Leavitt path and graph C^*-algebras: connections via traces

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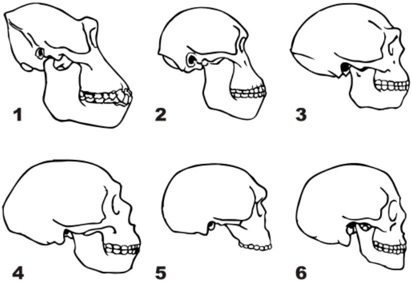
Making connections

Traces

Graph algebra evolution

1. **1950s:** Leavitt algebras as examples of rings with $R^m \cong R^n$.
2. **1970s:** Cuntz's algebras – C^*-algebras defined by analogous identities.
3. **1980s:** Cuntz-Krieger algebras – generalization of 2.



4. **1990s:** Graph C^*-algebras.
5. **2000s:** Leavitt path algebras as algebraic analog of 4. and generalization of 1.

Missing link?

Two worlds, two languages:

$$\begin{array}{ccc} \boxed{\text{Operator Theory world}} & \leftrightsquigarrow & \boxed{\text{Algebra world}} \\ \uparrow & & \uparrow \\ \text{Graph } C^*\text{-algebras} & \leftrightsquigarrow & \text{Leavitt Path Algebras} \end{array}$$

Gene Abrams: "Find a Rosetta stone".

This talk's agenda – two fold

1. The first agenda.

Traces of Leavitt path algebras

Traces of graph C^*-algebras

The second agenda

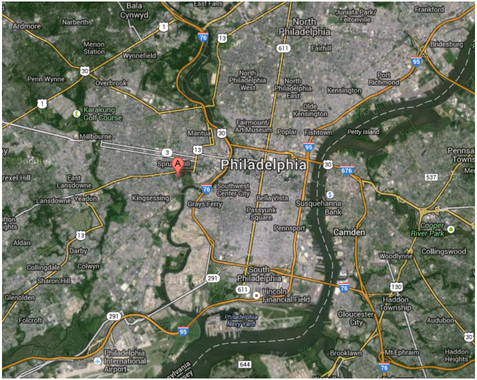
2. While working on 1. I ended up filling the blank below.

A LPA is **directly finite** iff the graph is ________.

Illustrate a more general method of

“localization”

in the sketch of the proof.



But first – the larger picture

Berberian 1972. “Von Neumann algebras are blessed with an excess of structure – algebraic, geometric, topological – so much, that one can easily obscure, through proof by overkill, what makes a particular theorem work.”

“If all the functional analysis is stripped away … what remains should (be) completely accessible through algebraic avenues”.

Two worlds – with more than one inhabitant each

Group Von Neumann algebras	$\leftrightsquigarrow$	Group rings
AW^*-algebras	$\leftrightsquigarrow$	Baer $*$-rings
Graph C^*-algebras	$\leftrightsquigarrow$	Leavitt Path Algebras

My algebraic avenues. 2000s: group VNAs $\to$ finite VNAs $\to$ Baer $*$-rings.

Need more Rosetta stones.

And then...

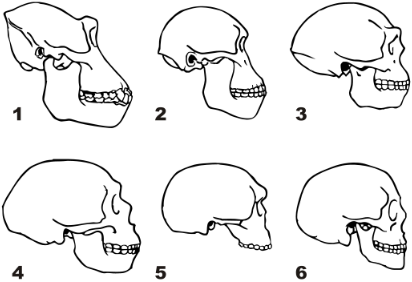
... I started hanging out with Gonzalo (Aranda Pino)...

- ... and noticed that the search of algebraic avenues motivates the study of Leavitt path algebras as well.

Trace evolution

1. The usual trace on $M_n(R)$.

   The **trace** of an idempotent/projection is related to the **dimension** of the corresponding projective submodule/subspace.



2. More generally, **traces of operators** of a Hilbert space $\rightarrow$ traces of von Neumann algebras $\rightarrow$ of C^*-algebras.
3. Tomforde (2002) Graph traces and tracial states of **graph C*-algebras**.
4. Traces on Leavitt path algebras?
5. Connections?

So, let us look at a trace...

... in the most general way.

Let R and T be rings. A

T-valued trace on R

is a map $t : R \to T$ which is

- **additive** and
- **central** i.e. $\boxed{t(xy) = t(yx)}$ for all $x, y \in R$

If R and T are K-algebras, we also want it to be

- K-**linear** i.e. $t(kx) = kt(x)$ for all $x \in R$ and $k \in K$.

Examples

1. **Kaplansky trace** on a group ring KG. $\sum a_g g \mapsto a_1$
2. **Augmentation map** on KG. $\sum a_g g \mapsto \sum a_g$.
3. **Standard trace** on matrix ring over K.
   Matrix ring $= \overline{KG}$ for $G =$ matrix units.

$\overline{KG}$ = contracted $KG = KG/K0$.

$KG = \overline{KG + 0}$ if G is without 0.

Traces on contracted semigroup rings with Zak (Mesyan).

Characterization of minimal traces.

- t is **minimal:** $t(x) = 0$ iff x is in the commutator.

Relevance to Leavitt path algebras?

$G = \{pq^* \mid p, q$ paths of a graph $E\}$. Here q^* is a **ghost path**.

G = graph inverse semigroup and

$\overline{KG}$ = **Cohn path algebra.**

Cohn path algebra + CK2 axiom = Leavitt path algebra.

V $vv = v$ and $vw = 0$ if $v \neq w$,

E1 $\mathbf{s}(e)e = e\mathbf{r}(e) = e$

E2 $\mathbf{r}(e)e^* = e^*\mathbf{s}(e) = e^*$

CK1 $e^*e = \mathbf{r}(e)$, $e^*f = 0$ if $e \neq f$

CK2 $v = \sum ee^*$ for $e \in \mathbf{s}^{-1}(v)$ if v is regular (= emits edges, finitely many).

Traces on Cohn and Leavitt path algebras

G = graph inverse semigroup.

Proposition [Zak-Lia].

traces on Cohn path algebra	$\leftrightsquigarrow$	central maps on G
traces on Leavitt path algebra	$\leftrightsquigarrow$	central maps on G which agree with CK2

A central map t on G $\boxed{\textbf{agrees with CK2}}$ iff

$$t(v) = t(\sum ee^*) = \sum t(ee^*) = \sum t(e^*e) = \sum t(\mathbf{r}(e))$$

for v regular with $e \in \mathbf{s}^{-1}(v)$.

Involution kicks in

x in $*$-ring is **positive** ($x \geq 0$) if $x =$ finite sum of yy^*.
R, T $*$-rings, $t : R \to T$ trace.

- t is **positive** if $x \geq 0$ implies $t(x) \geq 0$.
- t is **faithful** if $x > 0$ implies $t(x) > 0$.

If t is **positive** on a LPA, then

$$\boxed{\text{(P)} \quad t(v) \geq \sum_{e \in I} t(\mathbf{r}(e))}$$

for all v, and finite $I \subseteq \mathbf{s}^{-1}(v)$.

- $I = \emptyset \Rightarrow t(v) \geq 0$.
- v regular and $I = \mathbf{s}^{-1}(v) \Rightarrow \geq$ is $=$.

If t is **faithful** then $\boxed{\text{(F)} \quad t(v) > 0}$ for all v.

Are these meaningful?

Desirable properties.

1. (P) is **sufficient** for positivity and (F) for faithfulness.
2. Traces are determined by **values on vertices.**

1 fails. The $\mathbb{C}$-valued t on $\mathbb{C}[x, x^{-1}]$ (=LPA of a loop) given by

$$t(x^n) = i^n, t(x^{-n}) = i^n$$

has (P) and (F) but is not positive since

$$t((1+x)(1+x^{-1})) = 2 + 2i.$$

2 also fails. The map on vertices of the graph below

$$\bullet^1 \longleftarrow \bullet^3 \longrightarrow \bullet^1$$

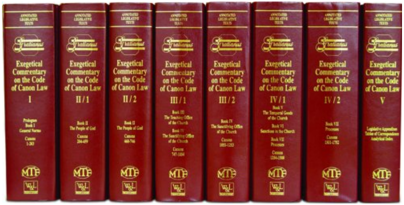
has (P) and (F) but does not extend to a trace: CK2 fails $(3 \neq 1 + 1)$.

Fixing 1 – Canonical traces

t = trace on $L_K(E)$, p, q = paths.

1. t is **canonical** if

$$t(pq^*) = 0, \text{ for } p \neq q \text{ and } t(pp^*) = t(\mathbf{r}(p)).$$



2. t is **gauge invariant** if

$$t(pq^*) = k^{|p|-|q|}t(pq^*) \quad \text{for any nonzero } k \in K.$$

Equivalent for char $K = 0$.

Harmony

Theorem 1 [Lia]. If t is a canonical trace on $L_K(E)$, then

t is positive	$\Longleftrightarrow$	(P) holds.
t is faithful	$\Longleftrightarrow$	(F) holds.

Fixing 2 – Graph Traces

A **graph trace** is a map δ on the set of vertices such that

- $\boxed{\delta(v) = \sum_{e \in I} \delta(\mathbf{r}(e))}$ for $I = \mathbf{s}^{-1}(v)$, and v regular.

It is

- **positive** if $\boxed{\delta(v) \geq \sum_{e \in I} \delta(\mathbf{r}(e))}$ for all v, and finite $I \subseteq \mathbf{s}^{-1}(v)$.
- **faithful** if positive and $\boxed{\delta(v) > 0}$ for all v.

Harmony continued

Theorem 2 [Lia].

canonical trace on $L_K(E)$	$\leftrightsquigarrow$	graph trace on E
positive, canonical trace on $L_K(E)$	$\leftrightsquigarrow$	positive graph trace on E
faithful, canonical trace on $L_K(E)$	$\leftrightsquigarrow$	faithful graph trace on E

Direct corollary of Theorem 1.

Instead of going over 6 pages of proof...

... let me tell you what my $\boxed{\textbf{driving force}}$ was.

1. Classification of von Neumann algebras via traces.
2. Results on traces of graph C^*-algebras.

Connecting with the C^*-algebra world

Theorem [Pask-Rennie, 2006]. E row-finite and countable. All maps are $\mathbb{C}$-valued.

faithful, semifinite, lower semicontinuous gauge-invariant trace on $C^*(E)$	$\leftrightsquigarrow$	faithful graph trace on E

semifinite $= \{x \in C^*(E)^+ | t(x) < \infty\}$ is norm dense in $C^*(E)^+$.

lower semicontinuous $= t(\lim_{n\to\infty} a_n) \le \liminf_{n\to\infty} t(a_n)$ for all $a_n \in C^*(E)^+$ norm convergent.

Let us better polish that Rosetta stone

Operator theory trace

Defined on the positive cone.

$\mathbf{t}(\mathbf{x}\mathbf{x}^*) = \mathbf{t}(\mathbf{x}^*\mathbf{x})$

Faithful if

$$t(xx^*) = 0 \Rightarrow x = 0.$$

Algebra trace

Defined everywhere.

Central.

Faithful if positive and

$$t\left(\sum xx^*\right) = 0 \Rightarrow \sum xx^* = 0.$$

Luckily, char $\mathbb{C} = 0$ so no Rosetta stone needed for:

$$\boxed{\textbf{canonical} = \textbf{gauge invariant.}}$$

Using Rosetta stone

Fixing the domain. Write $x = a + ib$ and $a = a^+ - a^-$, $b = b^+ - b^-$. Define

$$\boxed{t(x) = t(a^+) - t(a^-) + i(t(b^+) - t(b^-)).}$$

This is $\mathbb{C}$-linear and positive.

Fixing faithfulness. If R and T are $*$-rings, $t : R \to T$ a positive trace, and

1. T **positive definite** ($\sum_{i=1}^n x_i x_i^* = 0 \Rightarrow x_i = 0$ for all i, for all n),
2. R **proper** ($xx^* = 0 \Rightarrow x = 0$) then

$t(xx^*) = 0 \Rightarrow x = 0$	$\iff$	$t\left(\sum xx^*\right) = 0 \Rightarrow \sum xx^* = 0.$

Luckily, $\mathbb{C}$ is positive definite and any C^*-algebra is proper.

Connecting the worlds

Corollary [Lia]. E ~~row-finite and~~ countable. All maps are $\mathbb{C}$-valued.

semifinite, lower semicont., faithful, gauge-invariant trace on $C^*(E)$	$\leftrightsquigarrow$	faithful, canonical trace on $L_{\mathbb{C}}(E)$	$\leftrightsquigarrow$	faithful graph trace on E

Proof. We already have that (2) $=$ (3).
Every t as in (1) restricts to t as in (3) without using row-finiteness.
Every t as in (2) extends to t as in (1) using Gauge Invariant Uniqueness Theorem proven for countable graphs.

Where to next with this?

Remember my **driving force**:

A **von Neumann** algebra is	**finite**
iff	there is a finite, normal, faithful trace.

I wandered:

A **Leavitt path** algebra $L_K(E)$ is	**finite**
iff	there is a K-valued canonical, faithful trace (?)
iff	the graph is __________.

Recall that a $*$-ring is **finite** if

$$\boxed{xx^* = 1 \quad \text{implies} \quad x^*x = 1.}$$

Easy: the existence of a faithful trace implies finiteness.

$$xx^* = 1 \;\Rightarrow\; 1 - x^*x \geq 0 \text{ and } t(1 - xx^*) = 0 \text{ so}$$

$$t(1 - x^*x) = t(1 - xx^*) = 0 \;\Rightarrow\; 1 - x^*x = 0 \;\Rightarrow\; x^*x = 1.$$

Houston, we have a problem

$\boxed{\textbf{finite} \text{ iff } xx^* = 1 \ \Rightarrow \ x^*x = 1.}$

<u>What is "1" if E is not finite?</u>

There are still **local units**: for every finite set of elements, there is an idempotent acting like a unit.

A $*$-ring with local units R is $\boxed{\textbf{finite}}$ if for every x and an idempotent u with $xu = ux = x$,

$$\boxed{xx^* = u \quad \text{implies} \quad x^*x = u.}$$

In this case u is a projection (selfadjoint idempotent).

While we are at it...

A unital ring R is **directly (Dedekind) finite** if

$$\boxed{xy = 1 \quad \text{implies} \quad yx = 1.}$$

Equivalently: if no direct summand of R is isomorphic to R.

A ring with local units R is **directly finite** if for every x, y and an idempotent u with $xu = ux = x$ and $yu = uy = y$,

$$\boxed{xy = u \quad \text{implies} \quad yx = u.}$$

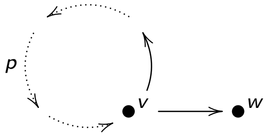
Finite	**Not Finite**
$M_n(K)$	Column finite matrices over K

Necessary condition for LPAs to be finite – no exits

If a cycle p has an **exit**,

then a LPA is

not (directly) finite.



Let $x = p + (1 - \delta_{v,w})w$, and $u = v + (1 - \delta_{v,w})w$. Then $x^*x = u$ and $xx^* \neq u$.

If E is finite, this is **sufficient too:**
E no-exit $\Rightarrow L_K(E)$ finite sum of matricial algebras over K or $K[x, x^{-1}] \Rightarrow L_K(E)$ is directly finite.

Idea for the converse

1. Start with x, y in $L_K(E)$ for some E no-exit.
2. Consider u, local unit for x and y, with $xy = u$. Want $yx = u$.
3. Consider a finite subgraph F determined by the paths appearing in x, y, u.
4. F is no-exit and so $L_K(F)$ is directly finite so $yx = u$. Done.

Problem: $L_K(F)$ may not be a subalgebra of $L_K(E)$.

So $yx = u$ in $L_K(F)$ does not mean $yx = u$ in $L_K(E)$.

Houston, can we "localize"?

Yes: using Cohn, Leavitt and everything in between

Cohn $\mathbf{C_K(E)}$	**Cohn-Leavitt** $\mathbf{CL_K(E, S)}$	**Leavitt** $\mathbf{L_K(E)}$
CK2 holds for **<u>no</u>** regular v's	CK2 holds for **<u>some</u>** regular v's $v \in S \Leftrightarrow$ CK2 holds	CK2 holds for **<u>all</u>** regular v's

Have their C^*-counterparts:
relative graph C^*-algebras

$\boxed{C^*(E, S)}$

No-exits for Cohn-Leavitt algebras over finite E

Not really that much larger class: $\boxed{CL_K(E,S) \cong L_K(E_S)}$

Using the above iso and no-exit characterization for finite graphs, we have that for E <u>**finite**</u>,

$CL_K(E,S)$ is **(directly) finite.**
iff
E is **no-exit** and
vertices of all cycles are in S.

Goodearl-Ara work

For every finite subgraph G of E, there are

- a finite subgraph F of E which contains G and
- a subset T of regular vertices of E such that

$$\mathbf{CL_K(F, T) \text{ is a subalgebra of } L_K(E).}$$

Proven in larger generality for **separated graphs.**

Original idea now works!

Same as originally:

1. Start with x, y in $L_K(E)$ for some E no-exit.
2. Consider a local unit u, local for x and y with $xy = u$. Want $yx = u$.
3. Consider a finite subgraph G determined by the paths appearing in x, y, u.

Different:

4. Look at finite F and its T such that $CL_K(F, T)$ is a subalgebra of $L_K(E)$.
5. F is no-exit and all the vertices of its cycles are in T by construction.
6. Thus $CL_K(F, T)$ is directly finite.
7. So $yx = u$ in $CL_K(F, T)$ and thus in $L_K(E)$ too. Done.

Corollaries

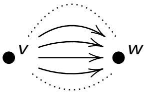
Idea of “localizing”: more general than just for finiteness.

For example. Proof of the Abrams-Rangaswamy result

$L_K(E)$ regular iff E acyclic.

Where will the trace take us next?

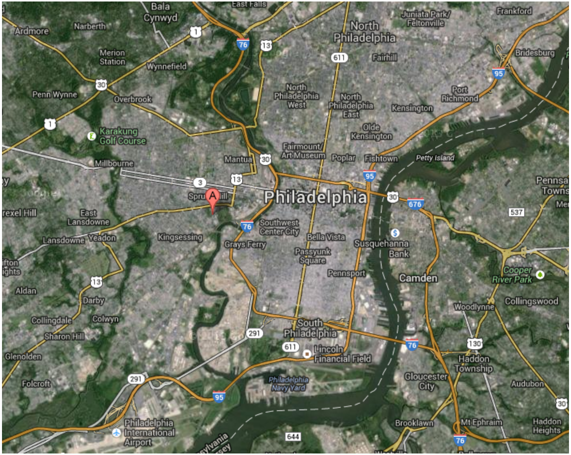
$$
\begin{array}{ccc}
\boxed{L_K(E) \text{ is (directly) finite}} & \begin{array}{c}\not\Rightarrow \\ \Leftarrow\end{array} & \boxed{L_K(E) \text{ has a faithful trace}} \\
\Updownarrow & & \Updownarrow \\
\boxed{E \text{ is no-exit}} & \Leftarrow & \boxed{E \text{ is no-exit and } \underline{\qquad\qquad} \text{ ?}}
\end{array}
$$



No exits here.

No trace since value of $t(v) \geq nt(w)$ for all n.

Local home



http://www.usciences.edu/∼lvas and arXiv.