GRADED CANCELLATION PROPERTIES OF GRADED RINGS, GRADED UNIT-REGULAR LEAVITT PATH ALGEBRAS, AND THE LPA-REALIZATION QUESTION

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Abstract. We raise the following general question regarding a ring graded by a group: "If P is a ring-theoretic property, how does one define the graded version P_{gr} of the property P in a meaningful way?". Some properties of rings have straightforward and unambiguous generalizations to their graded versions and these generalizations satisfy all the matching properties of the nongraded case. If P is either being unit-regular, having stable range 1 or being directly finite, that is not the case. The first half of the paper addresses this issue. Searching for appropriate generalizations, we consider graded versions of cancellation, internal cancellation, substitution, and module-theoretic direct finiteness.

In the second half of the paper, we turn to Leavitt path algebras. For Leavitt path algebras of finite graphs, we characterize graded unit-regularity and other cancellation properties in terms of the graph properties. Then we provide a complete description of graded matrix algebras over a trivially graded field which are graded isomorphic to Leavitt path algebras. As a consequence, we show that there are graded corners of Leavitt path algebras which are not graded isomorphic to Leavitt path algebras. This contrasts a recent result stating that every corner of a Leavitt path algebra of a finite graph is isomorphic to another Leavitt path algebra. If R is a finite direct sum of graded matricial algebras over a trivially graded field and over naturally graded fields of Laurent polynomials, we also present conditions under which R can be realized as a Leavitt path algebra.

0. Introduction

We raise the question "If P is a ring-theoretic property, how does one define the graded version P_{gr} of the property P in a meaningful way?" and consider it in the cases when P is unit-regularity, cancellability, stable range 1, and direct finiteness. To address these cases, we study graded generalizations of some of cancellation properties summarized in T.Y. Lam's "A crash course on stable range, cancellation, substitution and exchange" ([13]). We focus on these properties since it is not as obvious and straightforward to define their graded generalization as it is for some other properties and we elaborate on this in the introduction. We do not assume that the grade group Γ is abelian and study some properties previously considered only for abelian groups Γ. In the last part of the paper, we consider these properties for Leavitt path algebras of finite graphs and characterize them in terms of the properties of the graph. We also answer the LPA-Realization Question from [16].

A ring R is graded by a group Γ if $R = \bigoplus_{\gamma\in\Gamma} R_\gamma$ for additive subgroups R_γ and $R_\gamma R_\delta \subseteq R_{\gamma\delta}$ for all $\gamma, \delta \in \Gamma$. The elements of the set $H = \bigcup_{\gamma\in\Gamma} R_\gamma$ are said to be homogeneous. The grading is trivial if $R_\gamma = 0$ for every nonidentity $\gamma \in \Gamma$. Since every ring is graded by the trivial group, we

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can say that the class of graded rings generalizes the class of rings. Still, it is customary that a ring graded by the trivial group is referred to as a nongraded ring.

If a ring-theoretic property P is in its prenex form, the term *graded property* P has been used for the property P_{gr} obtained by replacing every $\forall x$ and $\exists x$ in P by the restricted versions $\forall x \in H$ and $\exists x \in H$. For example, if P is the property

$$\text{Reg:} \qquad (\forall x)(\exists y)(xyx = x)$$

defining a von Neumann regular (or regular for short) ring, then we say that a graded ring R is graded regular if $(\forall x \in H)(\exists y \in H)(xyx = x)$ and we denote this condition by Reg_{gr}.

If a property P has the form $(\forall x)(\exists y)\phi(x, y)$, let P^{w}_{gr} denote the statement $(\forall x \in H)(\exists y)\phi(x, y)$ which we call *the weak graded property* P. In some cases, P^{w}_{gr} and P_{gr} are equivalent. For example, if P is the property Reg, then P^{w}_{gr} and P_{gr} are equivalent. Indeed, if $\text{Reg}^{\text{w}}_{\text{gr}}$ holds and $x \in R_\gamma$, then there is y is such that $x = xyx$. Let $y_{\gamma^{-1}}$ be the γ^{-1}-component of y. Then $xy_{\gamma^{-1}}x = x$ showing that Reg_{gr} also holds. If P is the property that every nonzero element of R has a multiplicative inverse, then the equivalence of P^{w}_{gr} and P_{gr} can be shown similarly. If any of them holds, R is said to be a graded division ring (and a graded field if R is commutative).

The situation is trickier if P is a ring-theoretic property such that P^{w}_{gr} and P_{gr} are not equivalent. For example, consider the property

$$\text{UR:} \qquad (\forall x)(\exists u)(\exists v)(uv = vu = 1 \text{ and } x = xux)$$

defining a unit-regular ring. The conditions $\text{UR}^{\text{w}}_{\text{gr}}$ and UR_{gr} are not equivalent. Indeed, let K be a field trivially graded by the group of integers $\mathbb{Z}$ and R be the graded matrix ring $\mathbb{M}_2(K)(0, 1)$ (we review the definition in section 1). If $e_{ij}, i, j = 1, 2,$ denote the standard matrix units, then they are homogeneous and $e_{12}ue_{12} = e_{12}$ for no homogeneous invertible element u because all homogeneous invertible elements are diagonal. Thus, UR_{gr} fails. On the other hand, the fact that $\mathbb{M}_2(K)$ is unit-regular readily implies that $\mathbb{M}_2(K)(0, 1)$ satisfies $\text{UR}^{\text{w}}_{\text{gr}}$.

If P_{gr} and P^{w}_{gr} are not equivalent, the following anomalies can happen.

(1) If $P \Rightarrow Q$ holds for all rings, then it may happen that $P_{\text{gr}} \not\Rightarrow Q_{\text{gr}}$. For example, the graded ring $\mathbb{M}_2(K)(0, 1)$ from the previous example is graded semisimple (because it is a graded matrix ring over a graded field). Thus, $\mathbb{M}_2(K)(0, 1)$ is an example of a ring which is graded semisimple but not graded unit-regular. Note that $P \Rightarrow Q$ implies that $P^{\text{w}}_{\text{gr}} \Rightarrow Q^{\text{w}}_{\text{gr}}$ holds.
(2) If R is a graded ring which satisfies P, then R also satisfies P^{w}_{gr} while it may fail to have P_{gr}. For example, $\mathbb{M}_2(K)(0, 1)$ has UR so $\text{UR}^{\text{w}}_{\text{gr}}$ holds but, as we have seen, UR_{gr} fails.
(3) If a property P has a feature F, then the graded version F_{gr} may fail to hold for P_{gr}. For example, while UR is closed under formation of matrix algebras and corners, UR_{gr} is not closed under formation of graded matrix algebras (by example with $\mathbb{M}_2(K)(0, 1)$ above) and graded corners (by Example 2.8).

The above discussion seem to indicate that more than one aspect should be taken into consideration if looking for a meaningful way to generalize properties to graded rings. In some cases, a ring-theoretic definition may just be a convenient simplification of certain equivalent model-theoretic property. Sometimes the historical origin of a definition may provide a meaningful insight in the process of a generalization to graded rings. Considering all of these factors, we ask the following question, central for the motivation of the work in the first half of this paper:

Question 0.1. If P is a ring-theoretic property, how does one define the graded version P_{gr} of the property P in a meaningful way?

Unit-regularity, for example, originated as *a property of the endomorphism ring of a module, not the ring itself.* In the graded case, one considers graded homomorphisms instead of homomorphisms so, the graded component of the endomorphism ring corresponding to the group identity $\epsilon \in \Gamma$ has a special significance. Requiring this component of graded matrix rings (graded endomorphism rings of finitely generated graded free modules) to be unit-regular brings us to far less restrictive concept than the existing graded unit-regularity. Moreover, if a ring R is graded regular, in order for this last condition to hold, it is sufficient to assume that the ϵ-component R_ϵ of R is unit-regular. Thus, we consider the condition

$\mathrm{Reg}_{\mathrm{gr}}+\mathrm{UR}_\epsilon$: R is graded regular and R_ϵ is unit-regular.

It is less restrictive than $\mathrm{UR}_{\mathrm{gr}}$ but still strong enough to capture the relevant properties of unit-regularity in the graded case as we shall demonstrate.

After a review of prerequisites and some preliminary results in section 1, we consider graded versions of module-theoretic characterizations of unit-regularity in section 2. Let $P(A)$ be a property of a R-module A. In case when R is a graded ring and A a graded R-module, we let $P_{\mathrm{gr}}(A)$ denote the statement obtained from $P(A)$ if every instance of "module" in it is replaced by "graded module" and every instance of "homomorphism" by "graded homomorphism". In particular, we use $\cong_{\mathrm{gr}}$ to denote a graded isomorphism.

For nongraded rings, the internal cancellability of a module is equivalent with the unit-regularity of its endomorphism ring if the endomorphism ring is regular. If A is an R-module, we say that it satisfies *internal cancellation* (or that it is internally cancellable) if the condition

$\mathrm{IC}(A)$: $A = B \oplus C = D \oplus E$ and $B \cong D$ implies $C \cong E$

holds for all modules B, C, D, E. If A is a graded module, the subring $\mathrm{END}_R(A)$ of $\mathrm{End}_R(A)$, generated by graded homomorphisms of degree γ for all $\gamma \in \Gamma$, is naturally graded (and coincides with $\mathrm{End}_R(A)$ if A is finitely generated). If ϵ is the identity of Γ, the elements of $\mathrm{END}_R(A)_\epsilon$ are exactly the *graded* endomorphisms of A. Thus, the statement $\mathrm{IC}_{\mathrm{gr}}(A)$, the graded version of $\mathrm{IC}(A)$ obtained by the process we explained above, translates to a property of $\mathrm{END}_R(A)_\epsilon$ only, not the entire ring $\mathrm{End}_R(A)$. Indeed, if $\mathrm{END}_R(A)_\epsilon$ is regular, we show that $\mathrm{IC}_{\mathrm{gr}}(A)$ holds if and only if $\mathrm{END}_R(A)_\epsilon$ is unit-regular (Proposition 2.1). This fact enables us to relate the property below to a much less restrictive condition than $\mathrm{UR}_{\mathrm{gr}}$.

$\mathrm{IC}_{\mathrm{gr}}$: $\mathrm{IC}_{\mathrm{gr}}(P)$ holds for every finitely generated graded projective module P.

We say that R satisfies *graded internal cancellation* if $\mathrm{IC}_{\mathrm{gr}}$ holds. Note that this is a ring property, not a module property, and that the ring property $\mathrm{IC}_{\mathrm{gr}}$ is stronger than the property $\mathrm{IC}_{\mathrm{gr}}(R)$ of R as a graded R-module. By Proposition 2.2, $\mathrm{Reg}_{\mathrm{gr}}+$ $\mathrm{IC}_{\mathrm{gr}}$ is equivalent with the condition below.

Mat_ϵ: The ϵ-component of $\mathbb{M}_n(R)(\gamma_1, \ldots, \gamma_n)$ is unit-regular for every n and every $\gamma_1, \ldots, \gamma_n \in \Gamma$.

These conditions are also equivalent with the graded cancellability of R. Recall that an R-module A is *cancellable* in a category of R-modules $\mathcal{M}$ if the condition

$\mathrm{C}(A)$: $A \oplus B \cong A \oplus C$ implies $B \cong C$

holds for all modules B and C in $\mathcal{M}$. If R is regular, then R is unit-regular if and only if $\mathrm{C}(R)$ holds in the category of finitely generated projective modules. If R is a graded ring and A a graded module, we let $\mathrm{C_{gr}}(A)$ denote the *graded cancellability* obtained from $\mathrm{C}(A)$. Let $\mathcal{P}_{\mathrm{gr}}$ denote the category of finitely generated graded projective modules. By Theorem 2.9, if R is graded regular, the equivalent conditions $\mathrm{IC_{gr}}$ and Mat_ϵ are also equivalent with any of the following.

$\mathrm{C_{gr}}$: $\mathrm{C_{gr}}(P)$ holds in $\mathcal{P}_{\mathrm{gr}}$ for every object P of $\mathcal{P}_{\mathrm{gr}}$.
$\mathrm{C_{gr}}(R)$: R is graded cancellable in $\mathcal{P}_{\mathrm{gr}}$.
UR_ϵ: R_ϵ is unit-regular.

In particular, the requirement $\mathrm{Reg_{gr}}$+UR_ϵ is formulated only in terms of a ring R, without referring to any module. This condition is far less restrictive than $\mathrm{UR_{gr}}$ but, by Theorem 2.9, strong enough to guarantee that every module in category $\mathcal{P}_{\mathrm{gr}}$ is cancellable in $\mathcal{P}_{\mathrm{gr}}$. In addition, Corollary 2.10 shows that $\mathrm{Reg_{gr}}$+UR_ϵ is graded Morita invariant and the example with $M_2(K)(0,1)$ shows that $\mathrm{UR_{gr}}$ is not. So, $\mathrm{Reg_{gr}}$+UR_ϵ does not have the anomaly of $\mathrm{UR_{gr}}$ pointed out before.

We relate $\mathrm{UR^w_{gr}}$ and $\mathrm{UR_{gr}}$ with the following weak and strong internal cancellation properties respectively (Proposition 2.4).

$\mathrm{IC^w_{gr}}(R)$: $R = A \oplus B = C \oplus D$ and $A \cong_{\mathrm{gr}} (\gamma)C$ for some $\gamma \in \Gamma$ implies $B \cong D$.
$\mathrm{IC^s_{gr}}(R)$: $R = A \oplus B = C \oplus D$ and $A \cong_{\mathrm{gr}} (\gamma)C$ for some $\gamma \in \Gamma$ implies $B \cong_{\mathrm{gr}} (\gamma)D$.

Here $(\gamma)A$ denotes the shift of a right module A by γ (we review this concept in section 1). By Propositions 2.2 and 2.4 and Theorem 2.9, the properties we consider relate as follows.

$\mathrm{UR_{gr}}$ $\Leftrightarrow$
$\mathrm{Reg_{gr}}$+$\mathrm{IC^s_{gr}}(R)$

$\swarrow$ $\searrow$

$\mathrm{Reg_{gr}}$+$\mathrm{C_{gr}}(R)$ $\Leftrightarrow$ $\mathrm{Reg_{gr}}$+UR_ϵ $\Leftrightarrow$ $\mathrm{Reg_{gr}}$+$\mathrm{IC_{gr}}(R)$ $\mathrm{UR^w_{gr}}$ $\Leftrightarrow$ $\mathrm{Reg_{gr}}$+$\mathrm{IC^w_{gr}}(R)$

Both diagonal arrows are strict and $\mathrm{UR^w_{gr}}$ and $\mathrm{Reg_{gr}}$+UR_ϵ are not equivalent by Example 2.11.

Proposition 2.4 indicates a serious disadvantage of $\mathrm{UR^w_{gr}}$: $\mathrm{IC^w_{gr}}(\_)$ involves an isomorphism, not a *graded* isomorphism, of graded modules. So, the condition $\mathrm{UR^w_{gr}}$ ends up being outside of the category of graded modules. $\mathrm{Reg_{gr}}$+UR_ϵ, on the other hand, does not have this downside.

In section 3, we consider properties of having stable range 1 and being directly finite. Let $\mathrm{sr}(R) = 1$ denote the following condition and $\mathrm{sr_{gr}}(R) = 1$ its graded version if R is graded.

$\mathrm{sr}(R) = 1$: $(\forall x, y)(\exists z, u)(xR + yR = R \Rightarrow z = x + yu \text{ and } zR = R)$

The property $\mathrm{sr}(R) = 1$ also has its module-theoretic characterization related to cancellation. If A is an R-module, $\mathrm{sr}(\mathrm{End}_R(A)) = 1$ is equivalent with the property below, known as *substitution.*

$\mathrm{S}(A)$: If $A \oplus B = A' \oplus B' = M$ for some modules M, A', B, B' and $A \cong A'$, then there is a module C such that $B \oplus C = B' \oplus C = M$.

Let $\mathrm{S_{gr}}(A)$ denote the graded version of this property. By Theorem 3.5, a graded module A has substitution if and only if $\mathrm{sr}(\mathrm{END}_R(A)_\epsilon) = 1$. Since substitution implies cancellability, Theorem 3.5 has a corollary that $\mathrm{sr}(\mathrm{END}_R(A)_\epsilon) = 1$, much weaker condition that $\mathrm{sr_{gr}}(\mathrm{END}_R(A)) = 1$, can be required to show that A is graded cancellable even without the requirements that A is finitely generated and that Γ is abelian. This shows that the conclusion of the Graded Cancellation Theorem ([9, Theorem 1.8.4]) holds without these two requirements and with the weaker assumption $\mathrm{sr}(\mathrm{END}_R(A)_\epsilon) = 1$ instead of $\mathrm{sr_{gr}}(\mathrm{END}_R(A)) = 1$.

Direct finiteness can also be related to the other cancellability conditions. An R-module A is said to be directly finite (or Dedekind finite) if

DF(A): $A\oplus B\cong A$ implies $B=0$ for any module B.

If DF denotes the property of the ring R below,

DF: $(\forall x)(\forall y)(xy=1\Rightarrow yx=1)$

then DF(A) holds if and only if DF holds on $\mathrm{End}_R(A)$ (see [7, Lemma 5.1]). A ring R is said to be directly finite if R is a directly finite left (equivalently right) R-module and this requirement holds if and only if DF holds on R. If R is a graded ring and A a graded R-module, consider the graded versions of DF(A) and DF.

$\mathrm{DF_{gr}}(A)$: $A\oplus B\cong_{\mathrm{gr}} A$ implies $B=0$ for any graded module B.
$\mathrm{DF_{gr}}$: $(\forall x\in H)(\forall y\in H)(xy=1\Rightarrow yx=1)$

The condition $\mathrm{DF_{gr}}(A)$ is *not* equivalent to $\mathrm{END}_R(A)$ being graded directly finite, but to $\mathrm{END}_R(A)_\epsilon$ being directly finite. By [10, Proposition 3.2], $\mathrm{DF_{gr}}$ holds on $\mathrm{END}_R(A)$ if and only if the condition below holds.

$\mathrm{DF^s_{gr}}(A)$: $A\oplus B\cong_{\mathrm{gr}} (\gamma)A$ for some $\gamma\in\Gamma$ implies $B=0$ for any graded module B.

If $\mathrm{DF^s_{gr}}(A)$ holds, the authors of [10] say that A is *graded directly finite*. A graded ring R satisfies $\mathrm{DF_{gr}}$ if and only if any of the equivalent conditions $\mathrm{DF^s_{gr}}(R_R)$ and $\mathrm{DF^s_{gr}}({}_RR)$ holds. The ring R_ϵ satisfies DF if and only if any of the equivalent conditions $\mathrm{DF_{gr}}(R_R)$ and $\mathrm{DF_{gr}}({}_RR)$ holds. By [10, Example 3.3], $\mathrm{DF^s_{gr}}(A)$ is strictly stronger than $\mathrm{DF_{gr}}(A)$.

The condition DF(A) can be obtained by requiring that the first two terms of two decompositions in the condition IC(A) are isomorphic to A and the second term in one of the two decompositions is zero. Thus, IC(A) clearly implies DF(A). By the same argument, the implications in the two rows below hold.

$$\begin{array}{ccc}\mathrm{IC^s_{gr}}(\_) & \Longrightarrow & \mathrm{DF^s_{gr}}(\_)\\ \Downarrow & & \Downarrow\\ \mathrm{IC_{gr}}(\_) & \Longrightarrow & \mathrm{DF_{gr}}(\_)\end{array}$$

Thus, our use of s in the superscript is consistent: the absence of s indicates that the property is obtained only by replacing "module" by "graded module" and "homomorphism" by "graded homomorphism" without considering the graded module shifts. So, for any graded module A,

graded module-theoretic properties of A correspond to properties of $\mathrm{END}_R(A)_\epsilon$ and strong graded module-theoretic properties of A correspond to graded properties of $\mathrm{END}_R(A)$.

The properties considered so far are in the following relations which match the relations of the nongraded analogues in the diagram in [13, Formula (4.2)].

$$\mathrm{S_{gr}}(\_)\ \Longrightarrow\ \mathrm{C_{gr}}(\_)\ \Longrightarrow\ \mathrm{IC_{gr}}(\_)\Longrightarrow\ \mathrm{DF_{gr}}(\_)$$

We also have the following.

$$\mathrm{UR_{gr}}(R)\ \Longrightarrow\ \mathrm{sr_{gr}}(R)=1\ \Longrightarrow\ \mathrm{sr}(R_\epsilon)=1\ \Longrightarrow\ \mathrm{C_{gr}}(R)$$

In section 3.4, we present examples showing that each implication above is strict.

In section 3.5, we consider the cancellation properties of a strongly graded ring R (i.e. $R_\gamma R_\delta = R_{\gamma\delta}$ for all $\gamma,\delta\in\Gamma$). For such R, the category of graded R-modules is equivalent to the category of R_ϵ-modules. Given this equivalence, it is not surprising that

$$\mathrm{S_{gr}}(A) \Longleftrightarrow \mathrm{S}(A_\epsilon),\quad \mathrm{C_{gr}}(A) \Longleftrightarrow \mathrm{C}(A_\epsilon),\quad \mathrm{IC_{gr}}(A) \Longleftrightarrow \mathrm{IC}(A_\epsilon),\quad \text{and } \mathrm{DF_{gr}}(A) \Longleftrightarrow \mathrm{DF}(A_\epsilon),$$

for a graded R-module A as we show in Proposition 3.8. In contrast, we show that all three implications below are strict even if R is strongly graded.

$$R \text{ satisfies } \mathrm{UR_{gr}} \Rightarrow R_\epsilon \text{ satisfies UR},\ \mathrm{sr_{gr}}(R) = 1 \Rightarrow \mathrm{sr}(R_\epsilon) = 1,\ R \text{ satisfies } \mathrm{DF_{gr}} \Rightarrow R_\epsilon \text{ satisfies DF}.$$

In section 4, we turn to Leavitt path algebras and their graded cancellation properties. If K is a trivially graded field and E is an oriented graph, the Leavitt path algebra $L_K(E)$ is naturally graded by the ring of integers. While some of the (graded) properties of $L_K(E)$ have been characterized in terms of the properties of E, it has not been known which condition on E is equivalent with $L_K(E)$ being graded unit-regular. If E is finite, Theorem 4.2 presents such a condition. This condition critically depends on the lengths of paths to cycles making it stand out from other known graph conditions which characterize algebraic properties of $L_K(E)$. Our proof of Theorem 4.2 heavily relies on Proposition 2.6 which characterizes when a $\mathbb{Z}$-graded matrix algebra over a trivially graded field K or over naturally $\mathbb{Z}$-graded $K[x^m, x^{-m}]$ is graded unit-regular. If E is finite, Proposition 4.3 characterizes other cancellability properties of $L_K(E)$ considered in this paper. The diagram below summarizes these results and some already known characterizations.

$\mathrm{UR_{gr}}$, $\mathrm{IC_{gr}}$ $\mathrm{sr_{gr}} = 1$	=	E satisfies (2) of Thm 4.2	$\longrightarrow$	$\mathrm{UR^w_{gr}}$, $\mathrm{DF_{gr}}$, IC, C, DF	=	E is no-exit	$\longleftarrow$	UR, Reg, sr = 1	=	E is acyclic
				$\downarrow$						
				$\mathrm{Reg_{gr}}$+UR_ϵ, $\mathrm{S_{gr}}(L_K(E))$	=	E is any finite graph				

While every matrix algebra over a field K can be realized as a Leavitt path algebra, this is not the case for every *graded* matrix algebra over K ($\mathbb{Z}$-graded trivially) by [16, Proposition 3.7]. The LPA-Realization Question of [16, Section 3.3] is asking for characterization of those graded matrix algebras over K which can be realized as Leavitt path algebras. In section 5, we answer this question (Proposition 5.2). As a consequence, we show that there are graded corners of Leavitt path algebras which are not graded isomorphic to Leavitt path algebras (Example 5.5). This contrasts a recent result from [2] which states that every corner of a Leavitt path algebra of a finite graph is isomorphic to another Leavitt path algebra. If R is a finite direct sum of graded matricial algebras over K and over naturally $\mathbb{Z}$-graded fields of the form $K[x^m, x^{-m}]$ for positive integers m, we also characterize when R is graded isomorphic to a Leavitt path algebra (Proposition 5.4).

We finish the introduction by one last comment which provides further evidence that $\mathrm{Reg_{gr}}$+ UR_ϵ is better fitted to be the graded analogue of unit-regularity than $\mathrm{UR_{gr}}$. Namely, no example of a $*$-regular ring which is not unit-regular is currently known and Handelman's Conjecture stipulates that every $*$-regular ring is unit-regular. In the graded case, graded unit-regularity is so restrictive that it is not difficult to find an example of a graded $*$-regular ring which is not graded unit-regular. For example, as we have seen $R = \mathbb{M}_2(\mathbb{C})(0, 1)$ with $\mathbb{C}$ trivially graded by $\mathbb{Z}$ is not graded unit-regular. With the involution induced by the complex-conjugation, R is $*$-regular. So, we believe that more relevant graded version of Handelman's conjecture is asking the following.

Question 0.2. Is the ϵ-component of every graded $*$-regular ring unit-regular?

In particular, since a graded $*$-regular ring is such that R_ϵ is $*$-regular, this question really boils down to the original question asking whether a $*$-regular ring R_ϵ is unit-regular. If K is a positive definite field, then a Leavitt path algebra $L_K(E)$ is graded $*$-regular (by [11]). If E^0 is finite (so that $L_K(E)$ is unital), the ϵ-component of $L_K(E)$ is a matricial algebra over a field and hence it is unit-regular. So, Question 0.2 has an affirmative answer for the class of unital Leavitt path algebras.

1. Graded rings prerequisites

Throughout the paper, Γ denotes an arbitrary group and ϵ denotes its identity element. Rings are assumed to be associative. Unless stated otherwise, rings are assumed to be unital and a module is assumed to be a right module.

In the introduction, we recalled the definitions of a graded ring, homogeneous elements, trivial grading, and graded division ring. We adopt the standard definitions of graded ring homomorphisms and isomorphisms, graded left and right R-modules, graded module homomorphisms, graded algebras, graded left and right ideals, graded left and right free and projective modules as defined in [14] and [9]. In [9], it is assumed that Γ is abelian and the results without this assumption are stated just occasionally. We do not assume that Γ is abelian.

If M is a graded right R-module and $\gamma \in \Gamma$, the *γ-shifted or γ-suspended* graded right R-module $(\gamma)M$ is defined as the module M with the Γ-grading given by

$$(\gamma)M_\delta = M_{\gamma\delta}$$

for all $\delta \in \Gamma$. Analogously, if M is a graded left R-module, the γ-shifted left R-module $M(\gamma)$ is the module M with the Γ-grading given by $M(\gamma)_\delta = M_{\delta\gamma}$ for all $\delta \in \Gamma$. Any finitely generated graded free right R-module is of the form $(\gamma_1)R \oplus \ldots \oplus (\gamma_n)R$ for $\gamma_1, \ldots, \gamma_n \in \Gamma$ and an analogous statement holds for finitely generated graded free left R-modules (both [14] and [9] contain details).

If M and N are graded right R-modules and $\gamma \in \Gamma$, then $\operatorname{Hom}_R(M,N)_\gamma$ denotes the following

$$\operatorname{Hom}_R(M,N)_\gamma = \{f \in \operatorname{Hom}_R(M,N) \mid f(M_\delta) \subseteq N_{\gamma\delta}\},$$

$\operatorname{HOM}_R(M,N)$ denotes $\bigoplus_{\gamma\in\Gamma} \operatorname{Hom}_R(M,N)_\gamma$, and $\operatorname{END}_R(M)$ is used in the case if $M = N$. If M is finitely generated (which is the case we often consider), then $\operatorname{Hom}_R(M,N) = \operatorname{HOM}_R(M,N)$ for any N (both [14] and [9] contain details) and $\operatorname{End}_R(M) = \operatorname{END}_R(M,M)$ is a Γ-graded ring.

In [9], for a Γ-graded ring R and $\gamma_1, \ldots, \gamma_n \in \Gamma$, $\mathbb{M}_n(R)(\gamma_1, \ldots, \gamma_n)$ denotes the ring of matrices $\mathbb{M}_n(R)$ with the Γ-grading given by

$$(r_{ij}) \in \mathbb{M}_n(R)(\gamma_1, \ldots, \gamma_n)_\delta \quad \text{if} \quad r_{ij} \in R_{\gamma_i^{-1}\delta\gamma_j} \text{ for } i,j = 1, \ldots, n.$$

The definition of $\mathbb{M}_n(R)(\gamma_1, \ldots, \gamma_n)$ in [14] is different: $\mathbb{M}_n(R)(\gamma_1, \ldots, \gamma_n)$ in [14] corresponds to $\mathbb{M}_n(R)(\gamma_1^{-1}, \ldots, \gamma_n^{-1})$ in [9]. More details on the relations between two definitions can be found in [17, Section 1]. Although the definition from [14] has been in circulation longer, some matricial representations of Leavitt path algebras involve positive integers instead of negative integers making the definition from [9] more convenient for us. Since we deal extensively with Leavitt path algebras in section 4, we opt to use the definition from [9]. With this definition, if F is the graded free right module $(\gamma_1^{-1})R \oplus \cdots \oplus (\gamma_n^{-1})R$, then $\operatorname{Hom}_R(F,F) \cong_{\mathrm{gr}} \mathbb{M}_n(R)(\gamma_1, \ldots, \gamma_n)$ as graded rings.

We also recall [14, Remark 2.10.6] stating the first two parts in Lemma 1.1 and [9, Theorem 1.3.3] stating part (3) for Γ abelian. The proof of this statement generalizes to arbitrary Γ. The last sentence in lemma is the statement of [9, Proposition 1.4.4. and Theorem 1.4.5].

Lemma 1.1. [14, Remark 2.10.6], [9, Theorem 1.3.3, Proposition 1.4.4, and Theorem 1.4.5] *Let R be a Γ-graded ring and $\gamma_1, \ldots, \gamma_n \in \Gamma$.*

(1) If π a permutation of the set $\{1, \ldots, n\}$, then

$$\mathbb{M}_n(R)(\gamma_1, \gamma_2, \ldots, \gamma_n) \cong_{\mathrm{gr}} \mathbb{M}_n(R)(\gamma_{\pi(1)}, \gamma_{\pi(2)} \ldots, \gamma_{\pi(n)})$$

by the map $x \mapsto pxp^{-1}$ where p is the permutation matrix with 1 at the $(i, \pi(i))$-th spot for $i = 1, \ldots, n$ and zeros elsewhere.

(2) If δ in the center of Γ,

$$\mathbb{M}_n(R)(\gamma_1, \gamma_2, \ldots, \gamma_n) = \mathbb{M}_n(R)(\gamma_1\delta, \gamma_2\delta, \ldots, \gamma_n\delta).$$

(3) If $\delta \in \Gamma$ is such that there is an invertible element u_δ in R_δ, then

$$\mathbb{M}_n(R)(\gamma_1, \gamma_2, \ldots, \gamma_n) \cong_{\text{gr}} \mathbb{M}_n(R)(\gamma_1\delta, \gamma_2 \ldots, \gamma_n)$$

by the map $x \mapsto u^{-1}xu$ where u is the diagonal matrix with $u_\delta, 1, 1, \ldots, 1$ on the diagonal.

If Γ is abelian and R and S are Γ-graded division rings, then

$$\mathbb{M}_n(R)(\gamma_1, \gamma_2, \ldots, \gamma_n) \cong_{\text{gr}} \mathbb{M}_m(S)(\delta_1, \delta_2, \ldots, \delta_m)$$

implies that $R \cong_{\text{gr}} S$, that $m = n$, and that the graded isomorphism of the two algebra is a composition of finitely many operations from parts (1) to (3).

1.1. **Three lemmas.** Recall that two idempotents e, f of a ring R are said to be algebraically (or Murray-von Neumann) equivalent if there are $x, y \in R$ such that $xy = e$ and $yx = f$ in which case we write $e \sim f$. This condition is equivalent both to $eR \cong fR$ and to $Re \cong Rf$. In addition, one can require that $x \in eRf$ and $y \in fRe$. More details can be found in [6, Proposition 5.2]. The following lemma, needed for Proposition 2.4 shows the graded version of these equivalences.

Lemma 1.2. *Let R be a Γ-graded ring and e, f homogeneous idempotents of R. The following conditions are equivalent.*

(1) $eR \cong_{\text{gr}} (\gamma)fR$ for some $\gamma \in \Gamma$.
(2) $Re \cong_{\text{gr}} Rf(\gamma^{-1})$ for some $\gamma \in \Gamma$.
(3) There is $x \in R_{\gamma^{-1}}$ and $y \in R_\gamma$ such that $xy = e$ and $yx = f$.
(4) There is $x \in eR_{\gamma^{-1}}f$ and $y \in fR_\gamma e$ such that $xy = e$ and $yx = f$.

Proof. (1) $\Rightarrow$ (4). If $\phi : eR \cong_{\text{gr}} (\gamma)fR$, then $\phi \in \text{Hom}_R(eR, (\gamma)fR)_\epsilon$. Thus, $y = \phi(e) \in (\gamma)fR_\epsilon \subseteq R_\gamma$. Analogously, $x = \phi^{-1}(f) \in eR_{\gamma^{-1}} \subseteq R_{\gamma^{-1}}$. Moreover, $ye = \phi(e)e = \phi(ee) = \phi(e) = y$ so $y \in Re$ and $x \in Rf$ similarly. Then $yx = \phi(e)x = \phi(ex) = \phi(x) = \phi(\phi^{-1}(f)) = f$ and $xy = e$ similarly.

(4) $\Rightarrow$ (1). If L_x and L_y denote the left multiplications by x and y respectively, then $L_y \in \text{Hom}_R(R, R)_\gamma = \text{Hom}_R(R, (\gamma)R)_\epsilon$ and, similarly, $L_x \in \text{Hom}_R(R, R)_{\gamma^{-1}} = \text{Hom}_R((\gamma)R, R)_\epsilon$. The conditions $x \in eRf$ and $y \in fRe$ imply that L_y maps eR into $(\gamma)fR$ and L_x maps $(\gamma)fR$ into eR. The conditions $xy = e$ and $yx = f$ imply that L_x and L_y are mutually inverse so $L_y : eR \cong_{\text{gr}} (\gamma)fR$.

The equivalence (4) $\Leftrightarrow$ (2) can be shown analogously. The condition (3) implies (4) since if x, y are as in (3), then exf and fye are elements as in (4). The converse (4) $\Rightarrow$ (3) directly holds. □

We use the following lemma in the proofs of Propositions 2.1 and 2.4.

Lemma 1.3. *If R is a Γ-graded ring, A is a graded R-module, $S = \text{END}_R(A)$, $\gamma \in \Gamma$, and e, f homogeneous idempotents in S (thus necessarily in S_ϵ), then the following conditions are equivalent.*

(1) $eS \cong_{\text{gr}} (\gamma)fS$.
(2) $eS_\epsilon \cong ((\gamma)fS)_\epsilon = fS_\gamma$.
(3) $eA \cong_{\text{gr}} (\gamma)fA$.

Proof. The equality in condition (2) follows by definition. We show (1) $\Leftrightarrow$ (2) and (1) $\Leftrightarrow$ (3).

An isomorphism $\phi : eS \cong_{\text{gr}} (\gamma)fS$ restricts to $eS_\epsilon \cong ((\gamma)fS)_\epsilon = fS_\gamma$ so (1) implies (2). Conversely, if $\phi_\epsilon : eS_\epsilon \cong (\gamma)fS_\epsilon$, then ϕ, defined by $ex \mapsto \phi_\epsilon(e)x$, is a graded isomorphism $eS \cong_{\text{gr}} (\gamma)fS$.

If (1) holds, then $e = xy, f = yx$ for some $x \in eS_{\gamma^{-1}}f$ and $y \in fS_\gamma e$ by Lemma 1.2. So, y restricted on eA is a graded isomorphism $eA \cong_{\text{gr}} (\gamma)fA$ which shows (3). Conversely, if (3) holds and y is a graded isomorphism $eA \to (\gamma)fA$ with inverse x, then y can be extended to an element of S_γ by $y((1-e)A) = 0$. Similarly, x can be extended to an element of $S_{\gamma^{-1}}$ by $x((1-f)A) = 0$. Since $xy(a) = xye(a) = e(a)$, $yx = e$ and, similarly, $xy = f$. Thus, (1) holds by Lemma 1.2. □

We also intend to use Lemma 1.4 below. Let R be a Γ-graded ring, $x \in R_\gamma$, and let L_x denote the left multiplication by x. Then $\ker L_x$ is a graded right ideal of R and $L_x \in \mathrm{Hom}_R(R,R)_\gamma = \mathrm{Hom}_R(R,(\gamma)R)_\epsilon$. So, xR is a graded submodule of $(\gamma)R$ which implies that $(\gamma^{-1})xR$ is a graded right ideal of R. Thus, the following two are short exact sequences of graded right R-modules.

$$0 \longrightarrow \ker L_x \longrightarrow R \xrightarrow{L_x} xR \longrightarrow 0$$

$$0 \longrightarrow (\gamma^{-1})xR \longrightarrow R \longrightarrow (\gamma^{-1})\operatorname{coker} L_x \longrightarrow 0$$

Lemma 1.4. *If R is a Γ-graded ring, $x \in R_\gamma$ for $\gamma \in \Gamma$, L_x is the left multiplication by x, and $x = xyx$ for some homogeneous element y, then L_x is a graded isomorphism of yxR, $xR = (\gamma)xyR$, and $(1-yx)R \cong_{\text{gr}} (\gamma)(1-xy)R$ if and only if $\ker L_x \cong_{\text{gr}} \operatorname{coker} L_x$.*

Proof. The relation $x = xyx$ implies that $y \in R_{\gamma^{-1}}$, that $\ker L_x = (1-yx)R$ and $(\gamma^{-1})xR = xyR$, and that $L_x : yxR \to xyxR = xR$ is a graded isomorphism. So, $(\gamma^{-1})\operatorname{coker} L_x \cong_{\text{gr}} (1-xy)R$ and thus $\operatorname{coker} L_x \cong_{\text{gr}} (\gamma)(1-xy)R$. □

2. Graded unit-regular rings and graded cancellability

2.1. **Graded unit-regular rings.** As discussed in the introduction, the graded unit-regularity is a rather strong condition, too strong for many desirable properties to hold. Thus, in search for a better behaved graded analogue, we turn to the module-theoretic conditions equivalent to unit-regularity. This brings us to [7, Theorem 4.1] stating that the following conditions are equivalent for a ring R, a right R-module A, and $S = \mathrm{End}_R(A)$.

(1) S is unit-regular.
(2) S is regular and A satisfies internal cancellation.
(3) S is regular and $e \sim f$ implies $1-e \sim 1-f$ for all idempotents $e, f \in S$.

These equivalences generalize to Propositions 2.1 and 2.4.

Proposition 2.1. *Let R be a Γ-graded ring, A a finitely generated graded right R-module, and S_ϵ be the component of the graded ring $S = \mathrm{END}_R(A)$ corresponding to the identity $\epsilon \in \Gamma$. Then the following conditions are equivalent.*

(1) S_ϵ is unit-regular.
(2) S_ϵ is regular and A satisfies graded internal cancellation $IC_{\text{gr}}(A)$.
(3) S_ϵ is regular and $e \sim f$ implies $1-e \sim 1-f$ for all idempotents $e, f \in S_\epsilon$.

If A is finitely generated, the above statements hold for $S = \mathrm{End}_R(A)$.

Proof. To show (1) ⇒ (2), let $A = B \oplus C = D \oplus E$ and $x : B \cong_{\text{gr}} D$. Extend x to an element of S_ϵ by mapping C to 0. Let $u \in S_\epsilon$ be invertible and such that $xux = x$. Then, $(1-ux)A = \ker x = C$ and $uxA = uD$ so u maps $D = xA$ onto uxA and so u maps E onto $(1-ux)A = C$. Hence $C \cong_{\text{gr}} E$.

To show (2) $\Rightarrow$ (3), let $e, f \in S_\epsilon$ be idempotents such that $e \sim f$ so $eA \cong_{\text{gr}} fA$. By (2), $(1-e)A \cong_{\text{gr}} (1-f)A$ which implies that $1-e \sim 1-f$ as elements of S_ϵ by Lemma 1.3.

To show (3) $\Rightarrow$ (1), let $x \in S_\epsilon$ and $y \in S_\epsilon$ be such that $xyx = x$. Then $e = xy$ and $f = yx$ are idempotents of S_ϵ such that $e \sim f$. By the assumption, $1-e \sim 1-f$. So, there are $u \in (1-e)S_\epsilon(1-f), v \in (1-f)S_\epsilon(1-e)$ such that $uv = 1-e$ and $vu = 1-f$. Since $x \in eS_\epsilon f$ and $yxy \in fS_\epsilon e$, $yxy + v \in S_\epsilon$ is invertible with inverse $x + u$ and $x(yxy+v)x = x$. □

Recall the conditions Mat_ϵ and IC_{gr} from the introduction.

Proposition 2.2. *Let R be a Γ-graded ring. The following conditions are equivalent.*

(1) Mat_ϵ holds for R.
(2) Mat_ϵ holds for $\mathbb{M}_m(R)(\delta_1, \ldots, \delta_m)$ for every positive integer m and every $\delta_1, \ldots, \delta_m \in \Gamma$.

These two conditions imply the condition IC_{gr}. If $\mathbb{M}_n(R)(\gamma_1, \ldots, \gamma_n)_\epsilon$ is regular for every n and every $\gamma_1, \ldots, \gamma_n \in \Gamma$, then (1), (2) and IC_{gr} are equivalent.

Proof. Since $(\gamma^{-1})(\delta^{-1})R(\delta)(\gamma) = ((\gamma\delta)^{-1})R(\gamma\delta)$ for $\gamma, \delta \in \Gamma$, we have that

$$\mathbb{M}_n(\mathbb{M}_m(R)(\delta_1, \ldots, \delta_m))(\gamma_1, \ldots, \gamma_n) = \mathbb{M}_{nm}(R)(\gamma_1\delta_1, \ldots, \gamma_1\delta_m, \; \ldots\ldots\ldots, \; \gamma_n\delta_1, \ldots, \gamma_n\delta_m)$$

for all positive integers m and n and all $\gamma_1, \ldots, \gamma_n, \delta_1, \ldots, \delta_m \in \Gamma$. So, assuming (1) is sufficient for (2) and the converse trivially holds.

Since $\text{IC}_{\text{gr}}(\_)$ is preserved under formation of graded direct summands, IC_{gr} holds iff $\text{IC}_{\text{gr}}(F)$ holds for every finitely generated graded free module F. Every such module F is of the form $\bigoplus_{i=1}^n (\gamma_i^{-1})R$ for some n and some $\gamma_1, \ldots, \gamma_n$. Since $\text{End}_R(F) = \text{END}_R(F) = \mathbb{M}_n(R)(\gamma_1, \ldots, \gamma_n)$, if $\text{End}_R(F)_\epsilon$ is unit-regular then $\text{IC}_{\text{gr}}(F)$ holds by Proposition 2.1. If $\text{End}_R(F)_\epsilon$ is regular, then $\text{IC}_{\text{gr}}(F)$ implies that $\text{End}_R(F)_\epsilon$ is unit-regular also by Proposition 2.1. □

Remark 2.3. Note that the assumption in the last sentence of Proposition 2.2 is automatically satisfied if R is graded regular. Indeed, by the graded analogue of [7, Theorem 1.7], graded regularity is passed to graded matrix algebras. The proof is analogous to the nongraded case: if R is graded regular, then it is direct to check that $(\gamma^{-1})R(\gamma)$ is graded regular for every $\gamma \in \Gamma$. So, $(\gamma_i^{-1})R(\gamma_i) \cong_{\text{gr}} e_{ii}\,\mathbb{M}_n(R)(\gamma_1, \ldots, \gamma_n)e_{ii}$ is graded regular for all the standard matrix units e_{ii} for any n and $\gamma_1, \ldots, \gamma_n \in \Gamma$. Then one shows that $\mathbb{M}_n(R)(\gamma_1, \ldots, \gamma_n)$ is graded regular by induction analogously to the proof in the nongraded case (see [7, Lemma 1.6]). This shows that if R is graded regular, then $\mathbb{M}_n(R)(\gamma_1, \ldots, \gamma_n)$ is graded regular and, consequently, $\mathbb{M}_n(R)(\gamma_1, \ldots, \gamma_n)_\epsilon$ is regular.

In Proposition 2.4, we relate the properties $\text{UR}^{\text{w}}_{\text{gr}}$ and UR_{gr} of $\text{END}_R(A)$ for a graded module A with $\text{IC}^{\text{w}}_{\text{gr}}(A)$ and $\text{IC}^{\text{s}}_{\text{gr}}(A)$ respectively.

Proposition 2.4. *Let R be a Γ-graded ring, A be a graded right R-module, and $S = \text{END}_R(A)$. The following conditions are equivalent.*

(1^{w}) S is weakly graded unit-regular.
(2^{w}) S is graded regular and A satisfies weak graded internal cancellation $\text{IC}^{\text{w}}_{\text{gr}}(A)$.
(3^{w}) S is graded regular and $eA \cong_{\text{gr}} (\gamma)fA$ for some $\gamma \in \Gamma$, implies $(1-e)A \cong (1-f)A$ for all homogeneous idempotents $e, f \in S$.

The following conditions are also equivalent.

(1^{s}) S is graded unit-regular.
(2^{s}) S is graded regular and A satisfies strong graded internal cancellation $\text{IC}^{\text{s}}_{\text{gr}}(A)$.

(3^{s}) *S is graded regular and $eA \cong_{\mathrm{gr}} (\gamma)fA$ for some $\gamma \in \Gamma$, implies $(1-e)A \cong (\gamma)(1-f)A$ for all homogeneous idempotents $e, f \in S$.*

If A is finitely generated, then the above statements hold for $S = \mathrm{End}_R(A)$.

Proof. Let us show (1^{w})⇒(2^{w}) and (1^{s}) ⇒(2^{s}). Let $A = B \oplus C = D \oplus E$ and $x : B \cong_{\mathrm{gr}} (\gamma)D$. Extend x to A by $xC = 0$. So, $x \in \mathrm{HOM}_R(A, (\gamma)A)_\epsilon = \mathrm{END}_R(A)_\gamma = S_\gamma$. Assuming ($1^{\mathrm{w}}$), there is invertible $u \in S$ such that $x = xux$. By the proof of (1)⇒(2) of Proposition 2.1, we obtain $C \cong E$. Assuming (1^{s}), such u can be found in $S_{\gamma^{-1}}$. Then, $(1-ux)A = \ker x = C$ and $uxA = u(\gamma)D$ so u maps $(\gamma)D = xA$ onto uxA and so u maps $(\gamma)E$ onto $(1-ux)A = C$. Hence $C \cong_{\mathrm{gr}} (\gamma)E$.

Let us show (2^{w})⇒(3^{w}) and (2^{s})⇒(3^{s}). Assume that $eA \cong_{\mathrm{gr}} (\gamma)fA$ for some $\gamma \in \Gamma$. Condition (2^{w}) implies that $(1-e)A \cong (1-f)A$ and condition (2^{s}) implies that $(1-e)A \cong_{\mathrm{gr}} (\gamma)(1-f)A$.

Let us show (3^{w})⇒(1^{w}) and (3^{s})⇒(1^{s}). Let $x \in S_\gamma$. Under either (3^{w}) or (3^{s}), there is $y \in S_{\gamma^{-1}}$ such that $xyx = x$. Then $e = xy$ and $f = yx$ are homogeneous idempotents and $eA \cong_{\mathrm{gr}} (\gamma^{-1})fA$ by Lemmas 1.2 and 1.3. Condition (3^{w}) implies that $(1-e)A \cong (1-f)A$ and condition (3^{s}) that $(1-e)A \cong_{\mathrm{gr}} (\gamma^{-1})(1-f)A$. In the second case, there are $u \in (1-e)S_\gamma(1-f), v \in (1-f)S_{\gamma^{-1}}(1-e)$ such that $uv = 1-e$ and $vu = 1-f$ by Lemmas 1.2 and 1.3. Then $yxy + v \in S_{\gamma^{-1}}$ is invertible with inverse $x + u \in S_\gamma$ and $x(yxy+v)x = x$. In the first case, there are $u \in (1-e)S(1-f)$ and $v \in (1-f)S(1-e)$ such that $uv = 1-e$ and $vu = 1-f$ and the rest of the prior arguments show that $yxy + v$ is invertible and that $x(yxy+v)x = x$. □

The implication $\mathrm{UR}_{\mathrm{gr}} \Rightarrow \mathrm{UR}^{\mathrm{w}}_{\mathrm{gr}}$ is direct and it is strict by Example 2.11. It is also direct to see that $\mathrm{IC}^{\mathrm{s}}_{\mathrm{gr}}(\_)$ implies both $\mathrm{IC}^{\mathrm{w}}_{\mathrm{gr}}(\_)$ and $\mathrm{IC}_{\mathrm{gr}}(\_)$. Hence, $\mathrm{UR}_{\mathrm{gr}}$ implies $\mathrm{IC}_{\mathrm{gr}}(R)$. However, it is not direct to see that $\mathrm{UR}_{\mathrm{gr}} \Rightarrow \mathrm{IC}_{\mathrm{gr}}$. This implication follows from Theorem 2.9 of section 2.4.

2.2. **Graded unit-regularity of some $\mathbb{Z}$-graded matrix algebras.** In this and the next section, let K be a trivially $\mathbb{Z}$-graded field and let $K[x^m, x^{-m}]$ be the graded field of Laurent polynomials naturally $\mathbb{Z}$-graded by $K[x^m, x^{-m}]_{mk} = Kx^{mk}$ and $K[x^m, x^{-m}]_n = 0$ if m does not divide n. We use the additive notation for the group operation of $\mathbb{Z}$. The main objective of this section is to prove Proposition 2.6 which characterizes graded unit-regularity of graded matrix algebras over K and $K[x^m, x^{-m}]$. This lemma ends up being an essential part of characterization of graded unit-regular Leavitt path algebras in section 4. This lemma also generalizes the example from the introduction showing that $M_2(K)(0,1)$ is not graded unit-regular for any trivially graded field K.

Lemma 2.5 plays an important role in the proof of Proposition 2.6. It also shows that $\mathrm{UR}_{\mathrm{gr}}$ forces a rather strong requirement on the grading. If $0 \neq x \in R_\gamma$ and R is graded unit-regular, then there is a homogeneous invertible element u such that $x = xux$. This last condition forces u to be in $R_{\gamma^{-1}}$ and so its inverse is in R_γ. This shows the following.

Lemma 2.5. *If a graded ring R is graded unit-regular, then every nonzero component contains an invertible element.*

Proposition 2.6. *Let m and n be positive and $\gamma_1, \gamma_2, \ldots, \gamma_n$ arbitrary integers.*

(1) The algebra $\mathbb{M}_n(K)(\gamma_1, \gamma_2 \ldots, \gamma_n)$ is graded unit-regular if and only if $n = 1$ or $\gamma_1 = \gamma_2 = \ldots = \gamma_n$.

(2) If the list $\gamma_1, \ldots, \gamma_n$ is such that all of $0, 1, \ldots, m-1$ appear on it when it is considered modulo m, then $\mathbb{M}_n(K[x^m, x^{-m}])(\gamma_1, \ldots, \gamma_n)$ is graded unit-regular if and only if $n = km$ for some positive integer k and the list $\gamma_1, \ldots, \gamma_n$, considered modulo m, is such that each $i = 0, 1, \ldots, m-1$ appears exactly k times.

The following example shows the idea of the proof of part (2) of Proposition 2.6.

Example 2.7. If K is any trivially $\mathbb{Z}$-graded field, Proposition 2.6 states that

$$\mathbb{M}_9(K[x^3,x^{-3}])(0,0,0,1,1,1,2,2,2) \text{ is graded unit-regular and}$$
$$\mathbb{M}_9(K[x^3,x^{-3}])(0,0,0,0,1,1,1,2,2) \text{ is not graded unit-regular}$$

because (0,0,0,1,1,1,2,2,2) has equal number of 0, 1, and 2 and (0,0,0,0,1,1,1,2,2) does not.

If A is a homogeneous matrix in the first algebra, one can produce an invertible homogeneous matrix U such that $AUA = A$. We illustrate this idea for an arbitrary element A of the 1-component. This component consists of elements of the form

$$\left[\begin{array}{c|c|c} 0 & 0 & \mathbb{M}_3(K)x^3 \\ \hline \mathbb{M}_3(K) & 0 & 0 \\ \hline 0 & \mathbb{M}_3(K) & 0 \end{array}\right] = \left[\begin{array}{ccc|ccc|ccc} 0&0&0&0&0&0&Kx^3&Kx^3&Kx^3 \\ 0&0&0&0&0&0&Kx^3&Kx^3&Kx^3 \\ 0&0&0&0&0&0&Kx^3&Kx^3&Kx^3 \\ \hline K&K&K&0&0&0&0&0&0 \\ K&K&K&0&0&0&0&0&0 \\ K&K&K&0&0&0&0&0&0 \\ \hline 0&0&0&K&K&K&0&0&0 \\ 0&0&0&K&K&K&0&0&0 \\ 0&0&0&K&K&K&0&0&0 \end{array}\right].$$

If $A = \left[\begin{array}{c|c|c} 0 & 0 & A_{13}x^3 \\ \hline A_{21} & 0 & 0 \\ \hline 0 & A_{32} & 0 \end{array}\right]$ is in the 1-component, let $U = \left[\begin{array}{c|c|c} 0 & U_{21} & 0 \\ \hline 0 & 0 & U_{32} \\ \hline U_{13}x^{-3} & 0 & 0 \end{array}\right]$ where $U_{ij} \in \mathbb{M}_3(K)$ are invertible matrices such that $A_{ij}U_{ij}A_{ij} = A_{ij}$ for $(i,j) \in \{(1,3),(2,1),(3,2)\}$. So, U is in the -1-component $\left[\begin{array}{c|c|c} 0 & \mathbb{M}_3(K) & 0 \\ \hline 0 & 0 & \mathbb{M}_3(K) \\ \hline \mathbb{M}_3(K)x^{-3} & 0 & 0 \end{array}\right]$, $U^{-1} = \left[\begin{array}{c|c|c} 0 & 0 & U_{13}^{-1}x^3 \\ \hline U_{21}^{-1} & 0 & 0 \\ \hline 0 & U_{32}^{-1} & 0 \end{array}\right]$ and $AUA = A$.

The algebra $\mathbb{M}_9(K[x^3,x^{-3}])(0,0,0,0,1,1,1,2,2)$ is not graded unit-regular by Lemma 2.5 since every element A of the 1-component has determinant zero. Indeed, every A in the 1-component has the following form.

$$\left[\begin{array}{cccc|ccc|cc} 0&0&0&0&0&0&0&Kx^3&Kx^3 \\ 0&0&0&0&0&0&0&Kx^3&Kx^3 \\ 0&0&0&0&0&0&0&Kx^3&Kx^3 \\ 0&0&0&0&0&0&0&Kx^3&Kx^3 \\ \hline K&K&K&K&0&0&0&0&0 \\ K&K&K&K&0&0&0&0&0 \\ K&K&K&K&0&0&0&0&0 \\ \hline 0&0&0&0&K&K&K&0&0 \\ 0&0&0&0&K&K&K&0&0 \end{array}\right]$$

Let M_{18} and M_{19} denote the only two possibly nonzero minors of the determinant of A expanded along the first row. The minor M_{18} is the determinant of a matrix of the form

$$\left[\begin{array}{cccc|ccc|c} 0&0&0&0&0&0&0&Kx^3 \\ 0&0&0&0&0&0&0&Kx^3 \\ 0&0&0&0&0&0&0&Kx^3 \\ \hline K&K&K&K&0&0&0&0 \\ K&K&K&K&0&0&0&0 \\ K&K&K&K&0&0&0&0 \\ \hline 0&0&0&0&K&K&K&0 \\ 0&0&0&0&K&K&K&0 \end{array}\right].$$

Expand M_{18} with respect to the first row. Since the determinant of a matrix of the form

$$\left[\begin{array}{cccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline K & K & K & K & 0 & 0 & 0 \\ K & K & K & K & 0 & 0 & 0 \\ K & K & K & K & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & K & K & K \\ 0 & 0 & 0 & 0 & K & K & K \end{array}\right]$$

is zero, $M_{18} = 0$. One obtains that $M_{19} = 0$ similarly. Hence, the determinant of A is zero.

We prove Proposition 2.6 now.

Proof. (1) The algebras $\mathbb{M}_n(K)(\gamma_1, \gamma_1, \ldots, \gamma_1)$ and $\mathbb{M}_n(K)(0, 0, \ldots, 0)$ are equal by part (2) of Lemma 1.1. The algebra $\mathbb{M}_n(K)(0, 0, \ldots, 0)$ is graded unit-regular since it is trivially graded and $\mathbb{M}_n(K)$ is unit-regular. For the converse, assume that $n > 1$ and that not all $\gamma_1, \gamma_2 \ldots, \gamma_n$ are equal. If γ_i is the smallest of $\gamma_1 \ldots, \gamma_n$, then $\delta_1 = \gamma_1 - \gamma_i, \ldots, \delta_n = \gamma_n - \gamma_i$ is a list of nonnegative integers such that at least one is positive by the assumption that not all $\gamma_1, \ldots, \gamma_n$ are equal and at least one is zero by construction. By permuting the entries, we can assume that δ_1 is zero and δ_2 is positive. Consider the δ_2-component of $\mathbb{M}_n(K)(0, \delta_2, \ldots, \delta_n)$. It is nonzero since the matrix unit e_{21} is in it. The first row of any element of this component consists of zeros since $\delta_2 + \delta_i > 0$ and so $K_{-0+\delta_2+\delta_i} = 0$ for all $i = 1, \ldots, n$. Hence, the determinant of any matrix in the δ_2-component is zero and so $\mathbb{M}_n(K)(0, \delta_2, \ldots, \delta_n)_{\delta_2}$ does not contain an invertible element. By Lemma 2.5, $\mathbb{M}_n(K)(0, \delta_2, \ldots, \delta_n) \cong_{\text{gr}} \mathbb{M}_n(K)(\gamma_1, \gamma_2 \ldots, \gamma_n)$ is not graded unit-regular.

(2) Let $R_k = \mathbb{M}_{mk}(K[x^m, x^{-m}])(0, 0, \ldots, 0,\ 1, 1, \ldots, 1,\ \ldots\ldots\ldots,\ m-1, m-1, \ldots, m-1)$ where each $i = 0, 1, \ldots, m-1$ appears exactly k times in the list of the shifts above. Then

$$R_k = \mathbb{M}_m\left(\mathbb{M}_k(K[x^m, x^{-m}])(0, 0, \ldots, 0)\right)(0, 1, \ldots, m-1)$$

We show that R_k is graded unit-regular. An element A of the l-component for $l = k'm + i'$ with $0 \leq i' < m$ can be written as $m \times m$ blocks of $k \times k$ matrices $[[a_{st}]_{ij}]$ with exactly m possibly nonzero blocks. The blocks at the $(i'+1, 1), (i'+2, 2), \ldots (m, m-i')$ spots are elements of

$$\mathbb{M}_k(K[x^m, x^{-m}])(0, 0, \ldots, 0)_{k'm} = \mathbb{M}_k(K)x^{k'm}$$

and the blocks at $(1, m-i'+1), (2, m-i'+2), \ldots (i', m)$ spots are elements of

$$\mathbb{M}_k(K[x^m, x^{-m}])(0, 0, \ldots, 0)_{(k'+1)m} = \mathbb{M}_k(K)x^{(k'+1)m}.$$

For each $(i,j) \in \{(i'+1, 1), (i'+2, 2), \ldots (m, m-i')\}$, $a_{st} = b_{st}x^{k'm} \in Kx^{k'm}$ for all $s, t = 1, \ldots, k$. For such (i,j), let $[v_{st}]_{ij}$ be an invertible matrix in $\mathbb{M}_n(K)$ such that $[b_{st}]_{ij}[v_{st}]_{ij}[b_{st}]_{ij} = [b_{st}]_{ij}$ and let $u_{st} = v_{st}x^{-k'm}$ for $s, t = 1, \ldots, k$. For each $(i,j) \in \{(1, m-i'+1), (2, m-i'+2), \ldots (i', m)\}$, $a_{st} = b_{st}x^{(k'+1)m} \in Kx^{(k'+1)m}$ for all $s, t = 1, \ldots, k$. For such (i,j), let $[v_{st}]_{ij}$ be an invertible matrix in $\mathbb{M}_n(K)$ such that $[b_{st}]_{ij}[v_{st}]_{ij}[b_{st}]_{ij} = [b_{st}]_{ij}$ and let $u_{st} = v_{st}x^{-(k'+1)m}$ for $s, t = 1, \ldots, k$. For all other (i,j), let $[u_{st}]_{ij} = 0_{k \times k}$. Finally, let $U_{ij} = [u_{st}]_{ij}$ and $U = [U_{ji}]$. By construction, U is in the $-l$-component of R_k and $AUA = A$. If U_{ij}^{-1} is the inverse of U_{ij} for $U_{ij} \neq 0$ and $U_{ij}^{-1} = 0$ for $U_{ij} = 0$, then U is invertible with $U^{-1} = [U_{ij}^{-1}]$.

Let us prove the converse now. Assume that m does not divide n or that $n = km$ for some k but that the list $\gamma_1, \ldots, \gamma_n$, considered modulo m, is such that some $i, j = 0, 1, \ldots, m-1$ appear different number of times. We show that there is a nonzero component of $M_n = \mathbb{M}_n(K[x^m, x^{-m}])(\gamma_1, \ldots, \gamma_n)$

such that all its elements have determinant zero and, consequently, are not invertible. By Lemma 2.5, this shows that M_n is not graded unit-regular.

Using part (3) of Lemma 1.1, it is sufficient to consider the case $0 \leq \gamma_j < m$ for $j = 1, \ldots, n$. Let d_i be the number of times i appears on the list $\gamma_1, \ldots, \gamma_n$ for all $i = 0, \ldots, m-1$. Since every $i = 0, \ldots, m-1$ appears in the list by the assumption, $d_i > 0$ for every i. Let j be such that $d_j = \min\{d_0, \ldots, d_{m-1}\}$. Using part (2) of Lemma 1.1, we can add $m-j-1$ to all the entries of the list and use part (3) of Lemma 1.1 again to consider the elements in the new list modulo m again. By doing this, we can assume that $j = m-1$. Permuting the entries using part (1) of Lemma 1.1 and relabeling $d_0, \ldots, d_{m-1}$ if necessary, we can assume that M_n is

$$\mathbb{M}_n(K[x^m, x^{-m}])(0, 0 \ldots, 0,\ 1, 1, \ldots, 1,\ \ldots\ldots\ldots,\ m-1, m-1, \ldots, m-1)$$

where every i appears d_i times on the above list, $d_{m-1} \leq d_j$ for all $j = 0, \ldots, m-1$, and $d_{m-1} < d_i$ for at least one $i = 0, \ldots, m-1$.

The $(i+1)$-component is nonzero since the matrix unit e_{s1} where $s = 1 + \sum_{j=0}^{i} d_i$ is in it. An arbitrary element A of the $(i+1)$-component of M_n can be divided into $m \times m$ blocks of sizes $d_j \times d_l$ for $j, l = 0, \ldots, m-1$. All of the blocks are zero except possibly blocks

$$d_0 \times d_{m-1-i},\ d_1 \times d_{m-i},\ \ldots,\ d_i \times d_{m-1} \quad \text{and} \quad d_{i+1} \times d_0,\ d_{i+2} \times d_1,\ \ldots,\ d_{m-1} \times d_{m-2-i}.$$

Note that the block $d_i \times d_{m-1}$ is the only nonzero block in d_i rows of A and the last d_{m-1} columns of A and that there are more rows than columns in this block since $d_i > d_{m-1}$. Compute the determinant of A using expansion with respect to the row of A corresponding to the first row of this block. Continue computing the minors of this minor. In each step, use the row corresponding to the first row of the remaining portion of this block. The condition $d_i > d_{m-1}$ implies that every minor of the $d_i - d_{m-1}$-th step is zero. Thus, all the minors computed in the previous steps are zero also and, as a consequence, the determinant of A is zero as well. □

Note that Lemma 1.1 and Proposition 2.6 imply that $\mathbb{M}_n(K[x, x^{-1}])(\gamma_1, \ldots, \gamma_n)$ is graded unit-regular for any n and any $\gamma_1, \ldots, \gamma_n \in \mathbb{Z}$. Indeed, since $K[x, x^{-1}]$ has an invertible element in any graded component, $\mathbb{M}_n(K[x, x^{-1}])(\gamma_1, \ldots, \gamma_n) \cong_{\text{gr}} \mathbb{M}_n(K[x, x^{-1}])(0, \ldots, 0)$ by part (3) of Lemma 1.1. This last algebra is graded unit-regular by Proposition 2.6.

2.3. **Graded corners.** If R is a graded ring and e a homogeneous idempotent, the ring eRe is a *graded corner.*

The property of being unit-regular, being directly finite and having stable range 1 are passed to corners. The proofs of these facts involve consideration of an element $x + 1 - e$ of R for any element x of eRe (see [12, Theorem, §2] for unit-regularity, [19, Theorem 2.8] for stable range 1 and [6, 7.3] for direct finiteness). This is problematic for graded rings since if x is a homogeneous element in R_γ for $\gamma \neq \epsilon$ and if $e \neq 1$, then $x + 1 - e$ is not homogeneous so none of the proofs of the nongraded cases can be adjusted to the graded cases. The following example shows that graded unit-regularity is *not* necessarily passed to graded corners.

Example 2.8. Let $R = \mathbb{M}_3(K[x^3, x^{-3}])(0, 1, 2)$. By Proposition 2.6, R is graded unit-regular. Let $e = e_{11} + e_{22}$ where $e_{11} + e_{22}$ are the standard matrix units. The corner eRe is graded isomorphic to $\mathbb{M}_2(K[x^3, x^{-3}])(0, 1)$ which is not graded unit-regular by Proposition 2.6. So, eRe is not graded unit-regular also.

2.4. **Graded cancellability.** In Theorem 2.9, we relate the conditions $\mathrm{IC_{gr}}$, $\mathrm{C_{gr}}$ and UR_ϵ for a graded regular ring. The cancellation property has a favorable feature that a finite direct sum is cancellable if and only if each of its terms is cancellable (see [13, Proposition 3.3]). Relating $\mathrm{IC_{gr}}(\_)$ with $\mathrm{C_{gr}}(\_)$ in Theorem 2.9, we show that $\mathrm{IC_{gr}}(\_)$ is closed under the formation of direct sums of modules if the ring is graded regular. The conditions $\mathrm{IC_{gr}}(\_)$ and $\mathrm{IC^s_{gr}}(\_)$ alone are not closed for finite direct sums (consider [13, Example 3.2 (3)] and grade the ring trivially by any group).

If R is a Γ-graded ring, $\mathcal{P}_{\mathrm{gr}}$ the category of finitely generated graded projective modules, and A in $\mathcal{P}_{\mathrm{gr}}$, we consider $\mathrm{C_{gr}}(A)$ only in $\mathcal{P}_{\mathrm{gr}}$ so we abbreviate "$\mathrm{C_{gr}}(A)$ holds in $\mathcal{P}_{\mathrm{gr}}$" as "$\mathrm{C_{gr}}(A)$ holds". [1]

Note that $\mathrm{C_{gr}}(A)$ holds if and only if $\mathrm{C_{gr}}((\gamma)A)$ holds for any $\gamma \in \Gamma$. Indeed, if $(\gamma)A \oplus B \cong_{\mathrm{gr}} (\gamma)A \oplus C$ for some modules B, C, then $A \oplus (\gamma^{-1})B \cong_{\mathrm{gr}} A \oplus (\gamma^{-1})C$ and so $(\gamma^{-1})B \cong_{\mathrm{gr}} (\gamma^{-1})C$ if $\mathrm{C_{gr}}(A)$ holds. Hence, $B = (\gamma)(\gamma^{-1})B \cong_{\mathrm{gr}} (\gamma)(\gamma^{-1})C = C$.

Also, $\mathrm{C_{gr}}(A \oplus B)$ holds if and only if $\mathrm{C_{gr}}(A)$ and $\mathrm{C_{gr}}(B)$ hold. This can easily be checked (and the argument is completely analogously to the nongraded case, see [13, Proposition 3.3]). [2] Thus,

$$\mathrm{C_{gr}}(R) \text{ holds} \quad \text{if and only if} \quad \mathrm{C_{gr}}(P) \text{ holds for any } P \in \mathcal{P}_{\mathrm{gr}}.$$

Hence, $\mathrm{C_{gr}}(R)$ holds if and only if the Γ-monoid $\mathcal{V}^\Gamma(R)$ (see [17, Section 1.3]) is cancellative.

In the nongraded case, $\mathrm{C}(\_) \Rightarrow \mathrm{IC}(\_)$ and the converse holds if R is regular ([7, Theorem 4.5]). We show the graded versions of these statements and relate $\mathrm{C_{gr}}$ with UR_ϵ and $\mathrm{IC_{gr}}$.

Theorem 2.9. *Let R be a Γ-graded ring and $P \in \mathcal{P}_{\mathrm{gr}}$.*

(1) If $\mathrm{C_{gr}}(P)$ holds, then $\mathrm{IC_{gr}}(P)$ holds and the converse holds if R is graded regular.
(2) If $\mathrm{C_{gr}}(R)$ holds, then $\mathrm{IC_{gr}}$ holds and the converse holds if R is graded regular.
(3) If R is graded regular, then R_ϵ is unit-regular if and only if $\mathrm{C_{gr}}(R)$ holds. Hence, the conditions UR_ϵ, $\mathrm{C_{gr}}(R)$, $\mathrm{C_{gr}}$, $\mathrm{IC_{gr}}$, and Mat_ϵ are all equivalent for a graded regular ring R.

Proof. Assuming that $\mathrm{C_{gr}}(P)$ holds, let $P = A \oplus B = C \oplus D$ and $A \cong_{\mathrm{gr}} C$. Then $A \oplus B \cong_{\mathrm{gr}} A \oplus D$. Since $\mathrm{C_{gr}}(P)$ implies $\mathrm{C_{gr}}(A)$, we have that $B \cong_{\mathrm{gr}} D$.

Let R be graded regular and let $P \oplus A \cong_{\mathrm{gr}} P \oplus B$ for some $A, B \in \mathcal{P}_{\mathrm{gr}}$ now. By [7, Theorem 2.8], two direct sum decompositions of a finitely generated projective module over a regular ring have isomorphic refinements. The graded version of this statement can be shown by a proof completely analogous to the proof of [7, Theorem 2.8]. So, there are graded decompositions $P = P_1 \oplus P_2$ and $A = A_1 \oplus A_2$ such that $P_1 \oplus A_1 \cong_{\mathrm{gr}} P$ and $P_2 \oplus A_2 \cong_{\mathrm{gr}} B$. Hence $P_1 \oplus A_1 \cong_{\mathrm{gr}} P = P_1 \oplus P_2$ implies $A_1 \cong_{\mathrm{gr}} P_2$ by $\mathrm{IC_{gr}}(P)$. Thus, $A = A_1 \oplus A_2 \cong_{\mathrm{gr}} P_2 \oplus A_2 \cong_{\mathrm{gr}} B$.

To show (2), assume that $\mathrm{C_{gr}}(R)$ holds. Since $\mathrm{C_{gr}}(\_)$ is closed under taking finite direct sums and graded direct summands, $\mathrm{C_{gr}}(P)$ holds for any $P \in \mathcal{P}_{\mathrm{gr}}$. By statement (1), $\mathrm{IC_{gr}}$ holds and the converse holds if R is graded regular.

To show (3), note that if R is graded regular, then UR_ϵ and $\mathrm{IC_{gr}}(R)$ are equivalent by Proposition 2.1. By part (1), $\mathrm{IC_{gr}}(R)$ and $\mathrm{C_{gr}}(R)$ are equivalent. By part (2), $\mathrm{C_{gr}}(R)$ and $\mathrm{IC_{gr}}$ are equivalent.

[1] One could also consider the weak and strong graded cancellability of a module $A \in \mathcal{P}_{\mathrm{gr}}$ analogously to the weak and strong graded internal cancellation as follows.

$\mathrm{C^w_{gr}}(A)$: $\quad A \oplus B \cong_{\mathrm{gr}} (\gamma)A \oplus C$ implies $\quad B \cong \quad C$ for every $\gamma \in \Gamma$ and every $B, C \in \mathcal{P}_{\mathrm{gr}}$.
$\mathrm{C^s_{gr}}(A)$: $\quad A \oplus B \cong_{\mathrm{gr}} (\gamma)A \oplus C$ implies $(\gamma)B \cong_{\mathrm{gr}} C$ for every $\gamma \in \Gamma$ and every $B, C \in \mathcal{P}_{\mathrm{gr}}$.

It is direct to show that $\mathrm{C^s_{gr}}(\_) \Rightarrow \mathrm{C_{gr}}(\_)$ and that $\mathrm{C^s_{gr}}(\_) \Rightarrow \mathrm{C^w_{gr}}(\_)$. One can show that the conditions $\mathrm{C^w_{gr}}(\_)$ and $\mathrm{C^s_{gr}}(\_)$ do not share the nice addition and shift-invariant properties of $\mathrm{C_{gr}}(\_)$.

[2] All the statements made in this section so far are true if $\mathcal{P}_{\mathrm{gr}}$ is replaced by any category of graded modules.

The conditions $\mathrm{C_{gr}}(R)$ and $\mathrm{C_{gr}}$ are equivalent since $\mathrm{C_{gr}}(\_)$ is closed under taking finite direct sums and graded direct summands and the conditions $\mathrm{IC_{gr}}$ and Mat_ϵ are equivalent by Proposition 2.2. ☐

If a graded property P_{gr} is closed under formation of graded matrix algebras and graded corners, then it is *graded Morita invariant* and the converse also holds (see [9, Section 2 and Theorem 2.3.8]). While unit-regularity is Morita invariant, graded unit-regularity is not graded Morita invariant (by Proposition 2.6 and also by Example 2.8). On the other hand, $\mathrm{Reg_{gr}}$+UR_ϵ *is* graded Morita invariant by Corollary 2.10. This exhibits another advantage of $\mathrm{Reg_{gr}}$+UR_ϵ over $\mathrm{UR_{gr}}$.

Corollary 2.10. *The property $\mathrm{Reg_{gr}}$+UR_ϵ is graded Morita invariant.*

Proof. The property $\mathrm{Reg_{gr}}$ is closed under forming graded matrix algebras (see Remark 2.3) and graded corners (direct to check). By Theorem 2.9, if $\mathrm{Reg_{gr}}$ holds, then $\mathrm{UR}_\epsilon \Leftrightarrow \mathrm{Mat}_\epsilon$ so UR_ϵ is closed under formation of graded matrix algebras. Since the property UR is closed under formation of corners and $(eRe)_\epsilon = eR_\epsilon e$ if R is a graded ring and e a homogeneous idempotent, UR_ϵ is closed under formation of graded corners. ☐

Propositions 2.2 and 2.4 and Theorem 2.9 show the diagram from the introduction (also below).

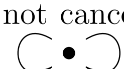
$$\begin{array}{ccc} & \mathrm{UR_{gr}} \Leftrightarrow \mathrm{Reg_{gr}}+\mathrm{IC^{s}_{gr}}(R) & \\ \swarrow & & \searrow \\ \mathrm{Reg_{gr}}+\mathrm{C_{gr}}(R) \Leftrightarrow \mathrm{Reg_{gr}}+\mathrm{UR}_\epsilon \Leftrightarrow \mathrm{Reg_{gr}}+\mathrm{IC_{gr}}(R) & & \mathrm{UR^{w}_{gr}} \Leftrightarrow \mathrm{Reg_{gr}}+\mathrm{IC^{w}_{gr}}(R) \end{array}$$

We present examples showing that both diagonal arrows are strict and that UR_ϵ and $\mathrm{UR^w_{gr}}$ are not equivalent even if a ring is graded regular. We also show that the following relations hold.

(1) $\mathrm{Reg_{gr}}+\mathrm{UR}_\epsilon \not\Rightarrow \mathrm{UR_{gr}}$, $\mathrm{UR} \not\Rightarrow \mathrm{UR_{gr}}$, $\mathrm{UR^w_{gr}} \not\Rightarrow \mathrm{UR_{gr}}$.
(2) $\mathrm{UR^w_{gr}} \not\Rightarrow \mathrm{UR}$, $\mathrm{UR_{gr}} \not\Rightarrow \mathrm{UR}$, $\mathrm{Reg_{gr}}+\mathrm{UR}_\epsilon \not\Rightarrow \mathrm{UR}$.
(3) $\mathrm{Reg_{gr}}+\mathrm{UR}_\epsilon \not\Rightarrow \mathrm{C}(R)$.
(4) $\mathrm{C}(R) \not\Rightarrow \mathrm{UR^w_{gr}}$ and $\mathrm{C_{gr}}(R) \not\Rightarrow \mathrm{UR^w_{gr}}$.

The examples for (1) and (2) also imply that the conditions UR and $\mathrm{UR_{gr}}$ are independent.

Example 2.11. In (1), (2) and (3) below, K is any field trivially graded by $\mathbb{Z}$.

(1) The graded ring $R = \mathbb{M}_2(K)(0,1)$ is not graded unit-regular by Proposition 2.6. Since $R_0 = \begin{bmatrix} K & 0 \\ 0 & K \end{bmatrix}$, R_0 is unit-regular. Graded regularity is passed to graded matrix algebras (see Remark 2.3) so R is graded regular. The ring $R = \mathbb{M}_2(K)$ is unit-regular and hence R is weakly graded unit-regular.
(2) Let $R = K[x, x^{-1}]$, $\mathbb{Z}$-graded as in section 2.2. Then R is a graded field so it is graded unit-regular, hence weakly graded unit-regular also. Since $R_0 = K$, R_0 is unit-regular. However, R is not unit-regular (consider $1 + x$ for example).
(3) Let R be the Leavitt algebra $L(1, 2)$ i.e. the universal example of a K-algebra R such that $R \oplus R \cong R$. Clearly, R is not cancellable. The algebra R can be represented as a Leavitt path algebra of the graph  and it is naturally graded by $\mathbb{Z}$ (see section 4). Since every Leavitt path algebra is graded regular and graded cancellable (by [8, Theorem 9] and [5, Corollary 5.8]), R is such too and hence $\mathrm{Reg_{gr}}$+UR_ϵ holds by Theorem 2.9.

(4) Let $R=\mathbb{Z}$. Then $\mathcal{V}(R)=\mathbb{Z}^+$ so R is cancellable. Consider R trivially graded by $\mathbb{Z}$. Then $\mathcal{V}^{\mathbb{Z}}(R)=\mathbb{Z}^+[x,x^{-1}]$ ([9, Example 3.1.5] has more details) so R is graded cancellable. The ring R is not regular, so it is not unit-regular and, since it is trivially graded, $\mathrm{UR}^{\mathrm{w}}_{\mathrm{gr}}$ fails.

3. Graded stable range 1 and graded direct finiteness

3.1. **Graded stable range 1.** A regular ring is unit-regular if and only if it has stable range 1. First, we review some related terminology and show the graded version of this statement.

A sequence of elements $a_1,\ldots,a_n$ of a ring R is said to be right unimodular if $a_1R+\ldots+a_nR=R$. If R is Γ-graded, a sequence of elements $a_1,\ldots,a_n$ with $\deg(a_i)=\gamma_i$, $i=1,\ldots,n$, is graded right unimodular if $(\gamma_1^{-1})a_1R+\ldots+(\gamma_n^{-1})a_nR=R$. Note that this last condition is equivalent with $\sum_{i=1}^n a_ix_i=1$ for some $x_1,\ldots,x_n$. However, by replacing x_i with its γ_i^{-1}-component y_i, we obtain *homogeneous* elements $y_1,\ldots,y_n$ such that $\sum_{i=1}^n a_iy_i=1$.

If R is nongraded, recall that a sequence of unimodular elements $a_1,\ldots,a_n$ of R is reducible if there are elements $b_1,\ldots,b_{n-1}$ such that $(a_1+a_nb_1)R+\ldots+(a_{n-1}+a_nb_{n-1})R=R$. As opposed to the conditions with weak and strong versions, there is just one level of graded reducibility since the following two conditions are equivalent for $n\geq 2$ and a graded unimodular sequence $a_1,\ldots,a_n$ of elements of R with $\deg(a_i)=\gamma_i, i=1,\ldots,n$.

(1) There are elements $b_1,\ldots,b_{n-1}$ such that $a_i+a_nb_i\in R_{\gamma_i}$ for $i=1,\ldots,n-1$ and $(\gamma_1^{-1})(a_1+a_nb_1)R+\ldots+(\gamma_{n-1}^{-1})(a_{n-1}+a_nb_{n-1})R=R$.
(2) There are *homogeneous* elements $b_1,\ldots,b_{n-1}$ such that $a_i+a_nb_i\in R_{\gamma_i}$ for $i=1,\ldots,n-1$ and $(\gamma_1^{-1})(a_1+a_nb_1)R+\ldots+(\gamma_{n-1}^{-1})(a_{n-1}+a_nb_{n-1})R=R$.

The first condition implies the second if we replace the elements b_i with their $\gamma_n^{-1}\gamma_i$-components and the converse clearly holds. If any of the above two conditions are satisfied, we say that the sequence $a_1,\ldots,a_n$ is *graded reducible.* The second definition was used in [9, Section 1.8].

Recall that the right stable range (or rank) of R is at most n, written $\mathrm{sr}^r(R)\leq n$, if any right unimodular sequence of more than n elements is reducible. If the smallest such n exists, $\mathrm{sr}^r(R)=n$. If the smallest such n does not exist, $\mathrm{sr}^r(R)=\infty$. The range function $\mathrm{sr}^r_{\mathrm{gr}}$ is defined analogously using graded reducibility instead of reducibility and the left-sided version $\mathrm{sr}^l_{\mathrm{gr}}$ is defined similarly.

In the nongraded case, $\mathrm{sr}^r(R)\leq n$ if and only if every right unimodular sequence of $n+1$ elements is reducible (originally in [18], see also [13, Proposition 1.3]). The proof of [13, Proposition 1.3] generalizes step-by-step to the graded case. So, $\mathrm{sr}^r_{\mathrm{gr}}(R)\leq n$ if and only if every graded right unimodular sequence of $n+1$ elements is graded reducible. One can also show that $\mathrm{sr}^r(R)=n$ iff $\mathrm{sr}^l(R)=n$ (see [18]), so one can denote sr^l and sr^r with sr only. We use the graded version of this result only in the case $n=1$ and include a proof for completeness.

Lemma 3.1. *If R is a Γ-graded ring, then $\mathrm{sr}^r_{\mathrm{gr}}(R)=1$ if and only if $\mathrm{sr}^l_{\mathrm{gr}}(R)=1$.*

Proof. We adapt the proof of [13, Theorem 1.8] to the graded case. Let $\mathrm{sr}^r_{\mathrm{gr}}(R)=1$ and let $b\in R_\gamma$ and $d\in R_\delta$ be such that $Rb(\gamma^{-1})+Rd(\delta^{-1})=R$. Thus, $ab+cd=1$ for some $a\in R_{\gamma^{-1}}$ and $c\in R_{\delta^{-1}}$ and so $(\gamma)aR+cdR=R$. Hence, there is $x\in R_{\gamma^{-1}}$ such that $u=a+cdx\in R_{\gamma^{-1}}$ is right invertible. By [9, Section 1.8], if $\mathrm{sr}^r_{\mathrm{gr}}(R)=1$, then a homogeneous element with a right inverse is invertible. Thus, u is invertible. Let $v\in R_\gamma$ be its inverse. If $w=a+x(1-ba)$ and $y=(1-bx)v$, then $w\in R_{\gamma^{-1}}$ and $y\in R_\gamma$. One checks that $w(1-bx)=(1-xb)u$ and $w(b+ycd)=1$ (for more details see [13, Theorem 1.8]). As $y\in R_\gamma$, $b+ycd$ is in R_γ also. Since $w(b+ycd)=1$, $R(b+ycd)(\gamma^{-1})=R$. ☐

This lemma allows us to shorten $\mathrm{sr}^r_{\mathrm{gr}}(R) = 1$ and $\mathrm{sr}^l_{\mathrm{gr}}(R) = 1$ to $\mathrm{sr}_{\mathrm{gr}}(R) = 1$ and we say that R has *graded stable range 1* in this case. The next proposition, stated without proof in [9, Example 1.8.8], relates this condition with graded unit-regularity.

Proposition 3.2. *If R is a Γ-graded ring then R is graded unit-regular if and only if R is graded regular and $\mathrm{sr}_{\mathrm{gr}}(R) = 1$.*

Proof. Assume that R is graded unit-regular and that $(\gamma^{-1})aR + (\delta^{-1})bR = R$ for some $a \in R_\gamma, b \in R_\delta$. Let $a = aua$ and $b = bvb$ for some $u \in R_{\gamma^{-1}}$ invertible and $v \in R_{\delta^{-1}}$. Then au and bv are homogeneous idempotents such that $(\gamma^{-1})aR = auR$, $(\delta^{-1})bR = bvR$ (see Lemma 1.4) so $auR+bvR = R$. Since $bvR/(auR\cap bvR) \cong_{\mathrm{gr}} R/auR \cong_{\mathrm{gr}} (1-au)R$, $auR\cap bvR$ is a graded summand of bvR. Let e be a homogeneous idempotent such that $bvR = (auR\cap bvR)\oplus eR$. Then $R = auR\oplus eR$. If $L_{u^{-1}}$ is the left multiplication by u^{-1}, then $L_{u^{-1}}$ restricted on uaR is $L_a : uaR \cong_{gr} aR = (\gamma)auR$. On $(1-ua)R$, $L_{u^{-1}}$ is $(1-ua)R \cong_{\mathrm{gr}} (\gamma)(1-au)R$ since $u^{-1}(1-ua)R = (1-au)u^{-1}R = (1-au)R$. So, $u^{-1} = L_{u^{-1}}(1) = L_{u^{-1}}(ua + 1 - ua) = a + u^{-1}(1 - ua) = a + (1 - au)x$ for some $x \in R_\gamma$. Since $(1 - au)x \in eR \subseteq bvR$, $(1 - au)x = bvy$ for some $y \in R_\gamma$. So, $a + bvy = u^{-1}$ is homogeneous and invertible.

Conversely, assume that $\mathrm{sr}_{\mathrm{gr}}(R) = 1$ and that R is graded regular. If a is in R_γ, then $a = aba$ for some $b \in R_{\gamma^{-1}}$ and so ab is a homogeneous idempotent. Since $1 = ab + 1 - ab$ and $abR = (\gamma^{-1})aR$, $R = (\gamma^{-1})aR + (1 - ab)R$. By the assumption that $\mathrm{sr}_{\mathrm{gr}}(R) = 1$, there is a homogeneous element y such that $a + (1 - ab)y$ is homogeneous and invertible. If u denotes its inverse, then $a = aba = ab(a + (1 - ab)y)ua = abaua = aua.$ □

In [9, Corollary 1.8.5], it is shown that if Γ is abelian and R a graded ring with $\mathrm{sr}_{\mathrm{gr}}(R) = 1$, then R is graded cancellable. In the proof, the relation $\mathrm{End}_R((\gamma)R) \cong_{\mathrm{gr}} R$ has been used. Since $\mathrm{End}_R((\gamma)R) = \mathbb{M}_1(R)(\gamma^{-1})$ and $R \cong_{\mathrm{gr}} \mathbb{M}_1(R)(\epsilon)$, this isomorphism follows from part (2) of Lemma 1.1 if γ is in the center of Γ. However, if Γ is nonabelian, then $\mathbb{M}_1(R)(\gamma^{-1}) \cong_{\mathrm{gr}} (\gamma)R(\gamma^{-1})$ may not be graded isomorphic to R. For example, let $\Gamma = D_3 = \langle a, b | a^3 = b^2 = 1, ba = a^2b\rangle$, $\Delta = \{1, b\}$ and let $R = K[\Delta]$ be Γ-graded by $R_\gamma = K\gamma$ if $\gamma \in \Delta$ and $R_\gamma = 0$ otherwise. Then $R_b = Kb$ and $((a)R(a^{-1}))_b = R_{aba^{-1}} = R_{a^2b} = 0$ so R and $(a)R(a^{-1})$ are not graded isomorphic.

The relation $\mathrm{End}_R((\gamma)R) \cong_{\mathrm{gr}} R$ of the proof of [9, Corollary 1.8.5] was used just for the following implication: $\mathrm{sr}_{\mathrm{gr}}(R) = 1 \Rightarrow \mathrm{sr}_{\mathrm{gr}}(\mathrm{End}_R((\gamma)R)) = 1$. We show that this implication holds without requiring that Γ is abelian.

Lemma 3.3. *If R is a Γ-graded ring and $\mathrm{sr}_{\mathrm{gr}}(R) = 1$, then $\mathrm{sr}_{\mathrm{gr}}(\mathrm{End}_R((\gamma)R)) = 1$ for every $\gamma \in \Gamma$.*

Proof. Since $\mathrm{End}_R((\gamma)R) \cong_{\mathrm{gr}} (\gamma)R(\gamma^{-1})$, we show that $\mathrm{sr}_{\mathrm{gr}}((\gamma)R(\gamma^{-1})) = 1$. Let a and b be homogeneous elements of $(\gamma)R(\gamma^{-1})$ (and hence of R as well) such that $ac + bd = 1$ for some homogeneous $c, d \in (\gamma)R(\gamma^{-1})$. So, a, b, c, d are homogeneous elements of R such that $ac + bd = 1$. By the assumption that $\mathrm{sr}_{\mathrm{gr}}(R) = 1$, there is a homogeneous element y such that $a + by$ is homogeneous and invertible. However, this also implies that y and $a + by$ are homogeneous as elements of $(\gamma)R(\gamma^{-1})$ and that $a + by$ is invertible as an element of $(\gamma)R(\gamma^{-1})$. □

As a direct corollary, we obtain that [9, Corollary 1.8.5] holds even if Γ is not abelian.

Corollary 3.4. *If R is a Γ-graded ring with $\mathrm{sr}_{\mathrm{gr}}(R) = 1$, then R is graded cancellable.*

In Corollary 3.6, we improve this statement by showing that the conclusion holds if the assumption $\mathrm{sr}_{\mathrm{gr}}(R) = 1$ is replaced by the weaker condition $\mathrm{sr}(R_\epsilon) = 1$. This shows that the conclusion of [9, Corollary 1.8.5] also holds under this weaker assumption and without assuming that Γ is abelian.

The implication $\mathrm{sr}_{\mathrm{gr}}(R) = 1 \Rightarrow \mathrm{sr}_{\mathrm{gr}}(\mathbb{M}_1(R)(\gamma)) = 1$ for any $\gamma \in \Gamma$ shown in Lemma 3.3 does not hold for graded matrix rings of sizes larger than one. Indeed, if R is $\mathbb{M}_2(K)(0,1)$ for a trivially $\mathbb{Z}$-graded field K, then $\mathrm{sr}_{\mathrm{gr}}(K) = 1$ and R is a graded regular ring which is not graded unit-regular so $\mathrm{sr}_{\mathrm{gr}}(R) > 1$. This property of $\mathrm{sr}_{\mathrm{gr}}$ differs from the well-known property of sr that $\mathrm{sr}(R) = 1 \Rightarrow \mathrm{sr}(\mathbb{M}_n(R)) = 1$. Thus,

$$\mathrm{sr}(R) = 1 \Rightarrow \mathrm{sr}(\mathbb{M}_n(R)) = 1 \quad \text{and} \quad \mathrm{sr}_{\mathrm{gr}}(R) = 1 \not\Rightarrow \mathrm{sr}_{\mathrm{gr}}(\mathbb{M}_n(R)(\gamma_1, \ldots, \gamma_n)) = 1.$$

3.2. **Substitution.** A module has substitution if and only if its endomorphism ring has stable range 1. We show the graded version of this statement in Theorem 3.5. This theorem enables us to weaken the conditions of Corollary 3.4 and the Graded Cancellation Theorem ([9, Theorem 1.8.4]).

Theorem 3.5. *Let R be a Γ-graded ring and A a graded R-module. Then* $\mathrm{sr}(\mathrm{END}_R(A)_\epsilon) = 1$ *if and only if A has graded substitution.*

Proof. We adapt the proof of the nongraded case (see, for example, [13, Theorem 4.4]). Assume that $\mathrm{sr}(\mathrm{END}_R(A)_\epsilon) = 1$ first, and let $A \oplus B = A' \oplus B' = M$ for some graded modules M, A', B, B' such that $A \cong_{\mathrm{gr}} A'$. Let ϕ and ψ denote the graded isomorphism $A \to A'$ and its inverse, π denote the natural graded projection $A \oplus B$ onto A and ι denote the natural graded injection $A \to A \oplus B$. Let (f, g) denote the projection π with respect to the decomposition $A' \oplus B'$ so that $\pi(a', b') = f(a') + g(b')$, and let $\begin{pmatrix} f' \\ g' \end{pmatrix}$ denote the injection ι with respect to the decomposition $A' \oplus B'$ so that $\iota(a) = (f'(a), g'(a))$. The relation $\pi\iota = 1_A$ implies that $f\phi\psi f' + gg' = ff' + gg' = 1_A$. By the assumption $\mathrm{sr}(\mathrm{END}_R(A)_\epsilon) = 1$, there are $h, u \in \mathrm{END}_R(A)_\epsilon$ such that u is invertible and $f\phi + gg'h = u$. Let $C = \{(\phi(a), g'h(a)) \in A' \oplus B' \mid a \in A\} = \mathrm{Im} \begin{pmatrix} \phi \\ g'h \end{pmatrix}$. Then C is a graded submodule of $A' \oplus B'$ such that $(a', b') = (\phi\psi(a'), g'h\psi(a')) + (0, b' - g'h\psi(a')) \in C \oplus B'$ for every $(a', b') \in A' \oplus B'$. On the other hand, $C \oplus B = A' \oplus B'$ also since $B = \ker(f, g)$ and $C = \{(a', b') | b' = g'h\psi(a')\}$ so that $(a', b') \in C$ implies that $0 = f(a') + g(b') = f(a') + gg'h\psi(a') = u\psi(a')$ iff $a' = 0$.

Conversely, if the relation $ff' + gg' = 1_A$ holds in $\mathrm{END}_R(A)_\epsilon$, then $\pi = (f, g) : A \oplus A \to A$ and $\iota = \begin{pmatrix} f' \\ g' \end{pmatrix} : A \to A \oplus A$ are graded homomorphisms such that $\pi\iota = 1_A$ so that $A \oplus A$ splits as $\ker \pi \oplus$ Im π. Since Im $\pi = A$ and A has graded substitution, there is a graded module C such that $A \oplus C = \ker \pi \oplus C$. Let ϕ be any graded isomorphism of A and C. View ϕ as a map $A \to C \subseteq C \oplus A$ and represent it by $\begin{pmatrix} f_1 \\ g_1 \end{pmatrix}$ for some graded maps $f_1 : A \to C, g_1 : A \to \ker \pi$. Since C is a complement of A, f_1 is invertible. Since C is a complement of $\ker \pi$, $\pi\phi$ is invertible. By construction, $\pi\phi = ff_1 + gg_1$ and so $\pi\phi f_1^{-1} = f + gg_1 f_1^{-1}$. Hence, if $h = g_1 f_1^{-1}$, then $f + gh$ is an invertible element of $\mathrm{END}_R(A)_\epsilon$. □

The Graded Cancellation Theorem ([9, Theorem 1.8.4]) states that $\mathrm{sr}_{\mathrm{gr}}(\mathrm{End}_R(A)) = 1$ implies $\mathrm{C}_{\mathrm{gr}}(A)$ if Γ is abelian and A finitely generated. Since graded substitution clearly implies graded cancellability, Theorem 3.5 shows that it is not necessary to require that Γ is abelian and if A is not finitely generated, $\mathrm{END}_R(A)$ can be considered instead of $\mathrm{End}_R(A)$. In addition, Theorem 3.5 shows that the conclusion of [9, Theorem 1.8.4] holds if the assumption $\mathrm{sr}_{\mathrm{gr}}(\mathrm{END}_R(A)) = 1$ is replaced by the weaker condition $\mathrm{sr}(\mathrm{END}_R(A)_\epsilon) = 1$.

Taking R for A in Theorem 3.5, we have that $\mathrm{sr}(R_\epsilon) = 1$ if and only if R has graded substitution. Thus, Theorem 3.5 has the following corollary.

Corollary 3.6. *If R is a Γ-graded ring with* $\operatorname{sr}(R_\epsilon)=1$, *then R is graded cancellable.*

Corollary 3.6 implies Corollary 3.4. Corollary 3.6 also shows that [9, Corollary 1.8.5] holds if the assumption $\operatorname{sr}_{\text{gr}}(R)=1$ is replaced by the weaker condition $\operatorname{sr}(R_\epsilon)=1$.

3.3. **Graded directly finite rings.** With the definitions of DF_{gr}, $\mathrm{DF}_{\text{gr}}(\_)$ and $\mathrm{DF}^{\text{s}}_{\text{gr}}(\_)$ as in the introduction, one can show that $\mathrm{END}_R(A)$ has DF_{gr} iff $\mathrm{DF}^{\text{s}}_{\text{gr}}(A)$ holds and that $\mathrm{END}_R(A)_\epsilon$ has DF iff $\mathrm{DF}_{\text{gr}}(A)$ holds. The first equivalence is shown in [10, Proposition 3.2] We sketch the proof of the second equivalence. If $\mathrm{DF}_{\text{gr}}(A)$ holds and $x,y\in\mathrm{END}_R(A)_\epsilon$ are such that $xy=1_A$, then $yA=yxA$ and y is a graded isomorphism of $A=xyA$ and $yxyA=yA$. Thus, $A=yA\oplus(1_A-yx)A$ implies that $(1_A-yx)A=0$ by $\mathrm{DF}_{\text{gr}}(A)$, and so $yx=1_A$. Conversely, if $\mathrm{END}_R(A)_\epsilon$ has DF, $A=B\oplus C$, and $y:A\to C$ is a graded isomorphism, then y^{-1} can be extended to an element x of $\mathrm{END}_R(A)_\epsilon$ by mapping B identically to zero. Since $xy=1_A$, yx is equal to 1_A by the assumption and so $b=yx(b)=y(0)=0$ for all $b\in B$.

The implications $\mathrm{IC}^{\text{s}}_{\text{gr}}(\_)\Rightarrow\mathrm{DF}^{\text{s}}_{\text{gr}}(\_)$ and $\mathrm{IC}_{\text{gr}}(\_)\Rightarrow\mathrm{DF}_{\text{gr}}(\_)$ can be shown analogously to $\mathrm{IC}(\_)\Rightarrow\mathrm{DF}(\_)$. However, $\mathrm{IC}^{\text{w}}_{\text{gr}}(\_)$ is sufficient to imply $\mathrm{DF}^{\text{s}}_{\text{gr}}(\_)$. Indeed, if $\mathrm{IC}^{\text{w}}_{\text{gr}}(A)$ holds for a graded module A and if $A\oplus B\cong_{\text{gr}}(\gamma)A$, then $(\gamma)(B)\cong 0$. So, $(\gamma)B=0$ and hence $B=0$.

We also note that while $\mathrm{IC}^{\text{s}}_{\text{gr}}(\_)\Rightarrow\mathrm{DF}^{\text{s}}_{\text{gr}}(\_)$ and $\mathrm{IC}_{\text{gr}}(\_)\Rightarrow\mathrm{DF}_{\text{gr}}(\_)$, we have that $\mathrm{IC}_{\text{gr}}\not\Rightarrow\mathrm{DF}_{\text{gr}}$. For example, consider the algebra R from part (3) of Example 2.11. By this example, $\mathrm{Reg}_{\text{gr}}+\mathrm{UR}_\epsilon$ holds and so IC_{gr} holds by Theorem 2.9. However, R is graded isomorphic to a Leavitt path algebra of a graph which has a cycle with an exit and so DF_{gr} fails by [10, Theorem 3.7].

Proposition 3.7. *If R is a Γ-graded ring with* $\operatorname{sr}_{\text{gr}}(R)=1$, *then R is graded directly finite.*

Proof. The proof is the graded version of the proof of [13, Lemma 1.7]. Let x,y be homogeneous elements such that $xy=1$ and let $e=1-yx$. If $\deg(y)=\gamma$, $(\gamma^{-1})yR=yxR$ and so $R=(\gamma^{-1})yR+eR$. By $\operatorname{sr}_{\text{gr}}(R)=1$, there is $z\in R$ such that $y+ez$ is homogeneous and invertible. Since $xe=0$, $x(y+ez)=xy=1$ which implies that $x=(y+ez)^{-1}$ is invertible. So, the condition $xy=1$ implies that y is the inverse of x and that $yx=1$. ☐

3.4. **Summary of relations.** The properties we considered can be related as follows.

$$\mathrm{S}_{\text{gr}}(\_)\quad\Longrightarrow\quad\mathrm{C}_{\text{gr}}(\_)\quad\Longrightarrow\quad\mathrm{IC}_{\text{gr}}(\_)\Longrightarrow\quad\mathrm{DF}_{\text{gr}}(\_)$$

Note that these relations match the relations of the nongraded analogues in the diagram in [13, Formula (4.2)]. Considering rings from [13, Examples 3.2(3) and 4.7] and [7, Example 5.10] and grading them trivially by any group shows that the implications are strict.

The implications below, which hold by Proposition 3.2 and Corollary 3.6, are also strict.

$$\mathrm{UR}_{\text{gr}}\quad\Longrightarrow\quad\operatorname{sr}_{\text{gr}}(R)=1\quad\Longrightarrow\quad\operatorname{sr}(R_\epsilon)=1\quad\Longrightarrow\quad\mathrm{C}_{\text{gr}}(R)$$

To see that the first implication is strict, consider any ring which has stable range 1 and which is not unit-regular (e.g. the ring $K[[x]]$ of power series of one variable over any field K, see [13, Examples 1.6]) and grade it trivially by any group. To see that the third implication is strict, consider the ring $R=\mathbb{Z}$ trivially graded by $\mathbb{Z}$. Then $\operatorname{sr}(R_0)=\operatorname{sr}(\mathbb{Z})=2>1$ and $\mathrm{C}_{\text{gr}}(R)$ holds by part (4) of Example 2.11. To see that the second implication is strict, consider any graded regular ring R which is not graded unit-regular and such that R_ϵ is unit-regular (e.g. the ring from part (1) of Example 2.11). This example also shows that the middle implication in the diagram below, which holds by Proposition 3.2 and Theorem 2.9, is strict.

$$\mathrm{Reg}_{\text{gr}}\quad\Longrightarrow\quad(\ \mathrm{UR}_{\text{gr}}\quad\Longleftrightarrow\quad\operatorname{sr}_{\text{gr}}(R)=1\quad\Longrightarrow\quad\operatorname{sr}(R_\epsilon)=1\quad\Longleftrightarrow\quad\mathrm{C}_{\text{gr}}(R)\ )$$

3.5. **Cancellation properties of strongly graded rings.** If R is a strongly graded ring (i.e. $R_\gamma R_\delta = R_{\gamma\delta}$ for all $\gamma, \delta \in \Gamma$), the category of graded right R-modules and the category of right R_ϵ-modules are equivalent under the equivalence

$$A \mapsto A_\epsilon \text{ with the inverse } B \mapsto B \otimes_{R_\epsilon} R.$$

We also have that $A \cong_{\text{gr}} A_\epsilon \otimes_{R_\epsilon} R$ and that $B \cong (B \otimes_{R_\epsilon} R)_\epsilon$ (see [9, Theorem 1.5.1]). This implies the following proposition.

Proposition 3.8. *Let R be a strongly Γ-graded ring and A a graded R-module. The following statements hold.*

(1) A is graded internally cancellable if and only if A_ϵ is internally cancellable.
(2) A is graded cancellable if and only if A_ϵ is cancellable.
(3) A has graded substitution if and only if A_ϵ has substitution.
(4) $DF_{\text{gr}}(A)$ holds if and only if A_ϵ is directly finite.

Proof. All four statements are shown similarly, using the equivalence of categories. We provide more details for the first condition and note that the proofs of (2), (3), and (4) are similar.

Assuming that $\text{IC}_{\text{gr}}(A)$ holds, $\text{IC}_{\text{gr}}(A_\epsilon \otimes_{R_\epsilon} R)$ also holds since A and $A_\epsilon \otimes_{R_\epsilon} R$ are graded isomorphic. If $A_\epsilon = B_\epsilon \oplus C_\epsilon = D_\epsilon \oplus E_\epsilon$ for some right R_ϵ-modules $B_\epsilon, C_\epsilon, D_\epsilon$, and E_ϵ with $f : B_\epsilon \cong D_\epsilon$, then $A_\epsilon \otimes_{R_\epsilon} R = B_\epsilon \otimes_{R_\epsilon} R \oplus C_\epsilon \otimes_{R_\epsilon} R = D_\epsilon \otimes_{R_\epsilon} R \oplus E_\epsilon \otimes_{R_\epsilon} R$ and f can be extended to the graded isomorphism $B_\epsilon \otimes_{R_\epsilon} R \to D_\epsilon \otimes_{R_\epsilon} R$ by $a \otimes r \mapsto f(a) \otimes r$. By $\text{IC}_{\text{gr}}(A)$, $C_\epsilon \otimes_{R_\epsilon} R \cong_{\text{gr}} E_\epsilon \otimes_{R_\epsilon} R$. Considering the ϵ-components, we obtain that $C_\epsilon \cong (C_\epsilon \otimes_{R_\epsilon} R)_\epsilon \cong (E_\epsilon \otimes_{R_\epsilon} R)_\epsilon \cong E_\epsilon$.

Assume that $\text{IC}(A_\epsilon)$ holds and that $A = B \oplus C = D \oplus E$ for some graded R-modules B, C, D, E with $B \cong_{\text{gr}} D$. Then $A_\epsilon = B_\epsilon \oplus C_\epsilon = D_\epsilon \oplus E_\epsilon$ and $B_\epsilon \cong D_\epsilon$. By $\text{IC}(A_\epsilon)$, $f : C_\epsilon \cong E_\epsilon$ for some f. Such f induces $\overline{f} : C_\epsilon \otimes_{R_\epsilon} R \cong_{\text{gr}} E_\epsilon \otimes_{R_\epsilon} R$ so that $C \cong_{\text{gr}} C_\epsilon \otimes_{R_\epsilon} R \cong_{\text{gr}} E_\epsilon \otimes_{R_\epsilon} R \cong_{\text{gr}} E$. □

The implications $\text{UR}_{\text{gr}} \Rightarrow \text{UR}_\epsilon$, $\text{sr}_{\text{gr}}(R) = 1 \Rightarrow \text{sr}(R_\epsilon) = 1$ and $\text{DF}_{\text{gr}} \Rightarrow$ (DF holds on R_ϵ) are strict even for strongly graded rings. Indeed, if R is the graded ring from part (3) of Example 2.11, then R is strongly graded by [9, Theorem 1.6.13]. By Example 2.11, R is graded regular and R_ϵ is unit-regular. Thus, $\text{sr}(R_\epsilon) = 1$ and R_ϵ is directly finite. However, DF_{gr} fails for R as we noted in section 3.3 (by [10, Theorem 3.7]). Hence, $\text{sr}_{\text{gr}}(R) > 1$ by Proposition 3.7 and so R is not graded unit-regular by Proposition 3.2.

4. Characterization of graded unit-regular Leavitt path algebras of finite graphs

We briefly review some relevant definitions. Let E be an oriented graph. The graph E is row-finite if every vertex emits finitely many edges and it is finite if it has finitely many vertices and edges. A sink of E is a vertex which does not emit edges. A vertex of E is regular if it is not a sink and if it emits finitely many edges. A cycle is a closed path such that different edges in the path have different sources. A cycle has an exit if a vertex on the cycle emits an edge outside of the cycle. The graph E is acyclic if there are no cycles. We say that graph E is no-exit if v emits just one edge for every vertex v of every cycle.

Let E^0 denote the set of vertices, E^1 the set of edges and $\mathbf{s}$ and $\mathbf{r}$ denote the source and range maps of a graph E. If K is any field, the *Leavitt path algebra* $L_K(E)$ of E over K is a free K-algebra generated by the set $E^0 \cup E^1 \cup \{e^* \mid e \in E^1\}$ such that for all vertices v, w and edges e, f,

(V) $vw = 0$ if $v \neq w$ and $vv = v$, (E1) $\mathbf{s}(e)e = e\mathbf{r}(e) = e$,
(E2) $\mathbf{r}(e)e^* = e^*\mathbf{s}(e) = e^*$, (CK1) $e^*f = 0$ if $e \neq f$ and $e^*e = \mathbf{r}(e)$,
(CK2) $v = \sum_{e\in \mathbf{s}^{-1}(v)} ee^*$ for each regular vertex v.

By the first four axioms, every element of $L_K(E)$ can be represented as a sum of the form $\sum_{i=1}^n a_ip_iq_i^*$ for some n, paths p_i and q_i, and elements $a_i \in K$, for $i = 1,\ldots,n$. Using this representation, it is direct to see that $L_K(E)$ is a unital ring if and only if E^0 is finite in which case the sum of all vertices is the identity. For more details on these basic properties, see [1].

A Leavitt path algebra is naturally graded by the group of integers $\mathbb{Z}$ so that the n-component $L_K(E)_n$ is the K-linear span of the elements pq^* for paths p, q with $|p| - |q| = n$ where $|p|$ denotes the length of a path p. While one can grade a Leavitt path algebra by any group Γ (see [9, Section 1.6.1]), we always consider the natural grading by $\mathbb{Z}$.

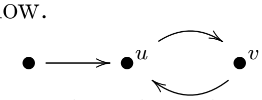
4.1. **Finite no-exit graphs.** If E is a finite no-exit graph, then $L_K(E)$ is graded isomorphic to

$$R = \bigoplus_{i=1}^{k} \mathbb{M}_{k_i}(K)(\gamma_{i1}\ldots,\gamma_{ik_i}) \oplus \bigoplus_{j=1}^{n} \mathbb{M}_{n_j}(K[x^{m_j}, x^{-m_j}])(\delta_{j1},\ldots,\delta_{jn_j})$$

where k is the number of sinks, k_i is the number of paths ending in the sink indexed by i for $i = 1,\ldots,k$, and γ_{il} is the length of the l-th path ending in the i-th sink for $l = 1,\ldots,k_i$ and $i = 1,\ldots,k$. In the second term, n is the number of cycles, m_j is the length of the j-th cycle for $j = 1,\ldots,n$, n_j is the number of paths which do not contain cycle indexed by j and which end in a fixed but arbitrarily chosen vertex of the cycle, and δ_{jl} is the length of the l-th path ending in the fixed vertex of the j-th cycle for $l = 1,\ldots,n_j$ and $j = 1,\ldots,n$.

Note that this representation is not necessarily unique as Example 4.1 shows, but it is unique up to a graded isomorphism. We refer to the graded algebra R above as a *graded matricial representation* of $L_K(E)$.

Example 4.1. Let E be the graph below.



If we form a graded matricial representation based on the number and lengths of paths which end at u, we obtain $\mathbb{M}_3(K[x^2, x^{-2}])(0,1,1)$. Using v, we obtain $\mathbb{M}_3(K[x^2, x^{-2}])(0,1,2)$. These two algebras are graded isomorphic by Lemma 1.1 since $(0,1,1) \to (0+1,1+1,1+1) \to (1,2,2-2) = (1,2,0) \to (0,1,2)$ where $\to$ denotes an application of an operation from Lemma 1.1 and results in a graded isomorphism of corresponding matrix algebras.

4.2. **Characterization of graded unit-regular Leavitt path algebras of finite graphs.** We use graded matricial representations and Proposition 2.6 to prove the main result of this section now.

Theorem 4.2. *If K is a field and E is a finite graph, the following conditions are equivalent.*

(1) $L_K(E)$ is graded unit-regular.
(2) E is a no-exit graph without sinks which receive edges such that the following condition holds.
(*) *For every cycle of length m, the lengths, considered modulo m, of all paths which do not contain the cycle and which end in an arbitrary vertex of the cycle, are*

$$0,0,\ldots,0,\ 1,1,\ldots,1,\ \ldots\ldots\ldots,\ m-1, m-1\ldots m-1$$

where each i is repeated the same number of times in the above list for $i = 0,\ldots,m-1$.

Proof. If (1) holds, then $L_K(E)$ is graded directly finite by Propositions 3.2 and 3.7. So, E is a no-exit graph by [10, Theorem 3.7]. Let R be a graded matricial representation

$$R = \bigoplus_{i=1}^{k} \mathbb{M}_{k_i}(K)(\gamma_{i1} \dots, \gamma_{ik_i}) \oplus \bigoplus_{j=1}^{n} \mathbb{M}_{n_j}(K[x^{m_j}, x^{-m_j}])(\delta_{j1}, \dots, \delta_{jn_j}).$$

Since R is graded unit-regular, each graded direct summand of R is graded unit-regular. If $k_i > 1$, then not all $\gamma_{i1} \dots, \gamma_{ik_i}$ are equal since one of them is zero (corresponding to the trivial path of length zero to the i-th sink) and the others are positive (corresponding to the lengths of nontrivial paths to the i-th sink). So, Proposition 2.6 and Lemma 1.1 imply that $k_i = 1$ for all $i = 1, \dots, k$ which means that the trivial path is the only one ending in the i-th sink for all $i = 1, \dots, k$.

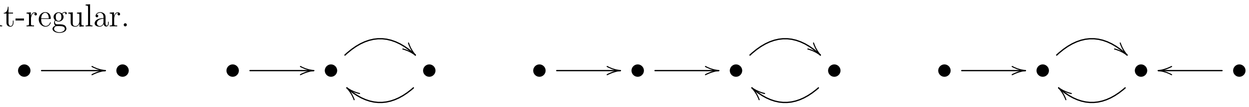
For every $j = 1, \dots, n$, each $l = 0, \dots m_j - 1$ appears on the list $\delta_{j1}, \dots, \delta_{jn_j}$ because there is a path of length l which is a subpath of the j-th cycle and which ends at the selected vertex v_j of the j-th cycle. Thus, Proposition 2.6 and Lemma 1.1 imply that n_j is a multiple of m_j and that the integers $\delta_{j1}, \dots, \delta_{jn_j}$, considered modulo m_j and permuted if necessary, produce a list as in condition (*). Thus, the lengths, considered modulo m_j, of paths which do not contain the j-th cycle and which end at v_j are as listed in condition (*).

Conversely, assume that E is such that (2) holds. Since E is no-exit, let R be a graded matricial representation of $L_K(E)$. and let R have the form as above. By the assumption that no sink receives an edge, $k_i = 1$ for every $i = 1, \dots, k$. By the assumption that (*) holds, we can apply Lemma 1.1 to permute the shifts and to replace each δ_{jl} by the remainder of the division by m_j for $l = 1, \dots, n_j$ and $j = 1, \dots, n$. This produces a graded isomorphism of R and the algebra

$$K^k \oplus \bigoplus_{j=1}^{n} \mathbb{M}_{k_j m_j}(K[x^{m_j}, x^{-m_j}])(0, 0, \dots, 0,\ 1, 1, \dots, 1,\ \dots\dots\dots,\ m_j - 1, m_j - 1, \dots, m_j - 1)$$

where each $i = 0, \dots, m_j - 1$ appears k_j times in the list of shifts above for all $j = 1, \dots, n$. By Proposition 2.6, every direct summand of this last algebra is graded unit-regular so R is graded unit-regular also. Hence, $L_K(E)$ is graded unit-regular. ☐

Theorem 4.2 enables one to readily conclude that the Leavitt path algebras of the first two graphs below are not graded unit-regular while the Leavitt path algebras of the last two graphs are graded unit-regular.



Indeed, the first graph has a sink which receives an edge so its Leavitt path algebra is not graded unit-regular. For the second graph, 0, 1, and 1 are the lengths (modulo 2) of paths which end at any vertex of the cycle and which do not contain the cycle. So, since the numbers of zeros and ones on this list are not equal, the Leavitt path algebra is not graded unit-regular. For the last two graphs, 0, 0, 1, and 1 are the lengths (modulo 2) of paths which end at any vertex of the cycle and which do not contain the cycle. So the Leavitt path algebras of the last two graphs are graded unit-regular.

4.3. **Characterizations of other cancellation properties of Leavitt path algebras.**

Proposition 4.3. *Let K be a field and E be a graph such that E^0 is finite. For part (1), (2) and (3), we also assume that E^1 is finite.*

(1) The following conditions are equivalent.

(a) $\mathrm{sr}_{\mathrm{gr}}(L_K(E)) = 1$.
(b) *IC_{gr} holds for $L_K(E)$.*
(c) Condition (2) of Theorem 4.2 holds.

(2) $\mathrm{sr}(L_K(E)) = 1$ *if and only if E is acyclic.*
(3) $L_K(E)$ is graded weakly unit-regular if and only if E is no-exit.
(4) IC holds for $L_K(E)$ if and only if E is no-exit.
(5) $L_K(E)$ has graded substitution and Reg_{gr}+UR_ϵ holds.

Proof. Note that the assumption that E^0 is finite ensures that $L_K(E)$ is unital. The assumption that E^1 is also finite in part (1) enables us to use Theorem 4.2 and in parts (2) and (3) ensures that a graded matricial representation of a no-exit graph has the form as in section 4.1.

(1) By [8, Theorem 9], $L_K(E)$ is graded regular. So, Propositions 3.2 and 2.4 and Theorem 4.2 imply that the conditions (a), (b) and (c) are each equivalent with the condition that $L_K(E)$ is graded unit-regular.

(2) If $\mathrm{sr}(L_K(E)) = 1$, then $L_K(E)$ is directly finite by Proposition 3.7 in the nongraded case. Thus, E is no-exit by [15, Theorem 4.12] and so $L_K(E)$ is isomorphic to a direct sum of matricial algebras over K and $K[x, x^{-1}]$ (which is a matricial representation if we ignore the grading). Since $\mathrm{sr}(K[x, x^{-1}]) > 1$ and $\mathrm{sr}(L_K(E)) = 1$, there cannot be matrix algebras over $K[x, x^{-1}]$ present. Thus, E is acyclic. Conversely, if E is acyclic, then $L_K(E)$ is unit-regular by [3, Theorem 2] and so $\mathrm{sr}(L_K(E)) = 1$ by Proposition 3.2 in the nongraded case.

(3) If $L_K(E)$ is graded weakly unit-regular, $\mathrm{IC}^{\mathrm{w}}_{\mathrm{gr}}(L_K(E))$ holds by Proposition 2.4. The condition $\mathrm{IC}^{\mathrm{w}}_{\mathrm{gr}}(L_K(E))$ implies $\mathrm{DF}^{\mathrm{s}}_{\mathrm{gr}}(L_K(E))$ as we showed in section 3.3. By the assumption that E^0 is finite, $L_K(E)$ is unital and so $L_K(E) \cong_{\mathrm{gr}} \mathrm{End}_{L_K(E)}(L_K(E))$. Thus, the condition $\mathrm{DF}^{\mathrm{s}}(L_K(E))$ implies that $L_K(E)$ is graded directly finite. By [10, Theorem 3.7], E is a no-exit graph. Conversely, if E is a no-exit graph, then a graded matricial representation of $L_K(E)$ is graded semisimple, and hence weakly graded unit-regular. Here we make use of the fact that the implication "semisimple $\Rightarrow$ UR" directly implies "graded semisimple $\Rightarrow$ $\mathrm{UR}^{\mathrm{w}}_{\mathrm{gr}}$". Thus, $L_K(E)$ is weakly graded unit-regular.

(4) If $\mathrm{IC}(P)$ holds for every finitely generated projective $L_K(E)$-module P, then $\mathrm{IC}(L_K(E))$ holds. Since $\mathrm{IC}(\_) \Rightarrow \mathrm{DF}(\_)$, $\mathrm{DF}(L_K(E))$ holds. Using the same argument as in the proof of (3), the assumption that E^0 is finite ensures that the condition $\mathrm{DF}(L_K(E))$ implies that $L_K(E)$ is directly finite. By [15, Theorem 4.12], E is a no-exit graph. Conversely, if E is no-exit, then $L_K(E)$ is cancellable by [5, Lemma 5.5]. So, $\mathrm{C}(P)$ holds for every finitely generated projective $L_K(E)$-module P which implies that $\mathrm{IC}(P)$ holds for every such module P.

(5) Since E^0 is finite, the algebra $L_K(E)_0$ is a matricial algebra over K (see [9, Section 3.9.3]). So, $L_K(E)_0$ is unit-regular and $\mathrm{sr}(L_K(E)_0) = 1$. Hence, UR_ϵ holds and $\mathrm{S}_{\mathrm{gr}}(L_K(E))$ holds by Theorem 3.5. $\mathrm{Reg}_{\mathrm{gr}}$ holds by [8, Theorem 9]. □

4.4. **Possible generalizations.** A local version of a ring-theoretic property P is typically obtained by requiring that for every finite set F, there is an idempotent e such that $F \subseteq eRe$ and eRe has property P. If R is non-unital, this definition enables one to consider local versions of properties whose definitions require the existence of the ring identity.

The properties of being unit-regular and directly finite can be generalized to non-unital rings in this way. This approach has been used in [3] for unit-regularity and in [15] for direct finiteness. While the condition $(\forall a, b)(aR + bR = R \Rightarrow (\exists x)(a + bx)R = R)$ does not specifically include the identity, it is just a shorter version of the condition $(\forall a, b)((\exists c, d) ac+bd = 1 \Rightarrow (\exists x, u)(a+bx)u = 1)$ where the identity does appear. So, $\mathrm{sr}(R) = 1$ should also be treated as a property of unital rings.

In the graded case, properties of unital graded rings can be generalized to non-unital case in the same way. In particular, a graded, possibly non-unital, ring R is *graded locally unit-regular* if for every finite set F, there is a homogeneous idempotent u such that $F \subseteq uRu$ and uRu is graded unit-regular. A graded ring having graded locally stable range 1, a graded locally directly finite ring, and a graded locally weakly unit-regular ring can be defined analogously.

Using these definitions, it is possible to consider graded local cancellability properties of Leavitt path algebras over graphs without any restrictions on the cardinality of vertices and edges. Given this fact, we wonder whether the requirements that E is finite can be dropped from the results of sections 4.2 and 4.3. In particular, we wonder about the following.

Question 4.4. What graph-theoretic condition is equivalent to the condition that the Leavitt path algebra of an arbitrary graph is graded locally unit-regular?

The answer to the above question would provide characterization of graded locally stable range 1 also because every Leavitt path algebra is graded regular. Graded regularity passes to graded corners so the local version of Proposition 3.2 holds.

4.5. **More on Question 0.2.** As mentioned at the end of the introduction, considering the graded version of Handelman's Conjecture provides further evidence that $\mathrm{Reg}_{\mathrm{gr}}+\mathrm{UR}_\epsilon$ is more suited as a graded analogue of unit-regularity than the current definition of graded unit-regularity. Recall that Handelman's Conjecture states that a ring with involution which is $*$-regular (see [6] or [4] for definition and basic properties) is necessarily directly finite and unit-regular. While the part on direct finiteness has been shown to hold, the part on unit-regularity is still open. In [4], the authors note that this conjecture holds for all Leavitt path algebras. In [11], the authors consider the graded version of $*$-regularity and note that every Leavitt path algebra over a field K with a positive definite involution (for any n and any $k_1, \ldots, k_n \in K$, $\sum_{i=1}^n k_i k_i^* = 0$ implies $k_i = 0$ for each $i = 1, \ldots, n$) is graded $*$-regular. The authors of [11] note that if E is the graph from part (3) of Example 2.11, then $L_K(E)$ is not graded unit-regular so the graded version of Handelman's Conjecture fails. However, as we have seen in this paper, graded unit-regularity is quite a restrictive condition. So, we stipulate that Question 0.2 from the introduction is more relevant as a graded version of Handelman's conjecture. For the class of unital Leavitt path algebras, the answer to this question is "yes" since every unital Leavitt path algebra satisfies $\mathrm{Reg}_{\mathrm{gr}}+\mathrm{UR}_\epsilon$ by Proposition 4.3.

5. LPA-Realization of graded matrix algebras

In the nongraded case, every matrix algebra over a field K or the ring $K[x, x^{-1}]$ is isomorphic to a Leavitt path algebra. Indeed, for any positive integer n, let L_n be the "line of length $n-1$", i.e. the graph with n vertices $v_1, v_2, \ldots, v_n$ and an edge from v_i to v_{i+1} for all $i = 1, \ldots, n-1$. Then $L_K(L_n) \cong M_n(K)$. Adding an edge from v_n to v_n to L_n produces a graph C_n such that $L_K(C_n) \cong M_n(K[x, x^{-1}])$. In contrast, not every graded matrix algebra over K is graded isomorphic to a Leavitt path algebra by [16, Proposition 3.7]. The LPA-Realization Question of [16, Section 3.3] is asking for characterization of those graded matrix algebras over K which can be realized as Leavitt path algebras. In this section, we answer this question. We also characterize when a graded matrix algebra over naturally $\mathbb{Z}$-graded $K[x^m, x^{-m}]$ for a positive integer m is graded isomorphic to a Leavitt path algebra. As a consequence, we present conditions under which a finite direct sum of graded matricial algebras over K and $K[x^m, x^{-m}]$ can be realized by a Leavitt path algebra.

To shorten the notation, if each $\gamma_i \in \mathbb{Z}, i = 1, \ldots, k$, appears d_i times in the list

$$\gamma_1, \gamma_1, \ldots, \gamma_1,\ \gamma_2, \gamma_2 \ldots, \gamma_2,\ \ldots\ldots\ldots,\ \gamma_k, \gamma_k, \ldots, \gamma_k,$$

we abbreviate this list as

$$d_1(\gamma_1), d_2(\gamma_2), \ldots, d_k(\gamma_k).$$

Also, if $S = K$ or $S = K[x^m, x^{-m}]$, we use the following abbreviation

$$\mathbb{M}_n(S)(\gamma_1, \gamma_1, \ldots, \gamma_1,\ \gamma_2, \gamma_2 \ldots, \gamma_2,\ \ldots\ldots\ldots,\ \gamma_k, \gamma_k, \ldots, \gamma_k) = \mathbb{M}_n(S)(d_1(\gamma_1), d_2(\gamma_2), \ldots, d_k(\gamma_k))$$

For example, the algebra $\mathbb{M}_9(K[x^3, x^{-3}])(0, 0, 0, 0, 1, 1, 1, 2, 2)$ from Example 2.7 is denoted shortly by $\mathbb{M}_9(K[x^3, x^{-3}])(4(0), 3(1), 2(2))$.

Lemma 5.1. *Let n and m be positive integers and $\gamma_1, \gamma_2, \ldots, \gamma_n$ be arbitrary integers.*

1. *If the smallest element is subtracted from the list $\gamma_1, \gamma_2, \ldots, \gamma_n$, the elements are permuted so that they are listed in a nondecreasing order, and if k is the largest element of the new list, the new list is $l_0(0), l_1(1), \ldots, l_k(k)$ for some nonnegative integers $l_1, \ldots, l_{k-1}$ and some positive l_0 and l_k such that $n = \sum_{i=0}^{k} l_i$. The integers k and $l_0, l_1, \ldots, l_k$ are unique for the graded isomorphism class of $\mathbb{M}_n(K)(\gamma_1, \gamma_2, \ldots, \gamma_n)$.*
2. *If the elements $\gamma_1, \gamma_2, \ldots, \gamma_n$ are considered modulo m and arranged in a nondecreasing order, the resulting list is $l_0(0), l_1(1), \ldots, l_{m-1}(m-1)$ for some nonnegative integers $l_0, l_1, \ldots, l_{m-1}$ such that $n = \sum_{i=0}^{m-1} l_i$. The integers $l_0, l_1, \ldots, l_{m-1}$ are unique for the graded isomorphism class of $\mathbb{M}_n(K[x^m, x^{-m}])(\gamma_1, \gamma_2, \ldots, \gamma_n)$ up to their order.*

Proof. (1) If k and $l_0, l_1, \ldots, l_k$ are obtained as in the statement of part (1), $\mathbb{M}_n(K)(\gamma_1, \gamma_2, \ldots, \gamma_n) \cong_{\text{gr}} \mathbb{M}_n(K)(l_0(0), l_1(1), \ldots, l_k(k))$ by Lemma 1.1. To show uniqueness, assume that

$$\mathbb{M}_n(K)(l_0(0), l_1(1), \ldots, l_k(k)) \cong_{\text{gr}} \mathbb{M}_n(K)(l'_0(0), l'_1(1), \ldots, l'_{k'}(k'))$$

for some nonnegative k' and $l'_1, \ldots, l'_{k'-1}$ and positive $l'_0, l'_{k'}$ such that $n = \sum_{i=0}^{k'} l'_i$. By Lemma 1.1, a graded isomorphism above is a finite composition of three types of operations from Lemma 1.1. Since the 0-component is the only nonzero component of K, the identity is the only feasible operation from part (3) of Lemma 1.1. If a positive element is added to the list $l_0(0), l_1(1), l_2(2), \ldots, l_k(k)$, the resulting list does not have 0 in it and if a negative element is added to the same list, the resulting list does not consist of nonnegative elements, hence an operations from part (2) of Lemma 1.1 is not present. This means that only an operation from part (1) of Lemma 1.1 can be performed, so $l'_0(0), l'_1(1), \ldots, l'_{k'}(k')$ is obtained by a permutation of $l_0(0), l_1(1), \ldots, l_k(k)$. However, since the elements are already listed in a nondecreasing order, this means that the lists are equal so $k = k'$ and $l_i = l'_i$ for all $i = 0, \ldots, k$.

(2) If $l_0, \ldots, l_{m-1}$ are obtained as in the statement of part (2), $\mathbb{M}_n(K[x^m, x^{-m}])(\gamma_1, \gamma_2, \ldots, \gamma_n) \cong_{\text{gr}} \mathbb{M}_n(K[x^m, x^{-m}])(l_0(0), l_1(1), \ldots, l_{m-1}(m-1))$ by Lemma 1.1. To show uniqueness, assume that

$$\mathbb{M}_n(K[x^m, x^{-m}])(l_0(0), l_1(1), \ldots, l_{m-1}(m-1)) \cong_{\text{gr}} \mathbb{M}_n(K)(l'_0(0), l'_1(1), \ldots, l'_{m-1}(m-1))$$

for some nonnegative $l'_0, l'_1, \ldots, l'_{m-1}$ such that $n = \sum_{i=0}^{m-1} l'_i$. By Lemma 1.1, a graded isomorphism above is a finite composition of three types of operations from Lemma 1.1. Since the elements in both lists of shifts are already in $\{0, 1, \ldots, m-1\}$, if an operation from part (2) of Lemma 1.1 is present, then the results are considered modulo m again using part (3) of Lemma 1.1. To obtain the resulting list in a nondecreasing order, the elements are permuted using part (1) of Lemma 1.1. This shows that there is an integer k such that $l'_i = l_{i +_m k}$ for all $i = 0, \ldots, m-1$ where $+_m$ denotes the operation of the cyclic abelian group $\mathbb{Z}/m\mathbb{Z}$ of order m. If we reorder the elements $l_0, \dots, l_{m-1}$ using the permutation of $\{0, \dots, m-1\}$ given by $i \mapsto i +_m k$, the list becomes $l_{0+_mk} = l'_0, \dots, l_{m-1+_mk} = l'_{m-1}$. □

We say that the nonnegative integers k and $l_0, l_1, \dots, l_k$ from part (1) of Lemma 5.1 are *representatives* of the graded isomorphism class of $\mathbb{M}_n(K)(\gamma_1, \gamma_2, \dots, \gamma_n)$. By Lemma 5.1, such representatives are unique. We also say that the nonnegative integers $l_0, l_1, \dots, l_{m-1}$ from part (2) of Lemma 5.1 are *representatives* of the graded isomorphism class of $\mathbb{M}_n(K[x^m, x^{-m}])(\gamma_1, \gamma_2, \dots, \gamma_n)$. By Lemma 5.1, such representatives are unique up to their order.

Proposition 5.2. *Let n be a positive integer, $\gamma_1, \gamma_2, \dots, \gamma_n$ be arbitrary integers, and R be the algebra $\mathbb{M}_n(K)(\gamma_1, \gamma_2, \dots, \gamma_n)$. The following conditions are equivalent.*

(1) R is graded isomorphic to a Leavitt path algebra.
(2) R is graded isomorphic to a Leavitt path algebra of a finite acyclic graph with a unique sink.
(3) R is graded isomorphic to $\mathbb{M}_n(K)(0, l_1(1), l_2(2), \dots, l_k(k))$ for some nonnegative k and positive integers $l_1, \dots, l_k$ such that $n = 1 + \sum_{i=1}^k l_i$.
(4) If k and $l_0, \dots, l_k$ are representatives of the graded isomorphism class of R, then l_i is positive *for all $i = 1, \dots, k$ and $l_0 = 1$.*

Proof. If $R \cong_{\text{gr}} L_K(E)$ for some graph E, then E is row-finite and acyclic by [16, Corollary 3.5]. Since R is unital, E has finitely many vertices. A row-finite graph with finitely many vertices is finite, so E is finite. The algebra R is graded simple (see the second paragraph of [9, Remark 1.4.8]), so E has only one sink since otherwise a graded matricial representation of $L_K(E)$ is not graded simple. This shows (1) ⇒ (2). The converse (2) ⇒ (1) is direct.

To show (2) ⇒ (3), let $R \cong_{\text{gr}} L_K(E)$ for some finite acyclic graph E with a unique sink v. Since the set of lengths of paths of E which end at v is finite, there is a maximal element k of this set and a path p to v of length k. Let l_i be the number of paths of length i to v for $i = 0, \dots, k$. Then $\mathbb{M}_{n'}(K)(l_0(0), l_1(1), l_2(2), \dots, l_k(k))$ where $n' = \sum_{i=0}^k l_i$ is graded isomorphic to a graded matricial representation of $L_K(E)$ and, hence, to R as well. The relation $n = n'$ holds by Lemma 1.1. The trivial path is the only one of length zero so $l_0 = 1$. The subpaths of p which end at v have lengths $0, 1, 2, \dots, k$, so l_i is positive for each $i = 0, \dots, k$.

To show (3) ⇒ (2), let k be any nonnegative integer and $l_1, \dots, l_k$ be positive integers such that $n = 1 + \sum_{i=1}^k l_i$. We construct a finite acyclic graph E with a unique sink such that $L_K(E) \cong_{\text{gr}} \mathbb{M}_n(K)(0, l_1(1), l_2(2), \dots, l_k(k))$. Let E_0 be an isolated vertex v_{01}. Obtain E_1 by adding l_1 new vertices $v_{11}, \dots, v_{1l_1}$ to E_0 and an edge from v_{1j} to v_{01} for all $j = 1, \dots, l_1$. If E_{i-1} is created, obtain E_i by adding l_i new vertices $v_{i1}, \dots, v_{il_i}$ to E_{i-1} and an edge from v_{ij} to $v_{(i-1)1}$ for all $j = 1, \dots, l_i$. After E_k is created, let $E = \bigcup_{i=0}^k E_i$. By construction, E is finite and acyclic and v_{01} is the only sink. The trivial path to v_{01} is the only one of length zero and E has exactly l_i paths of length i ending at v_{01} for all $i = 1, \dots, k$. So, $L_K(E) \cong_{\text{gr}} \mathbb{M}_n(K)(0, l_1(1), l_2(2), \dots, l_k(k))$.

Conditions (3) and (4) are equivalent by Lemma 5.1 since the representatives k and $l_0, \dots, l_k$ are unique. □

The key requirement in Proposition 5.2 is that the representatives $l_1, \dots, l_{k-1}$ of the graded isomorphism class of R are *positive*. This ensures that there are no "gaps" in the lengths of paths. For example, the algebra $\mathbb{M}_2(K)(0, 2)$ is graded isomorphic to no Leavitt path algebra since if there is a path of length 2 to a sink, then there has to be a path of length 1 to that sink also.

A graph is said to be a *comet* if every vertex connects to a unique cycle of the graph. Such graph is no-exit since if there is an exit e from the only cycle c, then the range of e connects to the cycle c implying the existence of another cycle containing e and a path from the range of e to some vertex of c. Since the cycle c is unique, no such e can exist.

Proposition 5.3. *Let m and n be positive integers, $\gamma_1, \gamma_2, \ldots, \gamma_n$ be arbitrary integers, and let $R = \mathbb{M}_n(K[x^m, x^{-m}])(\gamma_1, \gamma_2, \ldots, \gamma_n)$. The following conditions are equivalent.*

(1) R is graded isomorphic to a Leavitt path algebra.
(2) R is graded isomorphic to a Leavitt path algebra of a finite comet graph.
(3) R is graded isomorphic to $\mathbb{M}_n(K[x^m, x^{-m}])(l_0(0), l_1(1), \ldots, l_{m-1}(m-1))$ for some positive integers $l_0, l_1, \ldots, l_{m-1}$ such that $n = \sum_{i=0}^{m-1} l_i$.
(4) If $l_0, \ldots, l_{m-1}$ are representatives of the graded isomorphism class of R, then l_i is positive *for all $i = 0, \ldots, m-1$.*

Proof. To show (1) $\Rightarrow$ (2), assume that $R \cong_{\text{gr}} L_K(E)$ for some graph E. By [16, Corollary 3.6], E is a row-finite no-exit graph without sinks. Since R is unital, E has finitely many vertices so the condition that E is row-finite implies that E is finite. The algebra R is graded simple, so E has only one cycle since otherwise a graded matricial representation of $L_K(E)$ is not graded simple. Hence, E is a finite comet graph. The converse (2) $\Rightarrow$ (1) is direct.

To show (2) $\Rightarrow$ (3), let $R \cong_{\text{gr}} L_K(E)$ for some finite comet graph E. If m' is the length of the cycle of E, v is a vertex of the cycle, l_i is the number of paths to v of length i modulo m' which do not contain the cycle, and $n' = \sum_{i=0}^{m'-1} l_i$, then $\mathbb{M}_{n'}(K[x^{m'}, x^{-m'}])(l_0(0), l_1(1), \ldots, l_{m'-1}(m'-1))$ is graded isomorphic to a graded matricial representation of $L_K(E)$ and so to R also. By Lemma 1.1, $K[x^{m'}, x^{-m'}] \cong_{\text{gr}} K[x^m, x^{-m}]$. Assuming that $m' < m$, produces a contradiction by considering the m'-components. One shows that $m \geq m'$ similarly and so $m = m'$. By Lemma 1.1, $n = n'$. For $i = 0, \ldots, m-1$, l_i is positive since there is a subpath of the cycle which ends at v and which has length i.

To show (3) $\Rightarrow$ (2), consider any positive integers $l_0, \ldots, l_{m-1}$ such that $n = \sum_{i=0}^{m-1} l_i$. Construct a finite comet graph E as follows. Consider an isolated cycle of length m with vertices $v_0, \ldots, v_{m-1}$ ordered so that v_{i+1} emits an edge to v_i for $i = 0, \ldots, m-2$ and v_0 emits an edge to v_{m-1}. For each $i = 1, \ldots, m-1$, add $l_i - 1$ new vertices $v_{i1}, \ldots, v_{i(l_i-1)}$ and an edge from v_{ij} to v_{i-1} for each $j = 1, \ldots, l_i - 1$. Add also $l_0 - 1$ new vertices $v_{01}, \ldots, v_{0(l_0-1)}$ and an edge from v_{0j} to v_{m-1} for each $j = 1, \ldots, l_0 - 1$. The graph E obtained in this way is a finite comet graph with a cycle of length m. For each $i = 1, \ldots, m-1$, there are $l_i - 1$ paths to v_0 of length i which are not subpaths of the cycle and there is one path from v_i to v_0 inside of the cycle. There are $l_0 - 1$ paths to v_0 of length m which are not subpaths of the cycle and there is a trivial path to v_0. So, l_i is the number of paths to v_0 of length i modulo m which do not contain the cycle. Thus, $L_K(E) \cong_{\text{gr}} \mathbb{M}_n(K[x^m, x^{-m}])(l_0(0), l_1(1), \ldots, l_{m-1}(m-1))$.

Conditions (3) and (4) are equivalent by Lemma 5.1 since reordering a list of positive elements $l_0, \ldots, l_{m-1}$ produces a list where all elements are also positive. $\square$

Proposition 5.4. *Let k, n be nonnegative, k_i, n_j, m_j positive, and $\gamma_{i1} \ldots, \gamma_{ik_i}, \delta_{j1}, \ldots, \delta_{jn_j}$ arbitrary integers for $i = 1, \ldots, k, j = 1, \ldots, n$. If*

$$R = \bigoplus_{i=1}^{k} \mathbb{M}_{k_i}(K)(\gamma_{i1} \ldots, \gamma_{ik_i}) \oplus \bigoplus_{j=1}^{n} \mathbb{M}_{n_j}(K[x^{m_j}, x^{-m_j}])(\delta_{j1}, \ldots, \delta_{jn_j}),$$

then the following conditions are equivalent.

(1) R is graded isomorphic to a Leavitt path algebra.
(2) R is graded isomorphic to a Leavitt path algebra of a finite no-exit graph.
(3) There are some nonnegative integers k_i' and positive integers $l_{i1}, l_{i2}, \ldots, l_{ik_i'}, i = 1, \ldots, k,$ and $s_{j0}, s_{j1}, \ldots, s_{j(m_j-1)}, j = 1, \ldots, n$ such that $k_i = 1 + l_{i1} + l_{i2} + \ldots + l_{ik_i'}$, for all $i = 1, \ldots, k$, that $n_j = s_{j0} + s_{j1} + \ldots + s_{j(m_j-1)}$ for all $j = 1, \ldots, n$, and that R is graded isomorphic to

$$\bigoplus_{i=1}^{k} \mathbb{M}_{k_i}(K)(0, l_{i1}(1), l_{i2}(2), \ldots, l_{ik_i'}(k_i')) \oplus \bigoplus_{j=1}^{n} \mathbb{M}_{n_j}(K[x^{m_j}, x^{-m_j}])(s_{j0}(0), s_{j1}(1), \ldots, s_{j(m_j-1)}(m_j-1)).$$

(4) If k_i' and $l_{i0}, \ldots, l_{ik_i'}$ are representatives of the graded isomorphism class of the algebra $\mathbb{M}_{k_i}(K)(\gamma_{i1} \ldots, \gamma_{ik_i})$ for $i = 1, \ldots, k$ and if $s_{j0}, \ldots, s_{j(m_j-1)}$ are representatives of the graded isomorphism class of the algebra $\mathbb{M}_{n_j}(K[x^{m_j}, x^{-m_j}])(\delta_{j1}, \ldots, \delta_{jn_j})$ for $j = 1, \ldots, n$ then $l_{i0} = 1$ and $l_{i1}, \ldots, l_{ik_i'}$ are positive *for all $i = 1, \ldots, k$ and $s_{j0}, \ldots, s_{j(m_j-1)}$ are* positive *for all $j = 1, \ldots, n$.*

Proof. If $R \cong_{\text{gr}} L_K(E)$ for some graph E, E is row-finite and no-exit by [16, Corollary 3.4]. Since R is unital and E is row-finite, E is finite. This shows (1) ⇒ (2). The converse (2) ⇒ (1) is direct.

To show (2) ⇒ (3), let $R \cong_{\text{gr}} L_K(E)$ for some finite no-exit graph E. By the graded version of the Wedderburn-Artin Theorem (see [9, Remark 1.4.8]), by the argument that $K[x^{m'}, x^{-m'}] \cong_{\text{gr}} K[x^m, x^{-m}]$ implies that $m' = m$ shown in the proof of (2) ⇒ (3) of Proposition 5.3, and by reordering the terms of R if necessary, we can assume that a graded matricial representation M of $L_K(E)$ is $\bigoplus_{i=1}^k \mathbb{M}_{k_i}(K)(\gamma_{i1}' \ldots, \gamma_{ik_i}') \oplus \bigoplus_{j=1}^n \mathbb{M}_{n_j}(K[x^{m_j}, x^{-m_j}])(\delta_{j1}', \ldots, \delta_{jn_j}')$ for some integers $\gamma_{i1}' \ldots, \gamma_{ik_i}'$ and $\delta_{j1}', \ldots, \delta_{jn_j}'$. For each $i = 1, \ldots, k$, the proof of (2) ⇒ (3) in Proposition 5.2 implies that there is a nonnegative integer k_i' and positive integers $l_{i1}, \ldots, l_{ik_i'}$ such that $k_i = 1 + l_{i1} + \ldots + l_{ik_i'}$ and that there is $\phi_i : \mathbb{M}_{k_i}(K)(\gamma_{i1}' \ldots, \gamma_{ik_i}') \cong_{\text{gr}} \mathbb{M}_{k_i}(K)(0, l_{i1}(1), \ldots, l_{ik_i'}(k_i'))$. For each $j = 1, \ldots, n$, the proof of (2) ⇒ (3) in Proposition 5.3 implies that there are positive integers $s_{j0}, \ldots, s_{j(m_j-1)}$ such that $n_j = s_{j0} + \ldots + s_{j(m_j-1)}$ and that there is $\psi_j : \mathbb{M}_{n_j}(K[x^{m_j}, x^{-m_j}])(\delta_{j1}', \ldots, \delta_{jn_j}') \cong_{\text{gr}} \mathbb{M}_{n_j}(K[x^{m_j}, x^{-m_j}])(s_{j0}(0), \ldots, s_{j(m_j-1)}(m_j - 1))$. If ϕ is $\bigoplus_{i=1}^k \phi_i \oplus \bigoplus_{j=1}^n \psi_j$, then composing $R \cong_{\text{gr}} L_K(E)$ and $L_K(E) \cong_{\text{gr}} M$ with ϕ produces a graded isomorphism of R and a graded algebra as in condition (3).

To show (3) ⇒ (2), let k_i' be a nonnegative integer and let $l_{i1}, \ldots, l_{ik_i'}, s_{j0}, \ldots, s_{j(m_j-1)}$ be positive integers such that $k_i = 1 + l_{i1} + \ldots + l_{ik_i'}$ and that $n_j = s_{j0} + \ldots + s_{j(m_j-1)}$ for each $i = 1, \ldots, k$ and $j = 1, \ldots, n$. By Proposition 5.2, there is a finite acyclic graph E_i with a unique sink such that $L_K(E_i) \cong_{\text{gr}} \mathbb{M}_{k_i}(K)(0, l_{i1}(1), \ldots, l_{ik_i'}(k_i'))$ for every $i = 1, \ldots k$. By Proposition 5.3, there is a finite comet graph F_j such that $L_K(F_j) \cong_{\text{gr}} \mathbb{M}_{n_j}(K[x^{m_j}, x^{-m_j}])(s_{j0}(0), \ldots, s_{j(m_j-1)}(m_j - 1))$ for every $j = 1, \ldots, n$. Let E be the disjoint union of graphs $E_i, i = 1, \ldots, k$ and F_j, $j = 1, \ldots, n$ so that $L_K(E)$ is graded isomorphic to a graded algebra as in condition (3).

The equivalence of (3) and (4) holds by Lemma 5.1 since representatives of the graded isomorphism class of a matricial algebra over K are unique and representatives of the graded isomorphism class of a matricial algebra over $K[x^m, x^{-m}]$ are unique up to their order. □

By [2, Theorem 3.15], every corner of a Leavitt path algebra of a finite graph is isomorphic to another Leavitt path algebra. Using Proposition 5.2, example below shows that a *graded* corner of a Leavitt path algebra may not be *graded* isomorphic to another Leavitt path algebra.

Example 5.5. Let E be the graph below.

$$\bullet_u \xrightarrow{e} \bullet_v \xrightarrow{f} \bullet_w$$

Let ϕ be the graded isomorphism $L_K(E) \cong_{\text{gr}} \mathbb{M}_3(K)(0,1,2)$ described in section 4.1. So, ϕ maps the graded idempotent $u+w$ to the graded idempotent $e = e_{11}+e_{33}$. The graded corner $e\,\mathbb{M}_3(K)(0,1,2)e$ is graded isomorphic to the graded algebra $\mathbb{M}_2(K)(0,2)$. By Proposition 5.2, $\mathbb{M}_2(K)(0,2)$ is not graded isomorphic to any Leavitt path algebra.

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