

REALIZATION OF GRADED MATRIX ALGEBRAS AS LEAVITT PATH ALGEBRAS

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ABSTRACT. While every matrix algebra over a field K can be realized as a Leavitt path algebra, this is not the case for every *graded* matrix algebra over a graded field. We provide a complete description of graded matrix algebras over a field, trivially graded by the ring of integers, which are graded isomorphic to Leavitt path algebras. As a consequence, we show that there are graded corners of Leavitt path algebras which are not graded isomorphic to Leavitt path algebras. This contrasts a recent result stating that every corner of a Leavitt path algebra of a finite graph is isomorphic to a Leavitt path algebra. If R is a finite direct sum of graded matricial algebras over a trivially graded field and over naturally graded fields of Laurent polynomials, we also present conditions under which R can be realized as a Leavitt path algebra.

1. INTRODUCTION

Every matrix algebra over a field K or the ring $K[x, x^{-1}]$ is isomorphic to a Leavitt path algebra. In contrast, not every *graded* matrix algebra over a field is graded isomorphic to a Leavitt path algebra by [6, Proposition 3.7]. Here, a Leavitt path algebra is considered with the natural grading by the ring of integers \mathbb{Z} and the field K is considered to be trivially \mathbb{Z} -graded. The Leavitt Path Algebra Realization Question of [6, Section 3.3] asks for a characterization of those graded matrix algebras over K which can be realized as Leavitt path algebras. In Proposition 3.2, we answer this question by providing a complete characterization of graded matrix algebras over K which are graded isomorphic to Leavitt path algebras. In Proposition 3.4, we provide analogous characterization for graded matrix algebras over naturally \mathbb{Z} -graded $K[x^m, x^{-m}]$ for any positive integer m . These two results are used in Proposition 3.5 which presents conditions under which a finite direct sum of graded matricial algebras over K and $K[x^m, x^{-m}]$ can be realized by a Leavitt path algebra.

As a consequence of Proposition 3.2, we show that there are graded corners of Leavitt path algebras which are not graded isomorphic to Leavitt path algebras (Example 3.6). This contrasts a recent result from [2] which states that every corner of a Leavitt path algebra of a finite graph is isomorphic to another Leavitt path algebra.

2. PREREQUISITES

A ring R is graded by a group Γ if $R = \bigoplus_{\gamma \in \Gamma} R_\gamma$ for additive subgroups R_γ and $R_\gamma R_\delta \subseteq R_{\gamma\delta}$ for all $\gamma, \delta \in \Gamma$. The elements of the set $H = \bigcup_{\gamma \in \Gamma} R_\gamma$ are said to be homogeneous. The grading is trivial if $R_\gamma = 0$ for every nonidentity $\gamma \in \Gamma$. A graded ring R is a graded division ring if every nonzero homogeneous element has a multiplicative inverse. If a graded division ring R is commutative then R is a graded field.

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We adopt the standard definitions of graded ring homomorphisms and isomorphisms, graded left and right R -modules, graded module homomorphisms, and graded algebras as defined in [5] and [3]. We use \cong_{gr} to denote a graded ring isomorphism.

In [3], for a Γ -graded ring R and $\gamma_1, \dots, \gamma_n \in \Gamma$, $\mathbb{M}_n(R)(\gamma_1, \dots, \gamma_n)$ denotes the ring of matrices $\mathbb{M}_n(R)$ with the Γ -grading given by

$$(r_{ij}) \in \mathbb{M}_n(R)(\gamma_1, \dots, \gamma_n)_\delta \quad \text{if} \quad r_{ij} \in R_{\gamma_i^{-1}\delta\gamma_j} \quad \text{for } i, j = 1, \dots, n.$$

The definition of $\mathbb{M}_n(R)(\gamma_1, \dots, \gamma_n)$ in [5] is different: $\mathbb{M}_n(R)(\gamma_1, \dots, \gamma_n)$ in [5] corresponds to $\mathbb{M}_n(R)(\gamma_1^{-1}, \dots, \gamma_n^{-1})$ in [3]. More details on the relations between the two definitions can be found in [7, Section 1]. Although the definition from [5] has been in circulation longer, some matricial representations of Leavitt path algebras involve positive integers instead of negative integers making the definition from [3] more convenient when working with Leavitt path algebras. Because of this, we opt to use the definition from [3]. With this definition, if F is the graded free right module $(\gamma_1^{-1})R \oplus \dots \oplus (\gamma_n^{-1})R$,¹ then $\text{Hom}_R(F, F) \cong_{\text{gr}} \mathbb{M}_n(R)(\gamma_1, \dots, \gamma_n)$ as Γ -graded rings.

We also recall [5, Remark 2.10.6] stating the first two parts in Lemma 2.1 and [3, Theorem 1.3.3] stating part (3) for Γ abelian. Although we use these results in case when Γ is the ring of integers, we note that the proof [3, Theorem 1.3.3] generalizes to arbitrary Γ . The last sentence in the lemma is the statement of [3, Proposition 1.4.4. and Theorem 1.4.5].

Lemma 2.1. [5, Remark 2.10.6], [3, Theorem 1.3.3, Proposition 1.4.4, and Theorem 1.4.5] *Let R be a Γ -graded ring and $\gamma_1, \dots, \gamma_n \in \Gamma$.*

(1) *If π a permutation of the set $\{1, \dots, n\}$, then*

$$\mathbb{M}_n(R)(\gamma_1, \gamma_2, \dots, \gamma_n) \cong_{\text{gr}} \mathbb{M}_n(R)(\gamma_{\pi(1)}, \gamma_{\pi(2)}, \dots, \gamma_{\pi(n)})$$

by the map $x \mapsto pxp^{-1}$ where p is the permutation matrix with 1 at the $(i, \pi(i))$ -th spot for $i = 1, \dots, n$ and zeros elsewhere.

(2) *If δ in the center of Γ ,*

$$\mathbb{M}_n(R)(\gamma_1, \gamma_2, \dots, \gamma_n) = \mathbb{M}_n(R)(\gamma_1\delta, \gamma_2\delta, \dots, \gamma_n\delta).$$

(3) *If $\delta \in \Gamma$ is such that there is an invertible element u_δ in R_δ , then*

$$\mathbb{M}_n(R)(\gamma_1, \gamma_2, \dots, \gamma_n) \cong_{\text{gr}} \mathbb{M}_n(R)(\gamma_1\delta, \gamma_2\delta, \dots, \gamma_n\delta)$$

by the map $x \mapsto u^{-1}xu$ where u is the diagonal matrix with $u_\delta, 1, 1, \dots, 1$ on the diagonal.

If Γ is abelian and R and S are Γ -graded division rings, then

$$\mathbb{M}_n(R)(\gamma_1, \gamma_2, \dots, \gamma_n) \cong_{\text{gr}} \mathbb{M}_m(S)(\delta_1, \delta_2, \dots, \delta_m)$$

implies that $R \cong_{\text{gr}} S$, that $m = n$, and the list $\delta_1, \delta_2, \dots, \delta_m$ is obtained from the list $\gamma_1, \gamma_2, \dots, \gamma_n$ by a composition of finitely many operations as in parts (1) to (3).

To shorten the notation, if each $\gamma_i \in \Gamma, i = 1, \dots, k$, appears d_i times in the list

$$\gamma_1, \gamma_1, \dots, \gamma_1, \gamma_2, \gamma_2, \dots, \gamma_2, \dots, \dots, \gamma_k, \gamma_k, \dots, \gamma_k,$$

we abbreviate this list as

$$d_1(\gamma_1), d_2(\gamma_2), \dots, d_k(\gamma_k).$$

¹If M is a graded right R -module and $\gamma \in \Gamma$, the γ -shifted or γ -suspended graded right R -module $(\gamma)M$ is defined as the module M with the Γ -grading given by

$$(\gamma)M_\delta = M_{\gamma\delta}$$

for all $\delta \in \Gamma$. Any finitely generated graded free right R -module is of the form $(\gamma_1)R \oplus \dots \oplus (\gamma_n)R$ for $\gamma_1, \dots, \gamma_n \in \Gamma$ and $\text{Hom}_R(F, F)$ is a Γ -graded ring which is graded isomorphic to $\mathbb{M}_n(R)(\gamma_1, \dots, \gamma_n)$ (both [5] and [3] contain details).

So, if K is a graded field, we use the following abbreviation

$$\mathbb{M}_n(K)(\gamma_1, \gamma_1, \dots, \gamma_1, \gamma_2, \gamma_2 \dots, \gamma_2, \dots, \gamma_k, \gamma_k, \dots, \gamma_k) = \mathbb{M}_n(K)(d_1(\gamma_1), d_2(\gamma_2), \dots, d_k(\gamma_k))$$

For example, if K is a field trivially graded by the group of integers, we use $\mathbb{M}_9(K)(4(0), 3(1), 2(2))$ to shorten $\mathbb{M}_9(K)(0, 0, 0, 0, 0, 1, 1, 1, 2, 2)$.

2.1. Leavitt path algebras. Let E be a directed graph. The graph E is row-finite if every vertex emits finitely many edges and it is finite if it has finitely many vertices and edges. A sink of E is a vertex which does not emit edges. A vertex of E is regular if it is not a sink and if it emits finitely many edges. A cycle is a closed path such that different edges in the path have different sources. A cycle has an exit if a vertex on the cycle emits an edge outside of the cycle. The graph E is acyclic if there are no cycles. We say that graph E is no-exit if v emits just one edge for every vertex v of every cycle.

Let E^0 denote the set of vertices, E^1 the set of edges and \mathbf{s} and \mathbf{r} denote the source and range maps of a graph E . If K is any field, the *Leavitt path algebra* $L_K(E)$ of E over K is a free K -algebra generated by the set $E^0 \cup E^1 \cup \{e^* \mid e \in E^1\}$ such that for all vertices v, w and edges e, f ,

$$\begin{aligned} \text{(V)} \quad vw &= 0 \text{ if } v \neq w \text{ and } vv = v, & \text{(E1)} \quad \mathbf{s}(e)e &= e\mathbf{r}(e) = e, \\ \text{(E2)} \quad \mathbf{r}(e)e^* &= e^*\mathbf{s}(e) = e^*, & \text{(CK1)} \quad e^*f &= 0 \text{ if } e \neq f \text{ and } e^*e = \mathbf{r}(e), \\ \text{(CK2)} \quad v &= \sum_{e \in \mathbf{s}^{-1}(v)} ee^* \text{ for each regular vertex } v. \end{aligned}$$

By the first four axioms, every element of $L_K(E)$ can be represented as a sum of the form $\sum_{i=1}^n a_i p_i q_i^*$ for some n , paths p_i and q_i , and elements $a_i \in K$, for $i = 1, \dots, n$. Using this representation, it is direct to see that $L_K(E)$ is a unital ring if and only if E^0 is finite in which case the sum of all vertices is the identity. For more details on these basic properties, see [1].

A Leavitt path algebra is naturally graded by the group of integers \mathbb{Z} so that the n -component $L_K(E)_n$ is the K -linear span of the elements pq^* for paths p, q with $|p| - |q| = n$ where $|p|$ denotes the length of a path p . While one can grade a Leavitt path algebra by any group Γ (see [3, Section 1.6.1]), we always consider the natural grading by \mathbb{Z} .

2.2. Finite no-exit graphs. If K is a trivially \mathbb{Z} -graded field, let $K[x^m, x^{-m}]$ be the graded field of Laurent polynomials \mathbb{Z} -graded by $K[x^m, x^{-m}]_{mk} = Kx^{mk}$ and $K[x^m, x^{-m}]_n = 0$ if m does not divide n .

By [4, Proposition 5.1], if E is a finite no-exit graph, then $L_K(E)$ is graded isomorphic to

$$R = \bigoplus_{i=1}^k \mathbb{M}_{k_i}(K)(\gamma_{i1} \dots, \gamma_{ik_i}) \oplus \bigoplus_{j=1}^n \mathbb{M}_{n_j}(K[x^{m_j}, x^{-m_j}])(\delta_{j1}, \dots, \delta_{jn_j})$$

where k is the number of sinks, k_i is the number of paths ending in the sink indexed by i for $i = 1, \dots, k$, and γ_{il} is the length of the l -th path ending in the i -th sink for $l = 1, \dots, k_i$ and $i = 1, \dots, k$. In the second term, n is the number of cycles, m_j is the length of the j -th cycle for $j = 1, \dots, n$, n_j is the number of paths which do not contain the cycle indexed by j and which end in a fixed but arbitrarily chosen vertex of the cycle, and δ_{jl} is the length of the l -th path ending in the fixed vertex of the j -th cycle for $l = 1, \dots, n_j$ and $j = 1, \dots, n$.

Note that this representation is not necessarily unique as Example 2.2 shows, but it is unique up to a graded isomorphism. We refer to the graded algebra R above as a *graded matricial representation* of $L_K(E)$.

Example 2.2. Consider the graph below.



If we consider the number and lengths of paths which end at u , we obtain $\mathbb{M}_3(K[x^2, x^{-2}](0, 1, 1))$ as a graded matricial representation of the corresponding Leavitt path algebra. If we consider the paths ending at v , we obtain $\mathbb{M}_3(K[x^2, x^{-2}](0, 1, 2))$. These two algebras are graded isomorphic by Lemma 2.1 since $(0, 1, 1) \rightarrow (0 + 1, 1 + 1, 1 + 1) \rightarrow (1, 2, 2 - 2) = (1, 2, 0) \rightarrow (0, 1, 2)$ where \rightarrow denotes an application of an operation from Lemma 2.1 and results in a graded isomorphism of the corresponding matrix algebras.

3. REALIZATION OF GRADED MATRIX ALGEBRAS AS LEAVITT PATH ALGEBRAS

Every matrix algebra over a field K or the ring $K[x, x^{-1}]$ is isomorphic to a Leavitt path algebra. Indeed, for any positive integer n , let L_n be the “line of length $n - 1$ ”, i.e. the graph with n vertices v_1, v_2, \dots, v_n and an edge from v_i to v_{i+1} for all $i = 1, \dots, n - 1$. Then $L_K(L_n) \cong \mathbb{M}_n(K)$ ([1, Proposition 1.3.5] contains more details). Adding an edge from v_n to v_n to L_n produces a graph C_n such that $L_K(C_n) \cong \mathbb{M}_n(K[x, x^{-1}])$. In this section, we provide a complete description of graded matrix algebras over a trivially \mathbb{Z} -graded field K or over the Laurent polynomials $K[x^m, x^{-m}]$ (\mathbb{Z} -graded as in section 2.2) which are graded isomorphic to Leavitt path algebras. As a consequence, we also present conditions under which a finite direct sum of graded matricial algebras over K and over $K[x^m, x^{-m}]$ can be realized as a Leavitt path algebra.

Lemma 3.1. *Let n and m be positive integers and $\gamma_1, \gamma_2, \dots, \gamma_n$ be arbitrary integers.*

- (1) *If the smallest element is subtracted from the list $\gamma_1, \gamma_2, \dots, \gamma_n$, the elements are permuted so that they are listed in a nondecreasing order, and if k is the largest element of the new list, the new list is $l_0(0), l_1(1), \dots, l_k(k)$ for some nonnegative integers l_1, \dots, l_{k-1} and some positive l_0 and l_k such that $n = \sum_{i=0}^k l_i$. The integers k and l_0, l_1, \dots, l_k are unique for the graded isomorphism class of $\mathbb{M}_n(K)(\gamma_1, \gamma_2, \dots, \gamma_n)$.*
- (2) *If the elements $\gamma_1, \gamma_2, \dots, \gamma_n$ are considered modulo m and arranged in a nondecreasing order, the resulting list is $l_0(0), l_1(1), \dots, l_{m-1}(m - 1)$ for some nonnegative integers l_0, l_1, \dots, l_{m-1} such that $n = \sum_{i=0}^{m-1} l_i$. The integers l_0, l_1, \dots, l_{m-1} are unique for the graded isomorphism class of $\mathbb{M}_n(K[x^m, x^{-m}])(\gamma_1, \gamma_2, \dots, \gamma_n)$ up to their order.*

Proof. (1) If k and l_0, l_1, \dots, l_k are obtained as in the statement of part (1), $\mathbb{M}_n(K)(\gamma_1, \gamma_2, \dots, \gamma_n) \cong_{\text{gr}} \mathbb{M}_n(K)(l_0(0), l_1(1), \dots, l_k(k))$ by Lemma 2.1. To show uniqueness, assume that

$$\mathbb{M}_n(K)(l_0(0), l_1(1), \dots, l_k(k)) \cong_{\text{gr}} \mathbb{M}_n(K)(l'_0(0), l'_1(1), \dots, l'_{k'}(k'))$$

for some nonnegative k' and $l'_1, \dots, l'_{k'-1}$ and positive $l'_0, l'_{k'}$ such that $n = \sum_{i=0}^{k'} l'_i$. By Lemma 2.1, the list $l'_0(0), l'_1(1), \dots, l'_{k'}(k')$ is obtained from $l_0(0), l_1(1), \dots, l_k(k)$ by applying finitely many operations of the three types from Lemma 2.1. Since the 0-component is the only nonzero component of K , the only feasible operation as in part (3) of Lemma 2.1 does not change the list of shifts. If a positive element is added to the list $l_0(0), l_1(1), l_2(2), \dots, l_k(k)$, the resulting list does not have 0 in it and if a negative element is added to the same list, the resulting list does not consist of nonnegative elements, hence an operation from part (2) of Lemma 2.1 is not present. This means that only an operation from part (1) of Lemma 2.1 can be performed, so $l'_0(0), l'_1(1), \dots, l'_{k'}(k')$ is obtained by a permutation of $l_0(0), l_1(1), \dots, l_k(k)$. However, since the elements are already listed in a nondecreasing order, this means that the lists are equal so $k = k'$ and $l_i = l'_i$ for all $i = 0, \dots, k$.

(2) If l_0, \dots, l_{m-1} are obtained as in the statement of part (2), $\mathbb{M}_n(K[x^m, x^{-m}])(\gamma_1, \gamma_2, \dots, \gamma_n) \cong_{\text{gr}} \mathbb{M}_n(K[x^m, x^{-m}])(l_0(0), l_1(1), \dots, l_{m-1}(m-1))$ by Lemma 2.1. To show uniqueness, assume that

$$\mathbb{M}_n(K[x^m, x^{-m}])(l_0(0), l_1(1), \dots, l_{m-1}(m-1)) \cong_{\text{gr}} \mathbb{M}_n(K)(l'_0(0), l'_1(1), \dots, l'_{m-1}(m-1))$$

for some nonnegative $l'_0, l'_1, \dots, l'_{m-1}$ such that $n = \sum_{i=0}^{m-1} l'_i$. By Lemma 2.1, the list $l'_0(0), l'_1(1), \dots, l'_{m-1}(m-1)$ is obtained from $l_0(0), l_1(1), \dots, l_{m-1}(m-1)$ by applying finitely many operations of the three types from Lemma 2.1. Since the elements in both lists of shifts are already in $\{0, 1, \dots, m-1\}$, if an operation from part (2) of Lemma 2.1 is present, then the results are considered modulo m again using part (3) of Lemma 2.1. To obtain the resulting list in a nondecreasing order, the elements are permuted using part (1) of Lemma 2.1. This shows that there is an integer k such that $l'_i = l_{i+m k}$ for all $i = 0, \dots, m-1$ where $+_m$ denotes the operation of the cyclic abelian group $\mathbb{Z}/m\mathbb{Z}$ of order m . If we reorder the elements l_0, \dots, l_{m-1} using the permutation of $\{0, \dots, m-1\}$ given by $i \mapsto i +_m k$, the list becomes $l_{0+m k} = l'_0, \dots, l_{m-1+m k} = l'_{m-1}$. \square

We say that the nonnegative integers k and l_0, l_1, \dots, l_k from part (1) of Lemma 3.1 are *representatives* of the graded isomorphism class of $\mathbb{M}_n(K)(\gamma_1, \gamma_2, \dots, \gamma_n)$. By Lemma 3.1, such representatives are unique. We also say that the nonnegative integers l_0, l_1, \dots, l_{m-1} from part (2) of Lemma 3.1 are *representatives* of the graded isomorphism class of $\mathbb{M}_n(K[x^m, x^{-m}])(\gamma_1, \gamma_2, \dots, \gamma_n)$. By Lemma 3.1, such representatives are unique up to their order. For example, $m = 2$ and $l_0 = 1, l_1 = 2$ for the algebras $\mathbb{M}_3(K[x^2, x^{-2}])(0, 1, 1)$ and $\mathbb{M}_3(K[x^2, x^{-2}])(0, 1, 2)$ from Example 2.2.

Proposition 3.2. *Let n be a positive integer, $\gamma_1, \gamma_2, \dots, \gamma_n$ be arbitrary integers, and R be the algebra $\mathbb{M}_n(K)(\gamma_1, \gamma_2, \dots, \gamma_n)$. The following conditions are equivalent.*

- (1) R is graded isomorphic to a Leavitt path algebra.
- (2) R is graded isomorphic to a Leavitt path algebra of a finite acyclic graph with a unique sink.
- (3) R is graded isomorphic to $\mathbb{M}_n(K)(0, l_1(1), l_2(2), \dots, l_k(k))$ for some nonnegative k and positive integers l_1, \dots, l_k such that $n = 1 + \sum_{i=1}^k l_i$.
- (4) If k and l_0, \dots, l_k are representatives of the graded isomorphism class of R , then l_i is positive for all $i = 1, \dots, k$ and $l_0 = 1$.

Proof. If $R \cong_{\text{gr}} L_K(E)$ for some graph E , then E is row-finite and acyclic by [6, Corollary 3.5]. Since R is unital, E has finitely many vertices. A row-finite graph with finitely many vertices is finite, so E is finite. The algebra R is graded simple (see the second paragraph of [3, Remark 1.4.8]), so E has only one sink since otherwise a graded matricial representation of $L_K(E)$ is not graded simple. This shows (1) \Rightarrow (2). The converse (2) \Rightarrow (1) is direct.

To show (2) \Rightarrow (3), let $R \cong_{\text{gr}} L_K(E)$ for some finite acyclic graph E with a unique sink v . Since the set of lengths of paths of E which end at v is finite, there is a maximal element k of this set and a path p to v of length k . Let l_i be the number of paths of length i to v for $i = 0, \dots, k$. Then $\mathbb{M}_{n'}(K)(l_0(0), l_1(1), l_2(2), \dots, l_k(k))$ where $n' = \sum_{i=0}^k l_i$ is graded isomorphic to a graded matricial representation of $L_K(E)$ and, hence, to R as well. The relation $n = n'$ holds by Lemma 2.1. The trivial path is the only one of length zero so $l_0 = 1$. The subpaths of p which end at v have lengths $0, 1, 2, \dots, k$, so l_i is positive for each $i = 0, \dots, k$.

To show (3) \Rightarrow (2), let k be any nonnegative integer and l_1, \dots, l_k be positive integers such that $n = 1 + \sum_{i=1}^k l_i$. We construct a finite acyclic graph E with a unique sink such that $L_K(E) \cong_{\text{gr}} \mathbb{M}_n(K)(0, l_1(1), l_2(2), \dots, l_k(k))$. Let E_0 be an isolated vertex v_{01} . Obtain E_1 by adding l_1 new vertices v_{11}, \dots, v_{1l_1} to E_0 and an edge from v_{1j} to v_{01} for all $j = 1, \dots, l_1$. If E_{i-1} is created, obtain

E_i by adding l_i new vertices v_{i1}, \dots, v_{il_i} to E_{i-1} and an edge from v_{ij} to $v_{(i-1)1}$ for all $j = 1, \dots, l_i$. After E_k is created, let $E = \bigcup_{i=0}^k E_i$. By construction, E is finite and acyclic and v_{01} is the only sink. The trivial path to v_{01} is the only one of length zero and E has exactly l_i paths of length i ending at v_{01} for all $i = 1, \dots, k$. So, $L_K(E) \cong_{\text{gr}} \mathbb{M}_n(K)(0, l_1(1), l_2(2), \dots, l_k(k))$.

Conditions (3) and (4) are equivalent by Lemma 3.1 since the representatives k and l_0, \dots, l_k are unique. \square

Remark 3.3. The key requirement in Proposition 3.2 is that the representatives l_1, \dots, l_{k-1} of the graded isomorphism class of R are *positive*. This ensures that there are no ‘‘gaps’’ in the lengths of paths. For example, the algebra $\mathbb{M}_2(K)(0, 2)$ is graded isomorphic to no Leavitt path algebra since if there is a path of length 2 to a sink, then there has to be a path of length 1 to that sink also.

A graph is said to be a *comet* if every vertex connects to a unique cycle of the graph. Such graph is no-exit since if there is an exit e from the only cycle c , then the range of e connects to the cycle c implying the existence of another cycle containing e and a path from the range of e to some vertex of c . Since the cycle c is unique, no such e can exist.

Proposition 3.4. *Let m and n be positive integers, $\gamma_1, \gamma_2, \dots, \gamma_n$ be arbitrary integers, and let $R = \mathbb{M}_n(K[x^m, x^{-m}])(\gamma_1, \gamma_2, \dots, \gamma_n)$. The following conditions are equivalent.*

- (1) R is graded isomorphic to a Leavitt path algebra.
- (2) R is graded isomorphic to a Leavitt path algebra of a finite comet graph.
- (3) R is graded isomorphic to $\mathbb{M}_n(K[x^m, x^{-m}])(l_0(0), l_1(1), \dots, l_{m-1}(m-1))$ for some positive integers l_0, l_1, \dots, l_{m-1} such that $n = \sum_{i=0}^{m-1} l_i$.
- (4) If l_0, \dots, l_{m-1} are representatives of the graded isomorphism class of R , then l_i is positive for all $i = 0, \dots, m-1$.

Proof. To show (1) \Rightarrow (2), assume that $R \cong_{\text{gr}} L_K(E)$ for some graph E . By [6, Corollary 3.6], E is a row-finite no-exit graph without sinks. Since R is unital, E has finitely many vertices so the condition that E is row-finite implies that E is finite. The algebra R is graded simple, so E has only one cycle since otherwise a graded matricial representation of $L_K(E)$ is not graded simple. Hence, E is a finite comet graph. The converse (2) \Rightarrow (1) is direct.

To show (2) \Rightarrow (3), let $R \cong_{\text{gr}} L_K(E)$ for some finite comet graph E . If m' is the length of the cycle of E , v is a vertex of the cycle, l_i is the number of paths to v of length i modulo m' which do not contain the cycle, and $n' = \sum_{i=0}^{m'-1} l_i$, then $\mathbb{M}_{n'}(K[x^{m'}, x^{-m'}])(l_0(0), l_1(1), \dots, l_{m'-1}(m'-1))$ is graded isomorphic to a graded matricial representation of $L_K(E)$ and so to R also. By Lemma 2.1, $K[x^{m'}, x^{-m'}] \cong_{\text{gr}} K[x^m, x^{-m}]$. Assuming that $m' < m$, produces a contradiction by considering the m' -components. One shows that $m \geq m'$ similarly and so $m = m'$. By Lemma 2.1, $n = n'$. For $i = 0, \dots, m-1$, l_i is positive since there is a subpath of the cycle which ends at v and which has length i .

To show (3) \Rightarrow (2), consider any positive integers l_0, \dots, l_{m-1} such that $n = \sum_{i=0}^{m-1} l_i$. Construct a finite comet graph E as follows. Consider an isolated cycle of length m with vertices v_0, \dots, v_{m-1} ordered so that v_{i+1} emits an edge to v_i for $i = 0, \dots, m-2$ and v_0 emits an edge to v_{m-1} . For each $i = 1, \dots, m-1$, add $l_i - 1$ new vertices $v_{i1}, \dots, v_{i(l_i-1)}$ and an edge from v_{ij} to v_{i-1} for each $j = 1, \dots, l_i - 1$. Add also $l_0 - 1$ new vertices $v_{01}, \dots, v_{0(l_0-1)}$ and an edge from v_{0j} to v_{m-1} for each $j = 1, \dots, l_0 - 1$. The graph E obtained in this way is a finite comet graph with a cycle of length m . For each $i = 1, \dots, m-1$, there are $l_i - 1$ paths to v_0 of length i which are not subpaths of the cycle and there is one path from v_i to v_0 inside of the cycle. There are $l_0 - 1$

paths to v_0 of length m which are not subpaths of the cycle and there is a trivial path to v_0 . So, l_i is the number of paths to v_0 of length i modulo m which do not contain the cycle. Thus, $L_K(E) \cong_{\text{gr}} \mathbb{M}_n(K[x^m, x^{-m}])(l_0(0), l_1(1), \dots, l_{m-1}(m-1))$.

Conditions (3) and (4) are equivalent by Lemma 3.1 since reordering a list of positive elements l_0, \dots, l_{m-1} produces a list where all elements are also positive. \square

Proposition 3.5. *Let k, n be nonnegative, k_i, n_j, m_j positive, and $\gamma_{i1}, \dots, \gamma_{ik_i}, \delta_{j1}, \dots, \delta_{jn_j}$ arbitrary integers for $i = 1, \dots, k, j = 1, \dots, n$. If*

$$R = \bigoplus_{i=1}^k \mathbb{M}_{k_i}(K)(\gamma_{i1}, \dots, \gamma_{ik_i}) \oplus \bigoplus_{j=1}^n \mathbb{M}_{n_j}(K[x^{m_j}, x^{-m_j}])(\delta_{j1}, \dots, \delta_{jn_j}),$$

then the following conditions are equivalent.

- (1) R is graded isomorphic to a Leavitt path algebra.
- (2) R is graded isomorphic to a Leavitt path algebra of a finite no-exit graph.
- (3) There are some nonnegative integers k'_i and positive integers $l_{i1}, l_{i2}, \dots, l_{ik'_i}, i = 1, \dots, k$, and $s_{j0}, s_{j1}, \dots, s_{j(m_j-1)}, j = 1, \dots, n$ such that $k_i = 1 + l_{i1} + l_{i2} + \dots + l_{ik'_i}$, for all $i = 1, \dots, k$, that $n_j = s_{j0} + s_{j1} + \dots + s_{j(m_j-1)}$ for all $j = 1, \dots, n$, and that R is graded isomorphic to

$$\bigoplus_{i=1}^k \mathbb{M}_{k_i}(K)(0, l_{i1}(1), l_{i2}(2), \dots, l_{ik'_i}(k'_i)) \oplus \bigoplus_{j=1}^n \mathbb{M}_{n_j}(K[x^{m_j}, x^{-m_j}])(s_{j0}(0), s_{j1}(1), \dots, s_{j(m_j-1)}(m_j-1)).$$

- (4) If k'_i and $l_{i0}, \dots, l_{ik'_i}$ are representatives of the graded isomorphism class of the algebra $\mathbb{M}_{k_i}(K)(\gamma_{i1}, \dots, \gamma_{ik_i})$ for $i = 1, \dots, k$ and if $s_{j0}, \dots, s_{j(m_j-1)}$ are representatives of the graded isomorphism class of the algebra $\mathbb{M}_{n_j}(K[x^{m_j}, x^{-m_j}])(\delta_{j1}, \dots, \delta_{jn_j})$ for $j = 1, \dots, n$ then $l_{i0} = 1$ and $l_{i1}, \dots, l_{ik'_i}$ are positive for all $i = 1, \dots, k$ and $s_{j0}, \dots, s_{j(m_j-1)}$ are positive for all $j = 1, \dots, n$.

Proof. If $R \cong_{\text{gr}} L_K(E)$ for some graph E , E is row-finite and no-exit by [6, Corollary 3.4]. Since R is unital and E is row-finite, E is finite. This shows (1) \Rightarrow (2). The converse (2) \Rightarrow (1) is direct.

To show (2) \Rightarrow (3), let $R \cong_{\text{gr}} L_K(E)$ for some finite no-exit graph E . By the graded version of the Wedderburn-Artin Theorem (see [3, Remark 1.4.8]), by the argument that $K[x^{m'}, x^{-m'}] \cong_{\text{gr}} K[x^m, x^{-m}]$ implies that $m' = m$ shown in the proof of (2) \Rightarrow (3) of Proposition 3.4, and by reordering the terms of R if necessary, we can assume that a graded matricial representation M of $L_K(E)$ is $\bigoplus_{i=1}^k \mathbb{M}_{k_i}(K)(\gamma'_{i1}, \dots, \gamma'_{ik_i}) \oplus \bigoplus_{j=1}^n \mathbb{M}_{n_j}(K[x^{m_j}, x^{-m_j}])(\delta'_{j1}, \dots, \delta'_{jn_j})$ for some integers $\gamma'_{i1}, \dots, \gamma'_{ik_i}$ and $\delta'_{j1}, \dots, \delta'_{jn_j}$. For each $i = 1, \dots, k$, the proof of (2) \Rightarrow (3) in Proposition 3.2 implies that there is a nonnegative integer k'_i and positive integers $l_{i1}, \dots, l_{ik'_i}$ such that $k_i = 1 + l_{i1} + \dots + l_{ik'_i}$ and that there is $\phi_i : \mathbb{M}_{k_i}(K)(\gamma'_{i1}, \dots, \gamma'_{ik_i}) \cong_{\text{gr}} \mathbb{M}_{k_i}(K)(0, l_{i1}(1), \dots, l_{ik'_i}(k'_i))$. For each $j = 1, \dots, n$, the proof of (2) \Rightarrow (3) in Proposition 3.4 implies that there are positive integers $s_{j0}, \dots, s_{j(m_j-1)}$ such that $n_j = s_{j0} + \dots + s_{j(m_j-1)}$ and that there is $\psi_j : \mathbb{M}_{n_j}(K[x^{m_j}, x^{-m_j}])(\delta'_{j1}, \dots, \delta'_{jn_j}) \cong_{\text{gr}} \mathbb{M}_{n_j}(K[x^{m_j}, x^{-m_j}])(s_{j0}(0), \dots, s_{j(m_j-1)}(m_j-1))$. If ϕ is $\bigoplus_{i=1}^k \phi_i \oplus \bigoplus_{j=1}^n \psi_j$, then composing $R \cong_{\text{gr}} L_K(E)$ and $L_K(E) \cong_{\text{gr}} M$ with ϕ produces a graded isomorphism of R and a graded algebra as in condition (3).

To show (3) \Rightarrow (2), let k'_i be a nonnegative integer and let $l_{i1}, \dots, l_{ik'_i}, s_{j0}, \dots, s_{j(m_j-1)}$ be positive integers such that $k_i = 1 + l_{i1} + \dots + l_{ik'_i}$ and that $n_j = s_{j0} + \dots + s_{j(m_j-1)}$ for each $i = 1, \dots, k$ and $j = 1, \dots, n$. By Proposition 3.2, there is a finite acyclic graph E_i with a unique sink such

that $L_K(E_i) \cong_{\text{gr}} \mathbb{M}_{k_i}(K)(0, l_{i1}(1), \dots, l_{ik'_i}(k'_i))$ for every $i = 1, \dots, k$. By Proposition 3.4, there is a finite comet graph F_j such that $L_K(F_j) \cong_{\text{gr}} \mathbb{M}_{n_j}(K[x^{m_j}, x^{-m_j}])(s_{j0}(0), \dots, s_{j(m_j-1)}(m_j - 1))$ for every $j = 1, \dots, n$. Let E be the disjoint union of graphs $E_i, i = 1, \dots, k$ and $F_j, j = 1, \dots, n$ so that $L_K(E)$ is graded isomorphic to a graded algebra as in condition (3).

The equivalence of (3) and (4) holds by Lemma 3.1 since representatives of the graded isomorphism class of a matricial algebra over K are unique and representatives of the graded isomorphism class of a matricial algebra over $K[x^m, x^{-m}]$ are unique up to their order. \square

3.1. Graded corners of Leavitt path algebras. If R is a graded ring and e a homogeneous idempotent, the ring eRe is a *graded corner*. By [2, Theorem 3.15], every corner of a Leavitt path algebra of a finite graph is isomorphic to another Leavitt path algebra. Using Proposition 3.2, the example below shows that a *graded* corner of a Leavitt path algebra may not be *graded* isomorphic to another Leavitt path algebra.

Example 3.6. Let E be the graph below.

$$\bullet_u \xrightarrow{e} \bullet_v \xrightarrow{f} \bullet_w$$

If ϕ is the graded isomorphism $L_K(E) \cong_{\text{gr}} \mathbb{M}_3(K)(0, 1, 2)$ described in section 2.2, then ϕ maps the graded idempotent $u + w$ to the graded idempotent $e = e_{11} + e_{33}$ where e_{11} and e_{33} are the standard matrix units. So, the graded corner $e\mathbb{M}_3(K)(0, 1, 2)e$ is graded isomorphic to the graded algebra $\mathbb{M}_2(K)(0, 2)$. By Proposition 3.2, $\mathbb{M}_2(K)(0, 2)$ is not graded isomorphic to any Leavitt path algebra.

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