spine** is defined and the j-tails are defined.



$k_{21} = 0$ and $k_{31} = 1$



$k_{21} = 1$ and $k_{31} = 0$.

Exit-moves

Moving a $j1$-exit can create an $l1$-exit for some $l > j$.

Reduction,	tail-cutting,
feasible vertices,	canonical quotient,
and, finally, –	**a canonical form**

are defined analogously as when $n = 2$.

Outline of the proof for n-S-NE

- Use induction for E/H_1 and F/G_1 and obtain canonical forms with $E/H_1 \cong F/G_1$.
- Messing with H_2-to-H_1 part does not impact the quotient any more, only H_j-to-H_1 part. We make the H_2-to-H_1 part canonical and then ...
- ... proceed with the H_3-to-H_1 part and so on until we reach H_n-to-H_1.

Corollaries

- The GCC holds for graphs with disjoint cycles and finitely many vertices.
- All the results hold for the **graph C^*-algebras.**
- The graph operations **preserve the diagonal** (so $\approx$ leads to 111 relation in Eilers-Ruiz 3-bit codification).
- The **Graded Isomorphism Conjecture** holds:

1, 2, 3 are equivalent with any of these.

4, 5, 6. There is a (diagonal-preserving) graded algebra (*-)isomorphism $L_K(E) \to L_K(F)$.

7, 8, 9. There is a (diagonal-preserving) graded ring (*-)isomorphism $L_K(E) \to L_K(F)$.

10, 11. There is an equivariant (equiv. graded) isomorphism $C^*(E) \to C^*(F)$.

Possible roads from here

- Transfinite induction to shows the GCC for all S-NE graphs, not necessarily composition and not necessarily countable.
- Cofinal graphs with extreme cycles?
- Then apply induction methods of S-NE to show that GCC holds for all graphs with three original colors (including all graphs with finitely many vertices).

Arxiv link is in the abstract (and the announcement). It is an invitation for you to share your comments and thoughts.