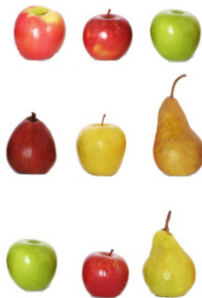


The graded structure of graph algebras (and their classification)

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classification



Our universe



graph C*-algebras



Groupoid C*-algebras



Leavitt path algebras

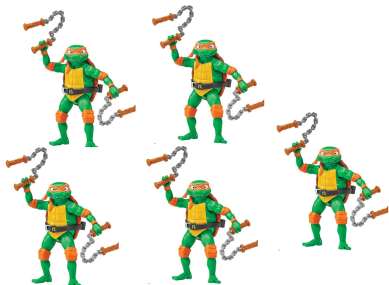


Steinberg algebras



symbolic dynamics applications...

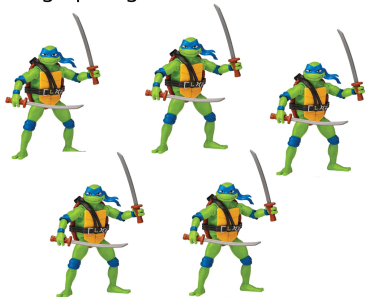
There also are mutants...



graph algebras



groupoid algebras



classification questions



various generalizations

My path to some of these algebras...

... and related questions may give you some indication of the flavor these talks will have.



graph algebras and
their classification

My current research story starts in late 2000s...

... when I met Gonzalo Aranda Pino...



who worked with **Leavitt path algebras** at that time.

and had similar interests in traveling



Soon after, I met another companion...

... who was really into **grading** of the algebras as well as their **classifications**.

Roozbeh Hazrat (middle) and his **Graded Classification Conjecture**:

Two graph algebras are graded isomorphic \Leftrightarrow their graded Grothendieck groups are pointed isomorphic.



The plan for three lectures

1



grading and



talented



Grothendieck

2



quotient



porcupine



porcupine-quotient

3



classification



disjoint cycles

Lecture 2 and a more general goal

The porcupine-quotient construction can be seen as defining E/F so that

$$\mathcal{A}(E/F) \cong_{\text{gr}} \mathcal{A}(E)/\mathcal{A}(F)$$

where $\mathcal{A}(_)$ stand for either $L_K(_)$ or for $C^*(_)$ (where $F \leq E$ is via admissible pairs).

A more general goal. Let \mathcal{A} be a functor from a category of combinatorial objects to the category of algebras.

If \mathcal{G} and \mathcal{H} are two combinatorial objects such that \mathcal{H} is a subobject of \mathcal{G} , strive to define the quotient \mathcal{G}/\mathcal{H} such that

$$\mathcal{A}(\mathcal{G}/\mathcal{H}) \cong \mathcal{A}(\mathcal{G})/\mathcal{A}(\mathcal{H})$$

holds.



Lecture 3 and a more general goal

We introduce an equivalence relation \approx on the class of graphs we consider which

Describe the graded isomorphism graph algebra class.

In other words, we have that

$$\mathcal{A}(E) \cong_{\text{gr}} \mathcal{A}(F) \text{ if and only if } E \approx F$$

A more general goal. Let \mathcal{A} be a functor from a category of combinatorial objects to the category of algebras.

Specify the equivalence relation \approx so that

$$\mathcal{A}(\mathcal{G}) \cong \mathcal{A}(\mathcal{H}) \Leftrightarrow \mathcal{G} \approx \mathcal{H}$$

for any pair of combinatorial objects \mathcal{G} and \mathcal{H} .



The Grothendieck group of a ring

1. Consider the isomorphism class $[P]$ of a finitely generated projective module P . Set of all such classes is a **monoid** (we add by

$$[P] + [Q] = [P \oplus Q]$$

and $[0]$ is the identity) usually denoted by $\mathcal{V}(R)$.

2. Force the cancellativity to hold on such a monoid, and
3. Complete to a group to get $K_0(R)$.

















Examples.

1. If K is a field, $\mathcal{V}(K) \cong \{0, 1, 2, \dots\}$ and so $K_0(K) \cong \mathbb{Z}$.

K_0 does not classify rings

2. If R is a ring such that $R \oplus R \cong R$, then $2[R] = [R]$ holds in $\mathcal{V}(R)$, so $[R] = 0$ when we force cancellativity. Hence, $K_0(R) = 0$ (and $R \not\cong 0$).

For **graph algebras**, K_0 can be *refined* by considering the grading and then there is a chance that this refined version **classifies** the algebras.

<p>1. Motorcycles 2 axles, 2 or 3 tires</p> 	<p>2. Passenger Cars 2 axles, can have 1- or 2-axle trailers</p> 	<p>3. Pickups, Panels, Vans 2 axles, 4-tire single units Can have 1 or 2 axle trailers</p> 	<p>4. Buses 2 or 3 axles, full length</p> 
<p>5. Single Unit 2-Axle Trucks 2 axles, 6 tires (3x2 rear tires), single-unit</p> 	<p>6. Single Unit 3-Axle Trucks 3 axles, single unit</p> 	<p>7. Single Unit 4 or More-Axle Trucks 4 or more axles, single unit</p> 	<p>8. Single Trailer 3- or 4-Axle Trucks 3 or 4 axles, single trailer</p> 
<p>9. Single Trailer 5-Axle Trucks 5 axles, single trailer</p> 	<p>10. Single Trailer 6 or More-Axle Trucks 6 or more axles, single trailer</p> 	<p>11. Multi-Trailer 5 or Less-Axle Trucks 5 or less axles, multiple trailers</p> 	<p>12. Multi-Trailer 6-Axle Trucks 6 axles, multiple trailers</p> 
<p>13. Multi-Trailer 7 or More-Axle Trucks 7 or more axles, multiple trailers</p> 	<p>14. Multi-Trailer 8 or More-Axle Trucks 8 or more axles, multiple trailers</p> 		

Graphs and paths

A directed graph E consists of a set of vertices E^0 , a set of edges E^1 , and the source and the range maps \mathbf{s} and \mathbf{r} defined on E^1 .

$$\bullet_{\mathbf{s}(e)} \xrightarrow{e} \bullet_{\mathbf{r}(e)}$$

A **path** of a directed graph E is a sequence $e_1 \dots e_n$ of edges such that

the range of e_i is the source
of e_{i+1}

for $i = 1, \dots, n - 1$.

$$\bullet \xrightarrow{e_1} \bullet \xrightarrow{e_2} \bullet \dots \bullet \xrightarrow{e_n} \bullet$$

Such path has the length n . A vertex \bullet has length zero.



Path algebra

Paths can be multiplied by **concatenation** and this yields a definition of the **path algebra** $P_K(E)$ of E over K .

Alternatively, $P_K(E)$ can be defined as a **free K -algebra** with vertices and edges as generators subject to

$$\begin{array}{l} V \quad vw = 0 \text{ if } v \neq w \text{ and } vv = v, \text{ and} \\ E1 \quad s(e)e = er(e) = e. \end{array}$$

$$s(e) \bullet \xrightarrow{e} \bullet r(e)$$

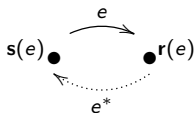
The path algebra is nice, but to get to a C^* -algebra we need to have an involution and a norm. We are adding more structure.

Ghost paths

1. Add the **ghost edges** – elements of the form e^* for $e \in E^1$.

2. Add $E_2 = E_1^*$

$$E_2 \quad r(e)e^* = e^*s(e) = e^*$$



The vertices are **selfadjoint**: $v^* = v$ for $v \in E^0$.

Example

If E is

$$\bullet_u \xrightarrow{e} \bullet_v \xrightarrow{f} \bullet_w$$

some of the “obvious” products are

$$e^*f = 0, \quad f^*u = 0, \quad ef^* = 0, \quad ue^* = 0.$$

There are some not so obvious products. For example,

What are e^*e and f^*f (if anything)?
What are ee^* and ff^* (if anything)?

To understand the answers, we briefly digress to...

... projections and partial isometries

In a $*$ -ring, an idempotent ($pp = p$) and selfadjoint ($p^* = p$) element is called a **projection**.



(So, the vertices are projections.)

An element x is a **partial isometry** if $xx^*x = x$. In this case, $p = xx^*$ and $q = x^*x$ are projections and

$$px = x \text{ and } xq = x$$

Isn't this just as $\mathbf{s}(e)e = e$ and $\mathbf{er}(e) = e$?

Because of this, one can think of p as “the source” and q as “the range” of x .

This leads us to the last two axioms. First, CK1.

1. One wants **edges to be partial isometries**. So, one requires that

$$e^*e = r(e)$$

since then $ee^*e = er(e) = e$ by E1.

In this case, e^* is also a partial isometry.

2. The edges have **mutually orthogonal** “sources”. This ends up being equivalent by requiring that $e^*f = 0$ for $e \neq f$.

$$e^*f = e^*ee^*ff^*f = e^*(ee^*)(ff^*)f = e^*0f = 0.$$

The two steps are combined in

$$\text{CK1} \quad e^*e = r(e) \text{ and } e^*f = 0 \text{ if } e \neq f.$$

Lastly, CK2.

3. We keep track of the number of other edges the source of an edge emits. So, we require that the following holds.

$$\text{CK2} \quad v = \sum ee^* \quad \text{where the sum is taken over } e \in \mathbf{s}^{-1}(v).$$

for every vertex v which emits at least one but finitely many edges. We say that such v is **regular**.

For example, in the graph



$$e^*e = v, \quad f^*f = w, \quad ee^* = u, \quad ff^* = v.$$

And in the graph



$$v = ee^* + ff^* \quad (\text{so } ee^* \neq v \text{ and } ff^* \neq v).$$

We got ourselves some algebras

K = field. The **Leavitt path algebra** $L_K(E)$ of E is a free K -algebra on vertices, edges and ghost edges subject to the following.

$$V \quad vv = v \text{ and } vw = 0 \text{ if } v \neq w,$$

$$E1 \quad e = s(e)e = er(e)$$

$$E2 \quad e^* = e^*s(e) = r(e)e^*$$

$$CK1 \quad e^*e = r(e), \text{ and } e^*f = 0 \text{ if } e \neq f$$

$$CK2 \quad v = \sum_{e \in s^{-1}(v)} ee^* \text{ if } v \text{ is regular.}$$

If $K = \mathbb{C}$, the **graph C^* -algebra** $C^*(E)$ of E is the completion of $L_{\mathbb{C}}(E)$. It is the universal C^* -algebra with

vertices = generating projections
edges = partial isometries and CK1, CK2, CK3.

(CK3 follows from E2 so we do not need to require it for LPAs.)

Example 1 – Matrices

The **path algebra** of $u \bullet \xrightarrow{e} \bullet v \xrightarrow{f} \bullet w$ is the algebra of **upper triangular** matrices by

$$\begin{bmatrix} u & e & ef \\ 0 & v & f \\ 0 & 0 & w \end{bmatrix}$$

And $L_K(E)$ corresponds to the set of **all** matrices over K .

$$\begin{bmatrix} u & e & ef \\ e^* & v & f \\ (ef)^* & f^* & w \end{bmatrix}$$

Graph C*-algebra: $M_3(\mathbb{C})$.

Generalizes to n -line.



and its graph algebra isomorphic to $M_n(K)$



Example 2 – Loop



Paths:

$$v = 1 = e^0, e = e^1, e^2, e^3, \dots$$

Path algebra: Polynomials with coefficients in K by $e \rightsquigarrow x$.

Ghost edge e^* and $e^* \rightsquigarrow x^{-1}$.

Leavitt path algebra: Laurent polynomials $K[x, x^{-1}]$.

Graph C^* -algebra: continuous functions on a circle $C(\mathbb{T})$.



Example 3 – two-petal rose



Path algebra: Free algebra on x and y (like polynomials but without $xy = yx$) where $e \rightsquigarrow x$ and $f \rightsquigarrow y$.

Ghost edges e^*, f^* . CK1 is $e^*e = f^*f = 1$

CK2 is $ee^* + ff^* = 1$ (so e and f have left inverses but not the right inverses). The pair of maps

$$a \mapsto (e^*a, f^*a) \quad \text{and}$$

$$(a, b) \mapsto ea + fb$$

are mutually inverse isomorphisms ensuring that

$$L_K(E) \oplus L_K(E) \cong L_K(E).$$

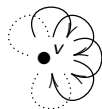


Example 3 – roses

Leavitt path algebra is known as the Leavitt algebra $L(1, 2)$.
It is a universal example of a ring R with $R^2 \cong R$.

Graph C^* -algebra: Cuntz algebra \mathcal{O}_2 .

Generalizes to n -rose.



Path algebra: free K algebra on n variables.

Leavitt path algebra: Leavitt algebra $L(1, n)$ – universal
example of a ring with $R^n \cong R$.

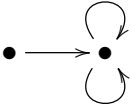
Graph C^* -algebra: Cuntz algebra \mathcal{O}_n

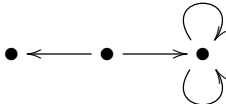
Some research trends. 1. Characterizations

For a given algebra property P , find a graph property Q so that
the algebra has a property $P \Leftrightarrow$ the graph has a property Q .

For example,

1. $L_K(E)$ has the identity $\Leftrightarrow E$ has finitely many vertices.
2. $L_K(E)$ is finite dimensional over $K \Leftrightarrow E$ is finite and has no cycles.
3. Characterization of $L_K(E)$ being **simple**.

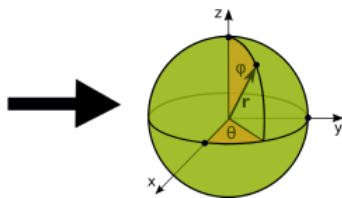
Gives us that  has a simple LPA and

 does not.



Research trends. 2. Generalizations

1. **Separated graphs, higher rank graphs, weighted graphs...**
2. **Non-field coefficients.** The coefficients may not have inverses.
3. **Steinberg algebras** – algebras over groupoid instead of graphs.



Research trends. 3. Classifications

1. **Field Dependence.** If $L_K(E) \cong L_K(F)$ is $L_{K'}(E) \cong L_{K'}(F)$?
2. **Isomorphism Conjecture.** $L_{\mathbb{C}}(E) \cong L_{\mathbb{C}}(F)$ as algebras (as rings) $\Leftrightarrow C^*(E) \cong C^*(F)$.
3. **Graded Classification Conjecture.** Graph algebras are graded isomorphic \Leftrightarrow their graded Grothendieck groups are (pointed) isomorphic.



classification



How to compute K_0 of a LPA?

Good news: you can do it “via graph” not “via projectives”.

For a row-finite E , define a monoid M_E , called the **graph monoid**, generated by the elements $[v]$ (\longleftrightarrow iso class of $vL_K(E)$) where v is a vertex, subject to relations

$$[v] = \sum_{e \in s^{-1}(v)} [r(e)]$$

whenever v is regular.

Why? Because left multiplication by e is an iso of fin. gen. proj. $r(e)L_K(E) = e^*eL_K(E)$ and $ee^*L_K(E)$. So, if v is regular, then

$$[v] = \left[\sum_{e \in s^{-1}(v)} ee^* \right] = \sum_{e \in s^{-1}(v)} [ee^*] = \sum_{e \in s^{-1}(v)} [e^*e] = \sum_{e \in s^{-1}(v)} [r(e)].$$

M_E and G_E

Then, form the Grothendieck group G_E of M_E and we have that

$$\begin{aligned}M_E &\cong \mathcal{V}(L_K(E)) \\ G_E &\cong K_0(L_K(E))\end{aligned}$$

For example, if $E = \begin{array}{c} \curvearrowright \\ \bullet^v \\ \curvearrowleft \end{array}$, then



$M_E = \langle v \mid v = v + v \rangle$. Its Grothendieck group is trivial since $v = v + v \Rightarrow 0 = v$.

Enter the grading to the rescue!

Graph algebras are naturally **graded** by \mathbb{Z} .



If Γ is a group, a ring R is Γ -**graded** if

$$R = \bigoplus_{\gamma \in \Gamma} R_{\gamma} \quad \text{such that} \quad R_{\gamma} R_{\delta} \subseteq R_{\gamma\delta}.$$

Grading of graph algebras...



ring



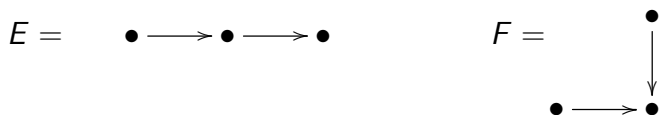
graded ring

For a $L_K(E)$, $L_K(E)_n = \text{span} \{pq^* \mid |p| - |q| = n\}$.

For $z \in \mathbb{T}$, γ_z^E is given by $\gamma_z^E(p_v) = p_v$ and $\gamma_z^E(s_e) = zs_e$. The **gauge action** γ^E on \mathbb{T} is given by $\gamma^E(z) = \gamma_z^E$. This action determines a \mathbb{Z} -grading of $C^*(E)$ so that

$$C^*(E)_n = \{x \in C^*(E) \mid \int_{\mathbb{T}} z^{-n} \gamma_z^E(x) dz = x\}.$$

Grading helps



$$L_K(E) \cong L_K(F) \cong \mathbb{M}_3(K)$$

as algebras. However,

$$L_K(E) \cong_{\text{gr}} \mathbb{M}_3(K)(0, 1, 2) \not\cong_{\text{gr}} L_K(F) \cong_{\text{gr}} \mathbb{M}_3(K)(0, 1, 1)$$

as graded algebras.

Appendix has more information on $\mathbb{M}_n(K)(\gamma_1, \dots, \gamma_n)$.

Graded version of the K_0 -group classifies better

If R is Γ -graded, replace “**projective**” by “**graded projective**” and repeat the construction for $\mathcal{V}(R)$, get $\mathcal{V}^\Gamma(R)$ with the Γ -action induced by $\gamma[P] = [P(\gamma)]$.

Then get the **Grothendieck Γ -group** $K_0^\Gamma(R)$.

$K_0^\Gamma(R)$ is a $\mathbb{Z}[\Gamma]$ -**module**.
Because of this additional structure, K_0^Γ

classifies better.



For graph algebras,
 $\Gamma = \langle t \rangle \cong \mathbb{Z}$, $\mathbb{Z}[\Gamma]$ is $\mathbb{Z}[t, t^{-1}]$.

$K_0^\Gamma(L_K(E))$ is a $\mathbb{Z}[t, t^{-1}]$ -module.

Graph-only approach

Recall that M_E and G_E are defined using a graph E only. We want the Γ -versions.

For a group Γ and a graph E , one wants a monoid M_E^Γ ,

the graph Γ -monoid.



a.k.a

the talented monoid.

The Grothendieck group of M_E^Γ is the **the graph Γ -group** G_E^Γ .

$$M_E^\Gamma \cong \mathcal{V}^\Gamma(L_K(E))$$

$$G_E^\Gamma \cong K_0^\Gamma(L_K(E))$$

Computing M_E^Γ and G_E^Γ

Let $\Gamma = \langle t \rangle$. M_E^Γ has the same generators $[v]$ as M_E and just one slight change in the defining relation. It is

$$[v] = \sum_{e \in S^{-1}(v)} t[r(e)]$$

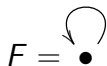
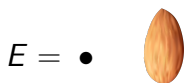
for all regular vertices v .

Why t ? Because $t = t^1$ is the length of the path e from v to $r(e)$ and the left multiplication by e gives

$$(-1)e^*eL_K(E) \cong_{\text{gr}} ee^*L_K(E) \Rightarrow t[e^*eL_K(E)] = [(-1)e^*eL_K(E)]$$

$$[v] = \left[\sum_{e \in S^{-1}(v)} ee^* \right] = \sum_{e \in S^{-1}(v)} [ee^*] = \sum_{e \in S^{-1}(v)} t[e^*e] = \sum_{e \in S^{-1}(v)} t[r(e)].$$

Examples



$$M_E^{\Gamma} \cong \mathbb{Z}^+[t, t^{-1}]$$

$$G_E^{\Gamma} \cong \mathbb{Z}[t, t^{-1}]$$

$$M_F^{\Gamma} \cong \mathbb{Z}^+[t, t^{-1}]/(t=1) = \mathbb{Z}^+$$

$$G_F^{\Gamma} \cong \mathbb{Z}$$

Just like in the examples with



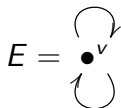
and



One more example

Let us compare M_E and M_E^Γ

for the rose



- ▶ $M_E = \langle v \mid v = v + v \rangle$ so $G_E = 0$.
- ▶ $M_E^\Gamma = \langle v \mid v = tv + tv \rangle$ and G_E^Γ is isomorphic to $\mathbb{Z}[\frac{1}{2}]$ if we identify v with 1 and t with $\frac{1}{2}$.

In general, M_E^Γ is **cancellative**.

All seems so good...

... that Roozbeh formed the following question (circa 2011):

Is for any two graphs E and F ,

$L_K(E) \cong_{\text{gr}} L_K(F)$
as graded algebras

iff

$G_E^\Gamma \cong G_F^\Gamma$

as pointed Γ -groups?



Classification

Let us look into “pointed” next...

Structure of G_E^Γ

- ▶ An abelian group
- ▶ with a Γ -action, and
- ▶ a **pre-order** \leq (from $x \leq y$ iff x is a summand of y).

If E^0 is finite, $u = \sum_{v \in E^0} [v]$ is an **order-unit** (meaning that for every $x \in G_E^\Gamma$, there is $a \in \mathbb{Z}^+[\Gamma]$ such that $-au \leq x \leq au$).

G_E^Γ considered with an order-unit is said to be **pointed**.



“Being pointed” matters

$$E = \bullet \longrightarrow \bullet \longrightarrow \bullet$$

$$F = \begin{array}{c} \bullet \\ \downarrow \\ \bullet \longrightarrow \bullet \end{array}$$

$$(G_E^\Gamma, [1]) \cong (\mathbb{Z}[t, t^{-1}], 1 + t + t^2) \text{ and} \\ (G_F^\Gamma, [1]) \cong (\mathbb{Z}[t, t^{-1}], 1 + t + t).$$

If f were to be an iso of pointed groups, we would have

$$1 + t + t^2 = (1 + t + t^2)f(1) = f(1 + t + t^2) = 1 + t + t \Rightarrow$$

$$t^2 = t \text{ holds in } \mathbb{Z}[t, t^{-1}]$$

So,

$$(G_E^\Gamma, [1]) \not\cong (G_F^\Gamma, [1]).$$