Lecture 2. Porcupine-quotient and the fourth primary color

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hedgehog

porcupine

porcupine-quotient

three primary colors the fourth one

Composition series

A very general question: given a **composition series**

$$0 = I_0 \lneq I_1 \lneq \ldots \lneq I_n = R$$

(so I_{i+1}/I_i is simple).

how to patch the information from its components to recover the information on R?

If $R =$ **finitely generated abelian group,** this goes really well.

We cross our fingers and consider R to be a **graph algebra** with its grading and $I_i = I(H_i, S_i)$ for an admissible pair (H_i, S_i).

Describing the ideals and their quotients

$H \subseteq E^0$ is **hereditary** if

$$u \in H \text{ and there is a path from } u \text{ to } v \quad \Rightarrow \quad v \in H.$$

H is **saturated** if

$$v \text{ is a regular vertex and } \mathbf{r}(\mathbf{s}^{-1}(v)) \subseteq H \quad \Rightarrow \quad v \in H.$$

For E row-finite,

I is graded iff $H = I \cap E^0$ is hereditary and saturated

When E has infinite emitters the set of **breaking vertices** of H is

$$B_H = \{v \in E^0 - H \mid v \text{ is an infinite emitter and}$$
$$0 < |\mathbf{s}^{-1}(v) \cap \mathbf{r}^{-1}(E^0 - H)| < \infty\}.$$

Admissible pairs

For $v \in B_H$, let

$$v^H = v - \sum_{e \in \mathbf{s}^{-1}(v) \cap \mathbf{r}^{-1}(E^0 - H)} ee^*$$

An **admissible pair** is a pair (H, S) where $H \subseteq E^0$ is hereditary and saturated and $S \subseteq B_H$.
For (H, S), the ideal $I(H, S)$ generated by $H \cup \{v^H \mid v \in S\}$ is graded and

I is graded iff $I = I(H, S)$ for $H = I \cap E^0$ and $S = \{v \in B_H \mid v^H \in I\}$.

The set of admissible pairs is a lattice with respect to

$$(H, S) \leq (G, T) \quad \text{if } H \subseteq G \text{ and } S \subseteq G \cup T.$$

The correspondence $(H, S) \mapsto I(H, S)$ is a lattice isomorphism.

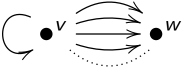
The quotient

Ara-Moreno-Pardo 2006 row-finite,

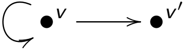
Tomforde 2007, non-row-finite.

$$\begin{aligned}(E/(H,S))^0 &= E^0 - H \cup \{v' \mid v \in B_H - S\},\\ (E/(H,S))^1 &= \{e \in E^1 \mid \mathbf{r}(e) \notin H\} \cup\\ &\quad \{e' \mid e \in E^1 \text{ and } \mathbf{r}(e) \in B_H - S\},\end{aligned}$$

and $\mathbf{s}(e') = \mathbf{s}(e)$, $\mathbf{r}(e') = \mathbf{r}(e)'$.

Examples. Let E be  and $H = \{w\}$. Then

$B_H = \{v\}$ and

$E/(H, B_H)$ is  and $E/(H, \emptyset)$ is  .

Getting rid of breaking vertices

Out-split. v s.t. $\mathbf{s}^{-1}(v) \neq \emptyset$, $\mathcal{P} = (\mathcal{E}_1, \ldots, \mathcal{E}_n)$ a partition of $\mathbf{s}^{-1}(v)$. $E_{v,\mathcal{P}}$ has

$$\text{vertices} = E^0 - \{v\} \cup \{v_1, \ldots, v_n\},$$

$$\text{edges} = \{f_1, \ldots, f_n \mid f \in E^1, \mathbf{r}(f) = v\} \cup \{f \in E^1 \mid \mathbf{r}(f) \neq v\},$$

$\mathbf{r}(f_i) = v_i$ and $\mathbf{s}(g)$ is

- v_i if $g = f_j \in \mathcal{E}_i$ (so $\mathbf{s}(f) = \mathbf{r}(f) = v$)
- v_i if $g = f \in \mathcal{E}_i$ (so $\mathbf{s}(f) = v$ and $\mathbf{r}(f) \neq v$)
- $\mathbf{s}(f)$ if $g = f_j$ and $\mathbf{s}(f) \neq v$ (so $\mathbf{r}(f) = v$)
- $\mathbf{s}(f)$ if $g = f$ and $\mathbf{s}(f) \neq v$ (so $\mathbf{r}(f) \neq v$).

$\phi : E^0 \cup E^1 \to L_K(E_{v,\mathcal{P}})$ is

$$\phi(v) = \sum_{i=1}^n v_i,\ \phi(w) = w \text{ for } w \in E^0 - \{v\},$$
$$\phi(f) = \sum_{i=1}^n f_i \text{ if } \mathbf{r}(f) = v \text{ and } \phi(f) = f \text{ otherwise},$$

Out-splits cont.

If v is regular or if it is an infinite emitter with all but one partition set finite (say the last can be infinite), then ϕ is onto and the inverse ψ of ϕ can be defined by

$$\psi(v_i) = \sum_{e \in \mathcal{E}_i} ee^* \text{ for } i = 1, \ldots, n-1, \quad \psi(v_n) = v - \sum_{i=1}^{n-1} \psi(v_i)$$

$$\psi(w) = w \text{ for } w \in E^0 - \{v_1, \ldots, v_n\}, \text{ and}$$

$$\psi(f) = f\psi(\mathbf{r}(f)).$$

For example,

When this does *not* work?

If v is an infinite emitter and more than one partition set is infinite, ϕ may not be onto.
For example,

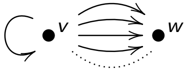
the first graph has the composition series length 3 and the second 4. So, the algebras cannot be graded isomorphic.

Getting rid of breaking vertices

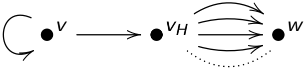
For $v \in B_H$, do the outsplit with the partition

$$\mathbf{s}^{-1}(v) \cap \mathbf{r}^{-1}(H) \text{ and } \mathbf{s}^{-1}(v) \cap \mathbf{r}^{-1}(E^0 - H).$$

Let us call the resulting graph E_v and the two vertices in which v splits v_H and v.

Example. $E =$ 

, $H = \{w\}$, $B_H = \{v\}$.

$E_v =$ 

So, if we are OK with changing graphs as long as their algebras stay in the same graded $*$-isomorphism classes,

we can assume there are no breaking vertices.

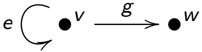
Hedgehog graph

$$F_1(H) = \{e_1 \dots e_n \text{ is a path of } E \mid \mathbf{r}(e_n) \in H, \mathbf{s}(e_n) \notin H\},$$
$$\overline{F_1}(H) \text{ is a copy of } F_1(H),$$

Then

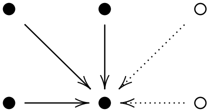
$$E_H^0 = H \cup F_1(H), \text{ and}$$
$$E_H^1 = \{e \in E^1 \mid \mathbf{s}(e) \in H\} \cup \overline{F_1}(H) \text{ with}$$

$$\mathbf{s}(\overline{p}) = p, \mathbf{r}(\overline{p}) = \mathbf{r}(p) \text{ for } \overline{p} \in \overline{F_1}(H).$$

Examples. Let E be 

and $H = \{w\}$. Then

$F_1 = \{e^n g | n = 0, 1, \dots\}$ and the hedgehog is



Some positives and some negatives

Positives

$$L_K(\text{quotient graph}) \cong_{\text{gr}} L_K(E)/I(H,S)$$

$$L_K(\text{hedgehog}) \cong I(H,S)$$

Negatives

$$L_K(\text{hedgehog}) \not\cong_{\text{gr}} I(H,S)$$

Indeed, if E is $\quad e \circlearrowleft \bullet^v \xrightarrow{g} \bullet^w$ and $H = \{w\}$,

the path $eeeg$ (of **length 4**) of $I(H)$ corresponds to an edge $\overline{eeeg}$ so it has **length 1** in the LPA of the hedgehog.

How to fix this?
Make the "spines" longer and get...

Porcupine graph P_H (2021)

Keep the definition of F_1.

For each $e \in F_1 \cap E^1$, let w^e be a **new vertex** and f^e a **new edge** such that $\mathbf{s}(f^e) = w^e$ and $\mathbf{r}(f^e) = \mathbf{r}(e)$.

For each path $p = eq$ where $q \in F_1$ and $|q| \geq 1$, add a **new vertex** w^p and a **new edge** f^p such that $\mathbf{s}(f^p) = w^p$ and $\mathbf{r}(f^p) = w^q$. Then let

$$P_H^0 = H \cup \{w^p \mid p \in F_1(H)\} \text{ and}$$

$$P_H^1 = \{e \in E^1 \mid \mathbf{s}(e) \in H\} \cup \{f^p \mid p \in F_1(H, S)\}$$

We get a graded iso by

$$w^p \leftrightsquigarrow pp^*, p \in F_1,$$

$$f^{ep} \leftrightsquigarrow epp^*, p \in F_1,$$

Example

Let E be $\quad e \circlearrowright \bullet^{v} \xrightarrow{g} \bullet^{w}$ and $H = \{w\}$. Then

$$F_1(H) = \{g, eg, eeg, eeeg, \dots\}$$

and the porcupine is

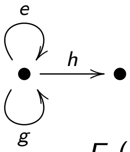
$$\cdots\cdots\to \bullet^{w^{e^2 g}} \xrightarrow{f^{e^2 g}} \bullet^{w^{eg}} \xrightarrow{f^{eg}} \bullet^{w^{g}} \xrightarrow{f^{g}} \bullet^{w}$$

We unroll the loop and make it into a single spine.

The graded iso is

$$eeeg \leftrightsquigarrow f^{e^3 g} f^{e^2 g} f^{eg} f^{g}.$$

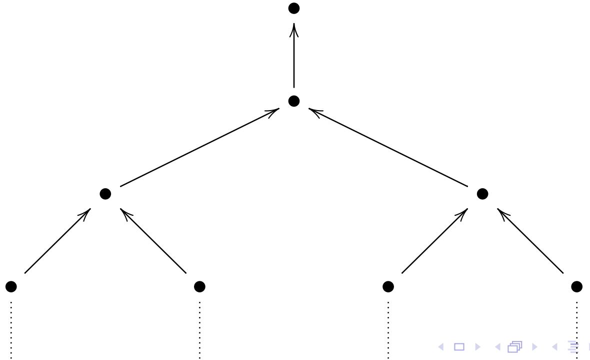
Another example



and $H = \{w\}$. Then

$$F_1(H) = \{h, eh, gh, eeh, egh, geh, ggh, eeeh, \ldots\}$$

and the **hedgehog** graph is the **same** as in the previous example. The, **porcupine**, on the other hand, is



Porcupine-quotient (2023)

Given $(H, S) \leq (G, T)$ (this means $H \subseteq G$ and $S \subseteq G \cup T$)

we want to do the **quotient** construction with (H, S) but **relative** to the **porcupine** graph of (G, T).

and we want to get

$$L_K((G, T)/(H, S)) \cong_{\text{gr}} I(G, T)/I(H, S).$$

The definition of G/H

$$F_1(G-H) = \{e_1 e_2 \dots e_n \text{ is a path of } E \mid \mathbf{r}(e_n) \in G-H,$$
$$\mathbf{s}(e_n) \notin (G-H)\}$$

The set of **vertices** is

$$(G-H) \cup \{w^p \mid p \in F_1(G-H)\}.$$

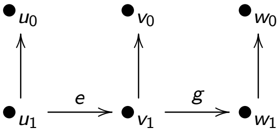
The set of **edges** is

$$\{e \in E^1 \mid \mathbf{r}(e) \in G-H \text{ and } \mathbf{s}(e) \in G-H\} \cup$$
$$\{f^p \mid p \in F_1(G-H)\}.$$

The **s** and **r** maps are defined analogously as for P_G.

Example 1

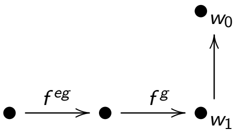
Let E be



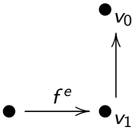
and let $H = \{w_0, w_1\}$ and $G = H \cup \{v_0, v_1\}$.

$\emptyset \leq H \leq G \leq E^0$ has "simple" porcupine-quotients.

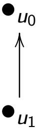
P_H is



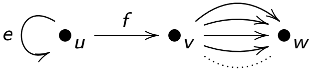
G/H is



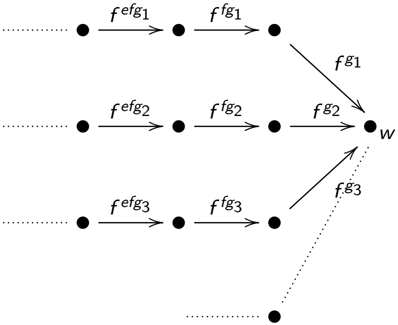
E/G is

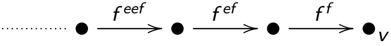


Example 2

Let E be 

If $H = \{w\}$ and $G = \{v, w\}$, then $\emptyset \leq H \leq G \leq E^0$ and

P_H is 

G/H is 

E/G is 

Everything agrees

If $(H, S) = (\emptyset, \emptyset)$,

porcupine-quotient is porcupine

If $G = E^0$ (so $T = \emptyset$),

porcupine-quotient is quotient

The best part

For graph monoids and pre-ordered monoid maps:

$$M_{(G,T)/(H,S)} \cong J(G,T)/J(H,S)$$

For talented monoids and pre-ordered Γ-monoid maps:

$$M^{\Gamma}_{(G,T)/(H,S)} \cong J^{\Gamma}(G,T)/J^{\Gamma}(H,S).$$

So, the requirements that a **composition series** of any of these exists are equivalent:

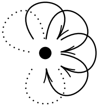
- admissible pairs of E,
- graded ideals of $L_K(E)$,
- order-ideals of M_E,
- Γ-order-ideals of M_E^{Γ}.

Graded simple LPAs

$L_K(E)$ is graded simple iff
no nontrivial and proper admissible pairs

i.e. composition series for E (equiv. $L_K(E)$) have **length 1**.

Three basic examples $\leftrightsquigarrow$ three primary colors.

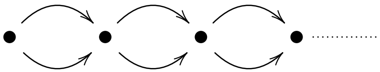


Three types of "terminal" vertices

1. A **sink** connects to no other vertex in the graph except, trivially, to itself.
2. The vertices on a **cycle without exits** do not connect to any vertices outside of the cycle.
3. An **extreme cycle** is a cycle such that the range of every exit from the cycle connects back to a vertex in the cycle. The vertices in such a cycle c connect only to the vertices on cycles in the same "cluster" as c.

The fourth type

The algebra of



is graded simple but E has no sinks, no-exit nor extreme c-s.

$T(V) =$ **tree** of V, vertices to which $v \in V$ emits paths,
$R(V) =$ **root** of V, vertices from which $v \in V$ receives paths.

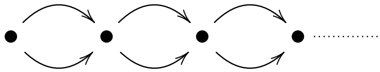
An infinite path α of E is **terminal** if

- no element of $T(\alpha^0)$ is an infinite emitter
- or on a cycle and
- for every infinite path β with $\mathbf{s}(\beta) \in \alpha^0$, $T(\beta^0) \subseteq R(\beta^0)$.

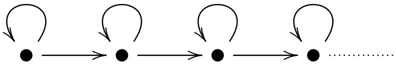
Everything that originates in α comes back to α.

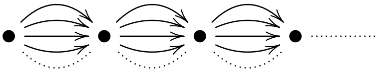
Examples

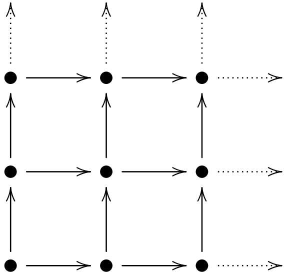
Each infinite path is terminal:



None of the infinite paths is terminal:







Four color characterization of graded simplicity

The graph algebra of E is **graded simple** (E is cofinal) iff exactly one of the following holds.

1. $E^0 = \overline{\{v\}}$ for v a sink. If so, E is row-finite and acyclic.
2. $E^0 = \overline{c^0}$ for a cycle c without exits. If so, E is row-finite and c is the only cycle.
3. $E^0 = \overline{c^0}$ for an extreme cycle c. If so, every cycle is extreme and every inf. emitter is on a cycle.
4. $E^0 = \overline{\alpha^0}$ for a terminal path α. If so, E is row-finite and acyclic.

Corollaries. $L_K(E)$ is **simple** iff 1, 3, or 4 hold.

$L_K(E)$ is **(graded) purely infinite simple** iff 3 holds.

Necessary cond. for having a composition series

Let

$$\mathrm{Ter}(E) = \overline{\text{terminal vertices}} =$$

$$\overline{\text{sinks}} \cup \overline{\text{no-exits}} \cup \overline{\text{extremes}} \cup \overline{\text{terminal paths}}.$$

If E^0 is finite, $I(\mathrm{Ter}(E))$ has been known as I_{lce}.

If a graph E has a composition series, the following hold.

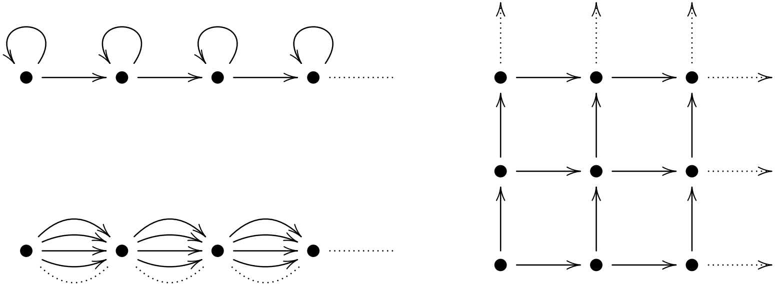
(a) $\mathrm{Ter}(E)$ is **nonempty**.

(b) The set of terminal vertices of E contains **finitely many clusters.**

(c) The set of breaking vertices of $\mathrm{Ter}(E)$ is **finite**.

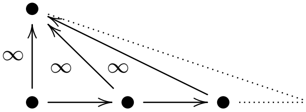
3 types of graphs not having a composition series

Ter(E) is empty:



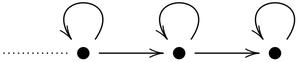
Infinitely many clusters: 

Infinitely many breaking vertices of Ter(E) :



Characterization of having a composition series

One more type of obstruction



Define the **composition quotients** F_n of E. Let $F_0 = E$. If $\mathrm{Ter}(F_n) \subsetneq F_n^0$, let

$$F_{n+1} = F_n/(\mathrm{Ter}(F_n), B_{\mathrm{Ter}(F_n)}).$$

If $\mathrm{Ter}(F_n) = F_n^0$, let $F_{n+1} = F_{n+2} = \ldots = \emptyset$.

The graph E **has a composition series** iff

1. Conditions (a), (b), and (c) hold for F_n for each n for which $F_n \neq \emptyset$.
2. There is $n \geq 0$ such that $F_{n+1} = \emptyset$ and $F_n \neq \emptyset$.

The proof is constructive. A corollary.

If $C_1, \ldots, C_n$ are clusters in $\mathrm{Ter}(E)$, start by

$$(\emptyset, \emptyset) \leq (\overline{C_1}, \emptyset) \leq (\overline{C_1} \cup \overline{C_2}, \emptyset) \leq \ldots (\mathrm{Ter}(E), \emptyset).$$

If $v_1, \ldots, v_m$ are breaking vertices of $\mathrm{Ter}(E)$, continue with

$$(\mathrm{Ter}(E), \emptyset) \leq (\mathrm{Ter}(E), \{v_1\}) \leq (\mathrm{Ter}(E), \{v_1, v_2\}) \leq \ldots$$
$$(\mathrm{Ter}(E), B_{\mathrm{Ter}(E)}).$$

"Extend" a series for F_1 by $(\mathrm{Ter}(E), B_{\mathrm{Ter}(E)})$ and append it to this.

Corollary.

Every **unital** graph algebra has a graded composition series.

Surprising? **Yes** – The algebras appear to be "wilder" than this.
No – there are fin many vertices and we are cutting nonzero many in each step.

Graph Γ-monoid and three colors

There are three types of elements of M_E^Γ : periodic, aperiodic and incomparable. If v is a vertex, the entries of the same row are equivalent.

$[v]$ is incomparable	v is not in the sat. closure of finitely many vertices on cycles.
$[v]$ is periodic	v is in the sat. closure of finitely many vertices on no-exit cycles.
$[v]$ is aperiodic	v is in the sat. closure of finitely many cycles, at least one with an exit.

M_E^Γ is periodic (resp. aperiodic, incomparable) if every nonzero elements is such.

Some corollaries

The following pairs of statements are equivalent for any (H, S) and (G, T) such that $(G, T)/(H, S)$ is cofinal.

1. $M^{\Gamma}_{(G,T)/(H,S)}$ is either periodic or incomparable.
2. The cycles of E are mutually disjoint.

1. $M^{\Gamma}_{(G,T)/(H,S)}$ is either aperiodic or incomparable.
2. Every cycle of E contains a vertex of another cycle of E.

1. $M^{\Gamma}_{(G,T)/(H,S)}$ is either periodic or aperiodic.
2. Every vertex of E is in the saturated closure of a finite set of vertices on cycles.

Finer characterization

The following pairs of statements are equivalent for any (H, S) and (G, T) such that $(G, T)/(H, S)$ is cofinal.

1. $M^{\Gamma}_{(G,T)/(H,S)}$ is periodic.
2. The cycles of E are mutually disjoint and every vertex is in the saturated closure of finitely many vertices on cycles.

1. $M^{\Gamma}_{(G,T)/(H,S)}$ is aperiodic.
2. Every cycle of E contains a vertex of another cycle of E and every vertex is in the saturated closure of a finite set of vertices on cycles.

1. $M^{\Gamma}_{(G,T)/(H,S)}$ is incomparable.
2. The graph E is acyclic.

Why am I so excited about this?

Neo

bullet

Neo moment

The more general goal mentioned last time

Let $\mathcal{A}$ be a functor from a category of combinatorial objects to the category of algebras.

If $\mathcal{G}$ and $\mathcal{H}$ are two combinatorial objects such that $\mathcal{H}$ is an subobject of $\mathcal{G}$, strive to define the quotient $\mathcal{G}/\mathcal{H}$ such that

$$\mathcal{A}(\mathcal{G}/\mathcal{H}) \cong \mathcal{A}(\mathcal{G})/\mathcal{A}(\mathcal{H})$$

holds.