Cleanness of Von-Neumann algebras and alike rings

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Mr. Clean meets the stars

How it all started?

At the conference at Ohio Univ. in Athens, March 2005.

T. Y. Lam asked a question...

Lam's Question

Which von Neumann algebras are clean as rings?

- Background on VNAs and VNA-like rings.
- Background on clean rings.
- Introducing stars: *-cleanness.
- Cleanness of a class of VNA- like rings.
- Generalizations and specializations: almost clean and strongly clean rings.
- Some more recent progress on Lam's question.

The story of von Neumann Algebra begins...

John von Neumann's dream – to capture abstractly the concept of an algebra of observables in quantum mechanics. He constructed algebras which

captured all the **types** of non-commutative measures that occur: (1) in classical theory, (2) in quantum systems (infinite in size or in degrees of freedom).

VNA - damsel in distress

H – Hilbert space

$\mathcal{B}(H)$ – bounded operators.

A **von Neumann algebra** $\mathcal{A}$ is a

1) $*$-closed unital subalgebra of $\mathcal{B}(H)$,

2a) equal to its double commutant $\mathcal{A}''$ (where $\mathcal{A}' = \{x \in \mathcal{B}(H) \mid ax = xa$ for all $a \in \mathcal{A}\}$)

equivalently

2b) weakly closed in $\mathcal{B}(H)$.

Five Types

finite, discrete	I_f	"sum" of I_n with μ on $\{1, 2, \ldots, n\}$
infinite, discrete	I_∞	μ on $\{1, 2, \ldots\}$
finite, continuous	II_1	μ on [0,1]
infinite, continuous	II_∞	μ on $\mathbb{R}$
"very" infinite	III	μ on $\{0, \infty\}$

Examples

I_n	$\mathcal{B}(H)$, $\dim(H) = n$ "finite matrices"
I_∞	$\mathcal{B}(H)$, $\dim(H) = \infty$ "infinite matrices"
II_1	group VNA for G "very infinite and nonabelian" (more on the next two slides)
II_∞	"infinite matrices" over type II_1

Types I_f and II_1 are **finite von Neumann algebras.**

Group von Neumann algebras

Recall: $l^2(G)$ = square summable complex valued functions over G. Hilbert space.

$$l^2(G) = \{ \sum_{g \in G} a_g g \mid \sum_{g \in G} |a_g|^2 < \infty \}.$$

The **group von Neumann algebra** $\mathcal{N}G$ is

- the space of G-invariant operators on $l^2(G)$ i.e. $f(xg) = f(x)g$ for $x \in l^2(G)$, $g \in G$.

equivalently

- Weak closure of $\mathbb{C}G$ in $l^2(G)$.

Types of group von Neumann algebras

$\mathcal{N}G$ is always finite (so either I_f, II_1 or a sum of the two types).

$\mathcal{N}G$ is I_f	G is virtually abelian
$\mathcal{N}G$ is II_1	G_f has infinite index

virtually abelian $=$ has an abelian subgroup of finite index

So, $\mathcal{N}G$ **is not** I_f **if** G **is "very infinite and nonabelian".**

$G_f = \{g \in G \mid g$ has finitely many elements in its conjugacy class $\}$.

Von Neumann Algebra – in distress

“Von Neumann algebras are blessed with an excess of structure – algebraic, geometric, topological – so much, that one can easily obscure, through proof by overkill, what makes a particular theorem work.”

“If all the functional analysis is stripped away ... what remains should (be) completely accessible through algebraic avenues”.

Berberian, S. K. Baer $*$-rings; Springer-Verlag, Berlin-Heidelberg-New York, 1972.

The overkill

The overkill that Berberian is referring to:

a mosquito

a machine gun

Law and Order – Enter the Rings

Kaplansky's dream: to axiomatize (at least part of) the theory of VNAs. Followed similar path as von Neumann (looked at projections, idempotents, annihilators) – ended up defining Baer $*$-rings and AW*-algebras.

idempotent: $ee = e$

projection: idempotent and $p = p^*$

($*$ is an involution: additive, $(xy)^* = y^*x^*$, and $(x^*)^* = x$)

right **annihilator** of a set X : $\{r \in R \mid Xr = 0\}$

The Knight in shining armor – Baer $*$-Ring

Baer ring – every right annihilator is generated by an idempotent.

Baer $*$-ring – every right annihilator is generated by a projection.

AW*-algebra – Baer $*$-ring that is also a C^*-algebra.

AW* generalizes VNA's; Baer $*$ generalizes AW*.

Finite "Von-Neumann-algebra-like" – Six Axioms

- A1 A Baer $*$-ring R is **finite** if $x^*x = 1$ implies $xx^* = 1$ for all $x \in R$.
- A2 R satisfies **existence of projections** and **unique positive square root** axioms.
- A3 Partial isometries are addable.
- A4 R is **symmetric**: for all $x \in R$, $1 + x^*x$ is invertible.
- A5 There is a central element $i \in R$ such that $i^2 = -1$ and $i^* = -i$.
- A6 R satisfies the **unitary spectral** axiom (if unitary u is such that $\mathrm{ann}_r(1 - u)$ is sufficiently small, then $1 - u$ is locally invertible in a sequence of subrings that converge to R).

What do A1 – A6 bring?

Berberian: R can be embedded in a

unit-regular ring Q

satisfying A1–A6, having

the same projections

as R.

regular: $(\forall x \in R)\ (\exists y \in R)\ x = xyx$
unit-regular: $(\forall x \in R)\ (\exists y \in R \text{ invertible})\ x = xyx$

Original Mr. Clean – Keith Nicholson

Nicholson introduced clean rings in 1977.

Ohio U., Zanesville, 2007.

Clean Rings

A ring R is **clean** if

every element = unit + idempotent

Additive version of unit-regular.

Examples: Unit-regular, local, semiperfect...

Non-examples: $\mathbb{Z}$, $R[x]$ for R commutative, not all regular ("Bergman example"), ...

Von-Neumann-algebra-like rings – “The Idea”

Recall that a VNA-like R has a **unit-regular** ring of quotients Q with **same projections**.

$$\begin{tikzcd}
\boxed{\begin{array}{c}\text{Go up:}\\ a \in Q\end{array}} \arrow[r] & \boxed{\begin{array}{c}\text{Have: } a = u + e,\ e \text{ idempotent}\\ \text{Want: } a = u + p,\ p \text{ projection}\end{array}} \arrow[d] \\
\boxed{\begin{array}{c}\textbf{Start here:}\\ a \in R\end{array}} \arrow[u] & \boxed{\begin{array}{c}p \in R \text{ and so } u = a - p \in R\\ \textbf{End with: } a = u + p \text{ in } R\end{array}}
\end{tikzcd}$$

(circa 2007)

Two issues here. The first...

We get an **idempotent** and want a **projection**.

Not that big of a deal: for $*$-rings projections take over the role of idempotents.

- Baer becomes Baer *-ring,
- regular becomes *-regular (regular: every principal ideal is gen by an idempotent)
- So **clean** should become...

***-clean:**

every element = unit + projection

The second issue

If $u \in R$ is **invertible in** Q, why should it be **invertible in** R also?

A ring R is **almost clean** if

element = non-zero-divisor + idempotent

Examples: clean, abelian Baer,...

$\mathbb{Z}$ is almost clean and not clean.

Non-examples: Couchot's paper.

Von-Neumann-algebra-like rings are almost clean

Type I_n Baer *-rings that satisfying A2:

- R *-isomorphic to $M_n(Z(R))$,
- $Z(Q)$ is abelian and *-regular so it is *-clean.
- Thus, $M_n(Z(Q)) \cong Q$ is *-clean.
- R is almost *-clean.

Type I_f Baer *-rings that satisfying A2–A6:

- There are central orthogonal projections p_n such that p_nR is of type I_n.
- Q is the direct product of p_nQ.
- Rings p_nQ are *-clean so Q is *-clean.
- R is **almost *-clean.**

Corollary: If R is regular, then $Q = R$ and R is *-clean.

Back to Lam's question

Corollary.

An AW^*-algebra (in particular von Neumann algebra) of type I_f is almost $*$-clean.

If it is regular, then it is $*$-clean.

Other types?

Example. Let $G = \prod_n G_n$, where G_n are finite.
Then $\mathcal{N}G$ is <u>*-clean</u> (because it is regular).

- If just finitely many G_n are not abelian, G is type I_f.
- If not, then $\mathcal{N}G$ is *-clean and **not of type** I_f.

How to extend "The Idea"?

It works for **any** ring that has a
clean overing with the same idempotents.

Start here:
$a \in R$

$\uparrow$

Go up:
$a \in Q$

$\longrightarrow$

Have: $a = u + e$, e idempotent, u unit

$\downarrow$

$e \in R$ and so $u = a - e \in R$
Thus, $a = u + e$, u non-zero-divisor.

Continuous, quasi-continuous, and CS

R – a ring, M – a right R-module.

<u>**Some axioms.**</u>

(C1) Every submodule of M is essential inside a summand of M.

(C2) Every submodule of M that is isomorphic to a summand of M is itself a summand of M.

(C3) If A and B are summands of M and $A \cap B = 0$, then $A \oplus B$ is also a summand of M.

<u>**Some definitions.**</u> R is

1. right **CS** ("complements are summands") if R_R satisfies (C1).
2. right **quasi-continuous** if R_R satisfies (C1) and (C3).
3. right **continuous** if R_R satisfies (C1) and (C2).

3. $\Rightarrow$ 2. $\Rightarrow$ 1.

Exploring "The Idea" with Evrim Akalan (in 2011)

- R r. quasi-continuous $\Rightarrow$

  $Q := E(R)$ and R have the same idempotents.

- R r. quasi-continuous + r. nonsingular $\Rightarrow$

  Q is clean and with the same idempotents as R. So,

R is almost clean.

Stronger statement. Corollary

Let $a \in R$ be arbitrary, $e \in R$ be idempotent.

$$\begin{array}{ccc}
\text{right continuous} & \longrightarrow & a = e + \text{ unit} \\
\downarrow & & \downarrow \\
\cancel{\text{r. quasi-continuous}} & \text{r. nonsing.} & \\
\text{right CS} & \longrightarrow & a = e + \text{non-zero-divisor}
\end{array}$$

Thus,

Right CS + right nonsingular $\Rightarrow$ almost clean.

Corollary:

Finite AW^*-algebras (thus II_1-type VNAs) are **almost clean.**

Special clean

Camillo-Khurana (2001):

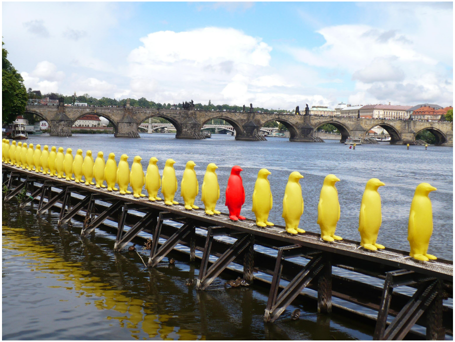
$$\boxed{\begin{array}{c}\text{unit-regular}\\ a = eu\end{array}} \quad \longleftrightarrow \quad \boxed{\begin{array}{c}\text{special clean}\\ a = e + u, \quad aR \cap eR = 0\end{array}}$$

Akalan-Vas (2013):

$$\boxed{\begin{array}{c}\text{Rickart}\\ a = er\end{array}} \quad \overset{\text{abelian}}{\longleftrightarrow} \quad \boxed{\begin{array}{c}\text{special almost clean}\\ a = e + r, \quad aR \cap eR = 0\end{array}}$$

"abelian" in the operator theory sense: all idempotents are central.

Uniqueness



Rickart $a = er$	abelian r. quasi-cont. $\longleftrightarrow$	uniquely special almost clean $a = e + r$ unique, $aR \cap eR = 0$
$\uparrow$		$\uparrow$
unit-regular $a = eu$	abelian $\longleftrightarrow$	uniquely special clean $a = e + u$ unique, $aR \cap eR = 0$

Adapting the results to *-rings

$$\begin{array}{ccc}
\boxed{\text{Rickart } *\text{-ring}} & \overset{\text{abelian}}{\longleftrightarrow} & \boxed{\text{special almost } *\text{-clean}} \\
\uparrow & & \uparrow \\
\boxed{*\text{-regular}} & \overset{\text{abelian}}{\longleftrightarrow} & \boxed{\text{special } *\text{-clean}}
\end{array}$$

Strongly clean

every element = idempotent e + invertible u
and $eu = ue$

More answers from 5 authors in 2022

Chui, Huang, Wu, Yuan, Zhang:

1. A VNA is **strongly clean** if and only if it is a finite direct sum of I_n types.
2. A VNA is **strongly *-clean** if and only if it is abelian (in the operator theory sense).
3. **All** finite VNAs are **clean**.
4. A VNA is **almost *-clean** if and only if it is finite.

Examples

G finite	$\mathcal{N}G$ is type I_f, regular (in fact semisimple), so ***-clean** (strongly iff abelian)
$G = \mathbb{Z}$	$\mathcal{N}G$ is not regular, type I_1 and **strongly *-clean**
$G = \mathbb{Z} \oplus D_3$	$\mathcal{N}G$ is type I_f, not regular, **strongly clean** but not **strongly *-clean**

Interesting: $\mathbb{C}\mathbb{Z}$ is not clean and it is dense in $\mathcal{N}(\mathbb{Z})$ which is (strongly *-) clean.

Another long-standing question

Is the Leavitt algebra $L(1, n)$ clean?

$L(1, 2)$ is the universal example of a ring R such that $R \oplus R \cong R$ (as modules). It can be defined via 4 generators x_1, x_2, y_1, y_2 satisfying the relations:

$$y_i x_i = 1, i = 1, 2, \quad y_i x_j = 0, i \neq j \quad \text{and} \quad x_1 y_1 + x_2 y_2 = 1$$

It can also be defined as the Leavitt **path** algebra of the graph

known as the rose with 2 petals.

Analogously, $L(1, n)$ can be represented as the LPA of the rose with n petals.

Is $L(1, n)$ clean?

Questions

1. **Other types?** Are type II_1 von Neumann algebras *-clean? (we know they are clean and almost *-clean)
   Possible start: consider $\mathcal{N}G$ for $G = \mathbb{Z} * \mathbb{Z}$.

2. Do the 5-authors' results generalize to AW*-algebras? To Baer $*$-rings?

Preprints of my papers are available on

www.liavas.net

and on **arXiv**.