Algebras of graphs

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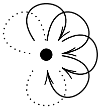
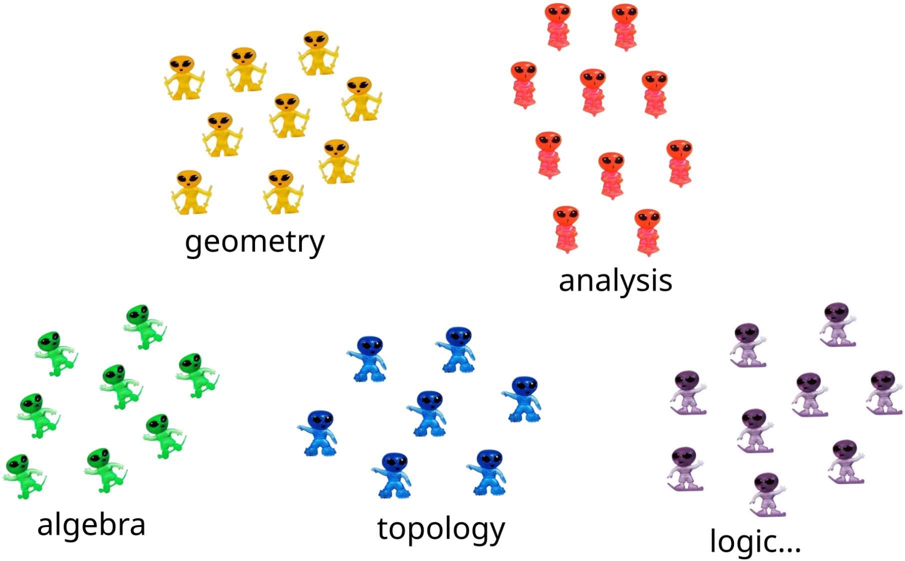
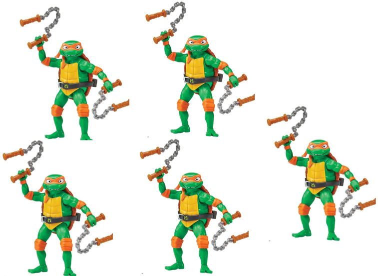


Modern mathematics...

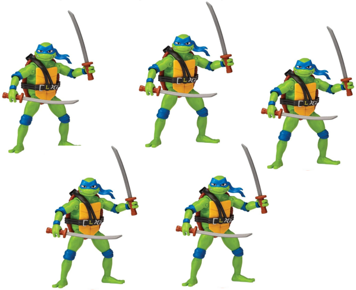


And then there are also mutants...



algebraic geometry

"continuous" algebra

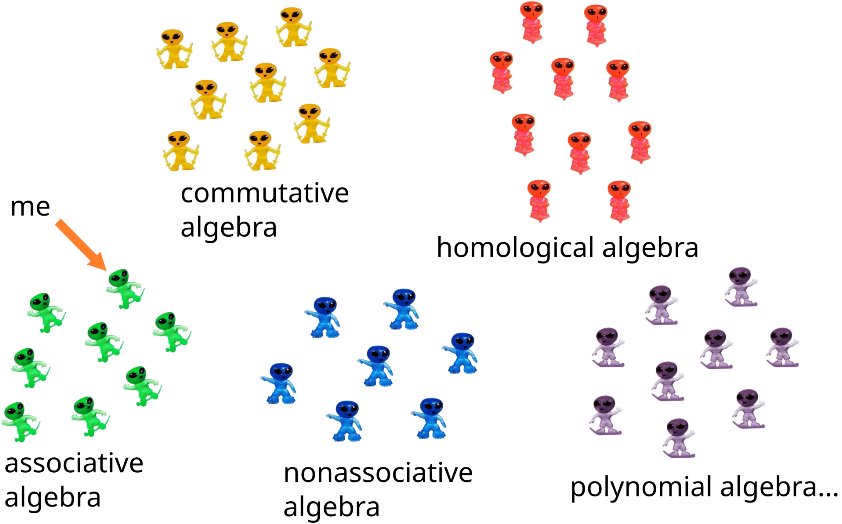


algebraic topology

universal algebra

Even within the same area...

... we may not speak the same language or understand each other.



Some bridges were built...

solving polynomials	Galois group $\leftrightsquigarrow$	groups
(some) differential equations	Laplace transform $\leftrightsquigarrow$	algebraic equations
topological structures	homology $\leftrightsquigarrow$	algebraic structures

So, building bridges is important...

... and I would like to talk about one of them.

Operator theory

Algebra

(or at least about one lane on the multi-lane highway of this bridge)

Let us start with the operator theory...

operators	$\leftrightsquigarrow$	generalized matrices

So, you can think of operator theory as the study of generalized matrices with the concept of **continuity** present.

Operator Theory	$\leftrightsquigarrow$	generalized Linear Algebra $+$ some Real Analysis present

It is a part of **functional analysis** which is the mathematical theory behind quantum mechanics and machine learning, among some other applications.

Hilbert spaces, von Neumann algebras

David Hilbert generalized the finite dimensional vector spaces to spaces possibly infinite dimensional today known as Hilbert spaces.
John von Neumann wanted to capture abstractly the concept of an algebra of observables in quantum mechanics. Studied algebras today known as von Neumann algebras.

We shall talk about an algebra which generalizes operators on Hilbert spaces and, also, generalizes von Neumann algebras.

C^*-algebra – damsel in distress

A C*-**algebra** is

- an **algebra** (meaning we can add and multiply with scalars, just as in a vector space, *and* we can multiply its elements)
- **with a norm** (meaning that we have a way to measure distances)
- **with an involution** $*$ (like taking adjoint and transpose for matrices)

  and

- the norm makes the algebra **complete** and all of the above **agrees** with each other.

Example – Matrices

Consider 2×2 matrices containing real numbers.

- We can **add and multiply** two such matrices and we can multiply a matrix with a real number.
- The vectors in $\mathbb{R}^2$ have a norm ($|\mathrm{x}| = \sqrt{x_1^2 + x_2^2}$), so one can take the **norm** to be

$$|A| = \sup_{\mathrm{x} \neq 0} \frac{|A(\mathrm{x})|}{|\mathrm{x}|}$$

- The **involution** is the transpose

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^* = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

The "agreeing"

The statements that all of the operations the algebra has **agree** with each other means that some relations as those below have to hold.

- $+$ and $\cdot$ $\quad a(b+c) = ab + ac$
- $|\ |$ and $+$ $\quad |a+b| \leq |a| + |b|$
- $|\ |$ and $\cdot$ $\quad |ab| \leq |a||b|$
- $+$ and * $\quad (a+b)^* = a^* + b^*$
- $\cdot$ and * $\quad (ab)^* = b^* a^*$
- $|\ |$ and * $\quad |a^*| = |a|$

More generally, if H is a Hilbert space, the algebra $\mathcal{B}(H)$ of **all** bounded (i.e. continuous) operators on H is a C^*-algebra.

Algebraization of Operator Theory

"Von Neumann algebras are blessed with an excess of structure – algebraic, geometric, topological – so much, that one can easily obscure, through proof by overkill, what makes a particular theorem work."

"If all the functional analysis is stripped away ... what remains should (be) completely accessible through algebraic avenues".

Berberian, S. K. Baer $*$-rings; Springer-Verlag, Berlin-Heidelberg-New York, 1972.

The overkill

The overkill that Berberian is referring to:

a mosquito

a machine gun

What structure do we need?

- **With** $+$ **and** $\cdot$ $\longrightarrow$

  a ring.

- **With an involution,**

  an additive map $*$
  with $(xy)^* = y^*x^*$
  and $(x^*)^* = x$ $\longrightarrow$

  a $*$-ring.

The Knight in shining armor

A $*$-ring with **enough**

projections

A **projection** p is a self-adjoint ($p^* = p$) idempotent ($pp = p$).

Traditional candidates:

- **Baer $*$-rings**,
- group rings (lead to **Hopf algebras**).

Novices:

- **Leavitt path algebras**.

Going back to C^*-algebras...

The examples of C^*-algebras were so vast and so diverse, that a need for their **classification** became evident.

This initiative is known as the

Elliott Classification Program.

Elliott completely classified one type of C^*-algebras (the "best behaved" type).

The algebra of a little graph

Some of the types more difficult to capture found a good representative in a C^*-algebra the graph below.

The algebra A over this little graph has a surprising property

$$A^2 \cong A.$$

A vector space of a finite dimension cannot have such a property (think how different $\mathbb{R}^2$ and $\mathbb{R}$ are), but with infinite dimensions, think that an ordered pair consisting of two vectors with countably infinitely many entries is

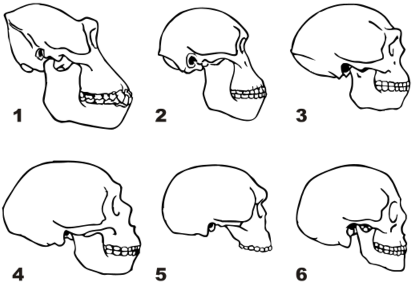
equally infinite

as the single such vector.

$$(1, 2, 3, \ldots, 1, 2, 3, \ldots) \quad \leftrightsquigarrow \quad (1, 1, 2, 2, 3, 3, \ldots)$$

Graph algebra evolution

1. **1950s:** Leavitt algebras as examples of rings R with $R^m \cong R^n$.
2. **1970s:** Cuntz algebras – C^*-algebras defined by analogous identities.
3. **1980s:** Cuntz-Krieger algebras – generalization of 2.



4. **1990s:** C^*-algebras of other graphs considered – the birth of graph C^*-algebras.
5. **2000s:** Leavitt path algebras introduced as the algebraic analog of 4. and a generalization of 1.

Graphs and paths

Start with a directed **graph**: vertices, edges, and source and range map, s and r.

$$\mathrm{s}(e)\bullet \xrightarrow{e} \bullet\,\mathrm{r}(e)$$

A **path** is a sequence $e_1 \dots e_n$ of edges such that

the range of e_i is the source of e_{i+1}

for $i = 1, \dots, n-1$ (such path has the length n) or a vertex (of length zero).

For example, the paths of $\bullet^u \xrightarrow{e} \bullet^v \xrightarrow{f} \bullet^w$ are u, v, w (length 0), e, f (length 1) and ef (length 2).

Adding and multiplying paths – path algebra

The addition of two paths p and q is $p + q$.

The product of p and q is the **concatenation.**

pq is $\bullet \overset{p}{\dashrightarrow} \bullet \overset{q}{\dashrightarrow} \bullet$ if $\mathrm{r}(p) = \mathrm{s}(q)$ and 0 otherwise.

For example, for $\bullet^u \xrightarrow{e} \bullet^v \xrightarrow{f} \bullet^w$, $e \cdot f = ef$ and $f \cdot e = 0$.

Form a vector space over your favorite coefficient field, say K, with the paths as the *basis*. This is the **path algebra** $P_K(E)$.

For example, an element of $P_{\mathbb{R}}(E)$ is a $\mathbb{R}$-*linear combination of the six paths*. So, $3e + \sqrt{5}\, ef$, and $2v - \frac{3}{4}f$ are some of the elements of $P_{\mathbb{R}}(E)$.

Example $\bullet^u \xrightarrow{e} \bullet^v \xrightarrow{f} \bullet^w$ continued

Let e_{ij} denote the **standard matrix unit** with 1 on (i,j) spot and 0 elsewhere. So, $e_{11} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $e_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ etc.

The map

$$u \mapsto e_{11}, \quad e \mapsto e_{12}, \quad ef \mapsto e_{13}, \quad v \mapsto e_{22}, \quad f \mapsto e_{23}, \quad \text{and} \quad w \mapsto e_{33}$$

extends to an isomorphism of $P_K(E)$ and the algebra of the **upper triangular matrices**. We represent this iso by

$$\begin{bmatrix} u & e & ef \\ 0 & v & f \\ 0 & 0 & w \end{bmatrix}$$

Another way to think of $P_K(E)$

Let E^0 be the set of vertices, E^1 the set of edges, and s and r the source and the range maps.

$P_K(E)$ is a **free K-algebra** with vertices and edges as **generators** subject to the two axioms below. For any $v, w \in E^0$ and $e \in E^1$,

V	$vw = 0$ if $v \neq w$ and $vv = v$, and
E1	$\mathrm{s}(e)e = e\mathrm{r}(e) = e$.

$$\mathrm{s}(e)\bullet \xrightarrow{e} \bullet\,\mathrm{r}(e)$$

The path algebra is nice, but to get to a C^*-algebra we need to have an involution and a norm. We are adding more structure.

Ghost paths

1. Add the **ghost edges** – elements of the form e^* for $e \in E^1$.

2. Add the "ghost version" of the axiom E1

   E2 $\quad r(e)e^* = e^*s(e) = e^*$

$$\begin{tikzcd} s(e) \arrow[r, bend left, "e"] & r(e) \arrow[l, bend left, dotted, "e^*"] \end{tikzcd}$$

The vertices are **selfadjoint:** $v^* = v$ for $v \in E^0$.

Example

In the example with E being

$$\bullet_u \xrightarrow{e} \bullet_v \xrightarrow{f} \bullet_w$$

some of the "obvious" products are

$$e^*f = 0, \quad f^*u = 0, \quad ef^* = 0, \quad ue^* = 0.$$

There are some not so obvious products. For example,

What are e^*e and f^*f (if anything)?
What are ee^* and ff^* (if anything)?

To understand the answers, we briefly digress to...

... projections and partial isometries

In a $*$-ring, an idempotent ($pp = p$) and selfadjoint ($p^* = p$) element is called a **projection**.

(So, the vertices are projections.)

An element x is a **partial isometry** if $xx^*x = x$. In this case, $p = xx^*$ and $q = x^*x$ are projections and

$$px = x \text{ and } xq = x$$

Isn't this just as $\mathrm{s}(e)e = e$ and $e\mathrm{r}(e) = e$?

Because of this, one can think of p as "the source" and q as "the range" of x.

This leads us to the last two axioms. First, CK1.

1. One wants **edges to be partial isometries**. So, one requires that

   $$\boxed{e^*e = \mathrm{r}(e)}$$

   since then $ee^*e = e\mathrm{r}(e) = e$ by E1.
   In this case, e^* is also a partial isometry.

2. The edges have **mutually orthogonal** "sources". This ends up being equivalent by requiring that $e^*f = 0$ for $e \neq f$.

   $$\boxed{e^*f = e^*ee^*ff^*f = e^*(ee^*)(ff^*)f = e^*0f = 0.}$$

The two steps are combined in

CK1 $\quad e^*e = \mathrm{r}(e)$ and $e^*f = 0$ if $e \neq f$.

Then, CK2.

3. We keep track of the number of other edges the source of an edge emits. So, we require that the following holds.

   CK2 $\quad v = \sum ee^* \quad$ where the sum is taken over $e \in s^{-1}(v)$.

   for every vertex v which emits at least one but finitely many edges. We say that such v is **regular**.

For example, in the graph $\quad \bullet_u \xrightarrow{e} \bullet_v \xrightarrow{f} \bullet_w$

$e^*e = v, \quad f^*f = w, \quad ee^* = u, \quad ff^* = v.$

And in the graph $\quad \bullet_u \xleftarrow{e} \bullet_v \xrightarrow{f} \bullet_w$

$v = ee^* + ff^* \quad$ (so $ee^* \neq v$ and $ff^* \neq v$).

We got ourselves some algebras

$K =$ field. The **Leavitt path algebra** $L_K(E)$ of E is a free K-algebra on vertices, edges and ghost edges subject to the following.

V	$vv = v$ and $vw = 0$ if $v \neq w$,
E1	$e = \mathrm{s}(e)e = e\mathrm{r}(e)$
E2	$e^* = e^*\mathrm{s}(e) = \mathrm{r}(e)e^*$
CK1	$e^*e = \mathrm{r}(e)$, and $e^*f = 0$ if $e \neq f$
CK2	$v = \sum_{e \in \mathrm{s}^{-1}(v)} ee^*$ if v is regular.

If $K = \mathbb{C}$, the **graph** C^***-algebra** $C^*(E)$ of E is the completion of $L_{\mathbb{C}}(E)$. It is the universal C^*-algebra with

$$\begin{array}{lcl} \text{vertices} & = & \text{generating projections} \\ \text{edges} & = & \text{partial isometries} \end{array} \quad \text{and CK1, CK2, CK3.}$$

(CK3 follows from E2 so we do not need to require it for LPAs.)

Example 1 – Matrices

Recall that the **path algebra** of ${}^u\bullet \xrightarrow{e} \bullet^v \xrightarrow{f} \bullet^w$ is the algebra of **upper triangular** matrices.

The correspondence $e \leftrightsquigarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ implies that

$$e^* \leftrightsquigarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}^* = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, $L_K(E)$ corresponds to the set of **all** matrices over K.

$$\begin{bmatrix} u & e & ef \\ e^* & v & f \\ (ef)^* & f^* & w \end{bmatrix}$$

Graph C*-**algebra:** all 3x3 matrices $M_3(\mathbb{C})$ – the bounded operators on $\mathbb{C}^3$.

$n \times n$ matrices

Generalizes to n-line.

$$\bullet \xrightarrow{e_1} \bullet \xrightarrow{e_2} \bullet \cdots\cdots\cdots \bullet \xrightarrow{e_{n-1}} \bullet$$

Path algebra: upper triangular nxn matrices
Leavitt path algebra: all nxn matrices

Graph C^***-algebra:** bounded operators on $\mathbb{C}^n$ (so all nxn matrices but considered with the norm).

Example 2 – Loop

$$\begin{tikzcd} \bullet_v \arrow[loop above, "e"] \end{tikzcd}$$

Paths: $v = 1 = e^0, e = e^1, e^2, e^3, \ldots$ Representation: $e = x$

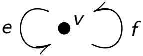
Path algebra: Polynomials with coefficients in K. Think that $3v + 5e^2 \leftrightsquigarrow 3 + 5x^2$.

Ghost edge e^*. Representation: $e^* = x^{-1}$

Leavitt path algebra: Laurent polynomials (like regular polynomials but with negative powers of x also). For example, $2e^* + 3v + 5e^2 \leftrightsquigarrow 2x^{-1} + 3 + 5x^2$.

Graph C***-algebra:** continuous functions on a circle $C(S^1)$.

Example 3 – two-petal rose



Paths: $v = 1, e, f, e^2, ef, fe, f^2, e^3, eef, \ldots$ All the products are defined. Representation: $e = x$, $f = y$ **Path algebra:** Free algebra on x and y (like polynomials but without $xy = yx$).

Ghost edges e^*, f^*. CK1 is $e^*e = f^*f = 1$
CK2 is $ee^* + ff^* = 1$ (so e and f have left inverses but not the right inverses). The pair of maps

$$a \mapsto (e^*a, f^*a) \quad \text{and}$$

$$(a, b) \mapsto ea + fb$$

are mutually inverse isomorphisms ensuring that

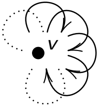
$$L_K(E)^2 \cong L_K(E).$$

Example 3 – roses

Leavitt path algebra is known as the Leavitt algebra $L(1,2)$. It is a universal example of a ring R with $R^2 \cong R$.

Graph C*-**algebra:** Cuntz algebra $\mathcal{O}_2$.

Generalizes to n-**rose.**



Path algebra: free K algebra on n variables.

Leavitt path algebra: Leavitt algebra $L(1,n)$ – universal example of a ring with $R^n \cong R$.

Graph C*-**algebra:** Cuntz algebra $\mathcal{O}_n$

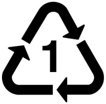
Some research trends. 1. Characterizations

For a given algebra property P, find a graph property Q so that

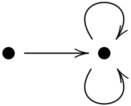
the algebra has a property $P \Leftrightarrow$ the graph has a property Q.

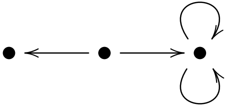
For example,

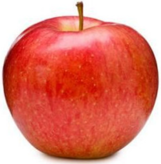
1. $L_K(E)$ has the identity $\Leftrightarrow$ E has finitely many vertices.



2. $L_K(E)$ is finite dimensional over K $\Leftrightarrow$ E is finite and has no cycles.
3. Characterization of $L_K(E)$ being **simple**.

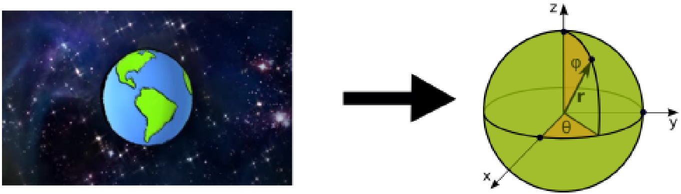
Gives us that  has a simple LPA and

 does not.



Research trends. 2. Generalizations

1. **Separated graphs**. Consider partitions of the set of edges a vertex emits and modify CK1 and CK2 accordingly.
2. **Non-field coefficients.** The coefficients may not have inverses.
3. **Steinbeg algebras** – algebras over groupoid instead of graphs.

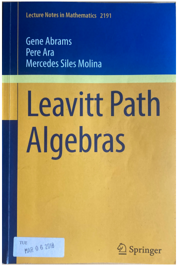


Research trends. 3. Classifications

1. **Field Dependence.** If $L_K(E) \cong L_K(F)$ is $L_{K'}(E) \cong L_{K'}(F)$?
2. **Isomorphism Conjecture.** $L_{\mathbb{C}}(E) \cong L_{\mathbb{C}}(F)$ as algebras (as rings) $\Leftrightarrow$ $C^*(E) \cong C^*(F)$.
3. **Graded Classification Conjecture.** $L_K(E) \cong L_K(F)$ as graded algebras $\Leftrightarrow$ the graded Grothendieck groups are (pointed) isomorphic.

Interested in more?

1. The book **Leavitt path algebras** by Gene Abrams, Pere Ara, and Mercedes Siles Molina.



2. The **Graph Algebra Problem Page** maintained by Mark Tomforde.

Also, you can talk to me or visit **http://liavas.net**