Disjoint Cycles

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The Graded Classification Conjecture (GCC)

Is for any two graphs E and F,

the graph algebras are
graded isomorphic
iff
the graph Γ-groups are
pointed isomorphic?

$\Gamma = \langle t \rangle \cong \mathbb{Z}$ and K_0^Γ is a $\mathbb{Z}[\Gamma]$-module.

Order-unit $=$ the class corresponding to sum of all vertices.
Generating interval $=$ summands of classes corresponding to sums of finitely many vertices.

The state of the GCC

Hazrat (2013) – GCC holds for finite **polycephaly** graphs (every path leads to a sink, a rose or a cycle with no exits).

Ara, Pardo (2014) – a weaker version of the GCC holds for finite graphs without sources and sinks.

Eilers, Ruiz, Sims (2020) – the GCC and its C^*-algebra version hold for countable "amplified" graphs.

Known classifications (continued)

Hazrat, Vaš (circa 2016) – the involutive version of the GCC holds for row-finite graphs in which every infinite path ends in a (finite or infinite) sink or a cycle without exits and the algebra is over a "nice enough" field (like $\mathbb{C}$, for example).

Eilers, Ruiz (2025) – the GCC holds for two subclasses of the class of graphs we consider: acyclic graphs with finitely many vertices and 2-S-NE graphs with finitely many vertices.

A cautionary tale...

Oberwolfach, March 2024

What came across...

March 2024 to January 2025 – no disjoint cycles

May 22, 2025 – v1 on arXiv

June 2025 to January 2026 – no disjoint cycles

January 2026

February 2026

February 28, 2026 – v3 on arXiv

The graphs...

... with

- disjoint cycles,
- each (right) infinite path ends in a cycle, and
- finitely many cycles, sinks, and infinite emitters.

These properties ensure

- no extreme cycles (no red),
- no terminal paths (no green), and
- finite composition series (so we can do induction).

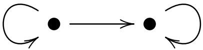
n-S-NE graphs

S-NE = every composition factor has either a **sink** ("S" is for a sink) or a **no-exit** cycle ("NE" is for no-exit).

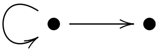
n = the number of cycles, sinks, and infinite emitters = the length of the **composition series**.

Presence of **n** $\Rightarrow$ we can do induction!

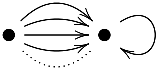
For example, four types of 2-S-NE graphs:



cycle-to-cycle



cycle-to-sink



infinite-emitter-to-cycle

infinite-emitter-to-sink

The general idea – canonical forms

Want a graph which "represents" well all graphs with algebras in the same graded isomorphism class:

a canonical form.

If we define $E \approx F$ by $E_{\text{can}} \cong F_{\text{can}}$, then we aim to have:

The main result. For composition S-NE graphs E and F, TFAE.

1. G^{Γ}s (equiv. talented monoids) are pointed isomorphic.
2. $E \approx F$
3. The algebras are graded $*$-isomorphic.

We **realize** any 1. isomorphism by a specific 3. isomorphism.

What are we doing?

We are describing the

graded $(*)$-isomorphism graph algebra class.

by finding a complete list of operations on graphs which extend to graded $(*)$-isomorphisms.

Some questions for the moves to get E_{can} out of E:

- **Are out-splits and in-split plus moves enough?**
- **If there is something else, can we even use "moves" to refer to such operations?**

The "moves"

Any $\phi : E^0 \cup E^1 \to L_K(F)$ such that

- $\phi(v)$ has degree 0 and $\phi(e)$ degree 1 (hence, the extension is **graded**),
- $\phi(v) \neq 0$ (hence, the extension is **injective**)

which we extend to $(E^1)^*$ by $\qquad \phi(e^*) = \phi(e)^*$
and such that the images of ϕ satisfy the axioms.

Then, we further extend ϕ to the entire algebra and have a **graded $*$-monomorphism.**

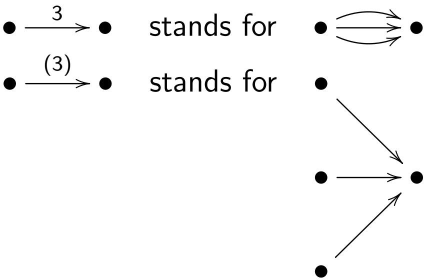
We also require that there is $\psi : F^0 \cup F^1 \to L_K(E)$ with the same properties such that ϕ and ψ compose to the identity maps of the vertex and edge sets. So, ϕ extends to a

graded $*$-isomorphism.

Examples: out-splits, in-split plus moves.

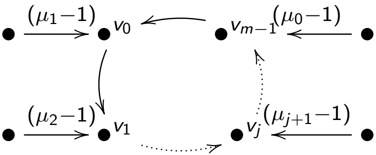
Canonical form of 1-S-NE graphs

Some notation:



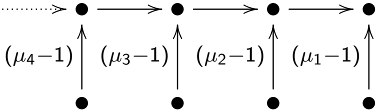
The NE case:

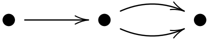
$L_K(E) \cong_{\text{gr}} \mathbb{M}_\kappa(K[x^m, x^{-m}])(\mu_0, \mu_1, \dots, \mu_{m-1})$, then E_{can} is

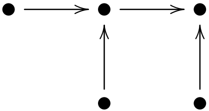


The sink case

The sink case: $L_K(E) \cong_{\text{gr}} \mathbb{M}_\kappa(K)(1, \mu_1, \mu_2, \ldots, \mu_k)$, then k=spine length and E_{can} is



$E =$  , $\quad L_K(E) \cong_{\text{gr}} \mathbb{M}_5(K)(1, 2, 2), \quad k = 2$

$E_{\text{can}} =$ 

What is the map $E \to E_{\text{can}}$?

The procedure for getting E_{can} out of a 1-S-NE E:

- (optional) Find the total out-split E_{tot} of E so that each regular vertex emits exactly one edge.
- Pick a vertex v_0 of the terminal cluster.
- Find a bijection ϕ of the paths ending in v_0 and not containing c and which is length preserving modulo $|c|$.

If p_v is unique path from v to v_0

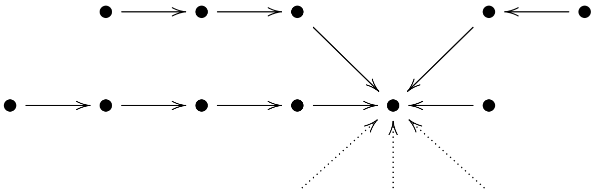
$$v \mapsto \phi(p_v)\phi(p_v)^* \quad \text{and} \quad e \mapsto \phi(ep_{\mathbf{r}(e)})\phi(p_{\mathbf{r}(e)})^*.$$

So, **all we need is a bijection of paths** ϕ.

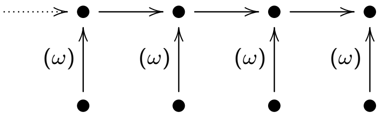
The main difference between this and O and I^+:
It is **global**, not local. If E^0 is infinite, you can be moving infinitely many elements simultaneously.

Path rearrangement – example

$E =$



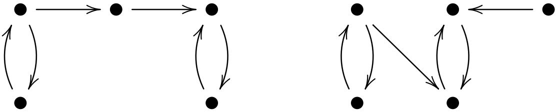
$E_{\mathrm{can}} =$



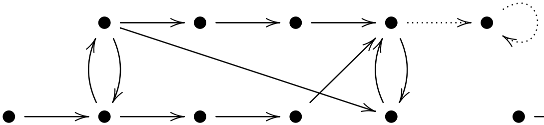
Relative construction

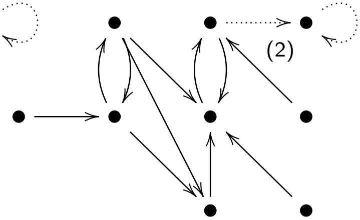
Let $V \subseteq E^0$. We repeat the E_{can} construction but without impacting the root $R(V) = \{u \in E^0 \mid u \geq v \text{ for some } v \in V\}$ and have $E_{\text{can},V}$.

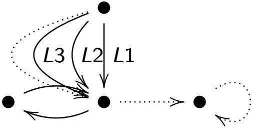
The main application: E= n-S-NE graph, H = nontrivial and proper her and sat set, and $V = R(V)$ = vertices of the porcupine of H which are not in H.

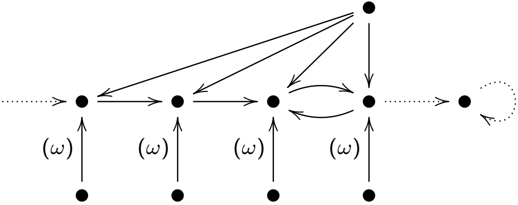


Examples









Four (or three?) "basic" types of operations

1. **Out-splits** and their inverses.

2. **In-split plus**.

3. **Path rearrangement** and its generalization on n-S-NE graphs. Should this be counted as I^+?

4. **Cut maps.** Defined only for graphs with infinitely many vertices. Not in $\langle O, I^+ \rangle$.

Out-split

If v is a vertex which emits edges, let $\mathcal{E}_1, \dots \mathcal{E}_n$ be a partition $\mathcal{P}$ of $\mathbf{s}^{-1}(v)$. The **out-split graph** $E_{v,\mathcal{P}}$

$$E_v^0 = E^0 - \{v\} \cup \{v_1, \dots, v_n\},$$

$$E_v^1 = \{f_1, \dots, f_n \mid f \in E^1, \mathbf{r}(f) = v\} \cup \{f \in E^1 \mid \mathbf{r}(f) \neq v\},$$

the range of f_i in $E_{v,\mathcal{P}}$ is v_i and the range of f in $E_{v,\mathcal{P}}$ is $\mathbf{r}(f)$, the source of $g \in E^1_{v,\mathcal{P}}$ is

- v_i if $g = f_j \in \mathcal{E}_i$ (so $\mathbf{s}(f) = \mathbf{r}(f) = v$)
- v_i if $g = f \in \mathcal{E}_i$ (so $\mathbf{s}(f) = v$ and $\mathbf{r}(f) \neq v$)
- $\mathbf{s}(f)$ if $g = f_j$ and $\mathbf{s}(f) \neq v$ (so $\mathbf{r}(f) = v$)
- $\mathbf{s}(f)$ if $g = f$ and $\mathbf{s}(f) \neq v$ (so $\mathbf{r}(f) \neq v$).

The map $v \mapsto \sum_{i=1}^n v_i$, $w \mapsto w$ for $w \in E^0 - \{v\}$, $f \mapsto \sum_{i=1}^n f_i$ if $\mathbf{r}(f) = v$ and $f \mapsto f$ otherwise, extends to a graded $*$-monomorphism $L_K(E) \to L_K(E_{v,\mathcal{P}})$.
Invertible if v is regular or all but one set $\mathcal{E}_1, \dots \mathcal{E}_n$ finite.

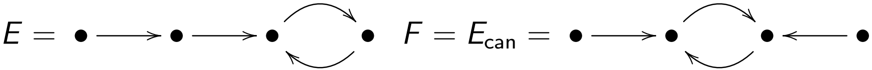
In-split plus

in-split = switch **r** and **s** in the definition of an out-split.

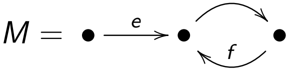
in-split minus = allow partitions to be empty.

in-split plus = E and F are *in-split plus* equivalent if there is a graph M (the "mother graph") such that both E and F are in-split minuses of E using the same vertex and partitions with the same number of sets.

For example,



are in-split plus equivalent – the mother is:



and $M \overset{I}{\hookrightarrow} E$ with $e|f$ and $M \overset{I}{\hookrightarrow} F$ with $ef|\emptyset$.

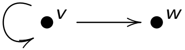
General idea

1. Define a canonical form E_{can} for any n-S-NE E. Let
$$E \approx F \quad \text{iff} \quad E_{\text{can}} \cong F_{\text{can}}.$$
2. Since there are specific operations
$$\phi_E : E \to E_{\text{can}},\ \phi_F : F \to F_{\text{can}},\ \text{and}\ \iota : E_{\text{can}} \cong F_{\text{can}},$$
E and F are "relatable" on the graph level by
$$\phi_F^{-1}\iota\phi_E.$$
3. If f is a talented monoid pointed iso, we can **find** E_{can} and F_{can} such that $f = \overline{\phi_F^{-1}\iota\phi_E}$.

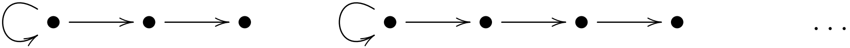
So, we can **realize** f.

Approach to defining a canonical form

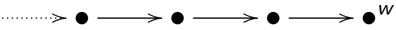
1. Make the connecting spines as short as possible.
2. Make all the tails canonical.
3. Make the graphs "reduced" with respect to out-splits. If you cannot, make them out-split invariant.
4. Make the number of tails minimal by reductions and cuts.

For example, the out-splits of 

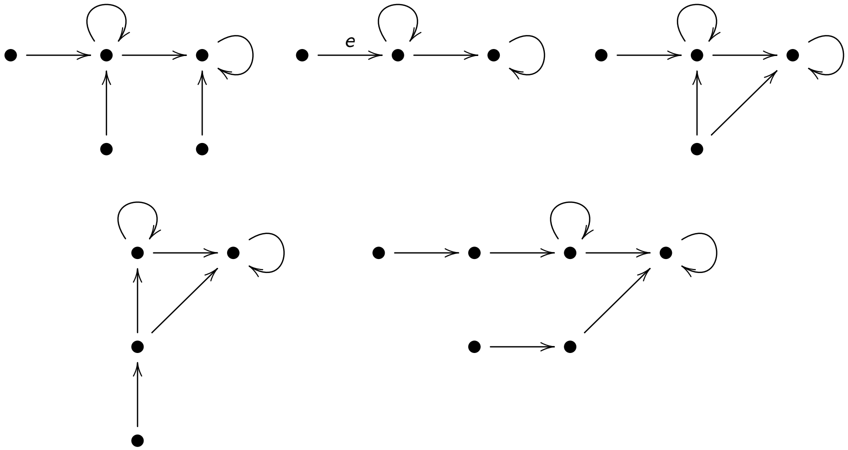
, produce



All these graphs have the same porcupine and quotient.

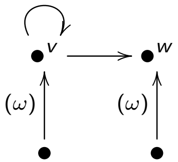


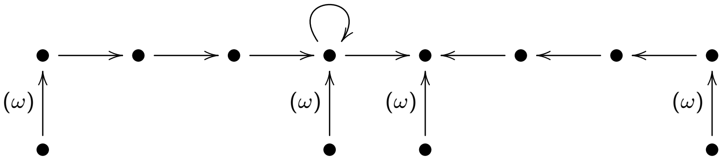
Extending the spine



When we ran out of tails, we cannot extend the spine any more.

Graph that permits arbitrary spine extending...



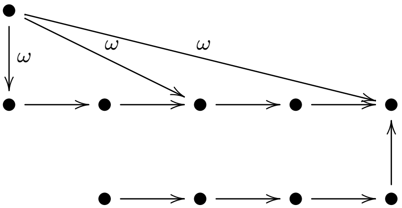


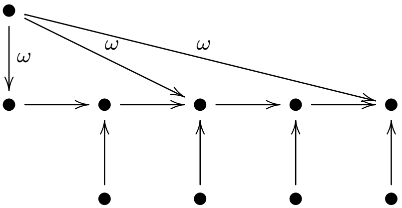
If we use v and w for the two matching vertices in F, a monoid iso f is the "identity" since

$$f([v]) = [v] \text{ and } f([w]) = [w],$$

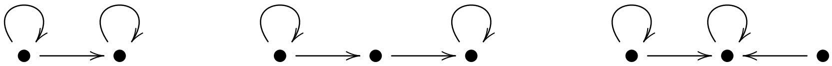
so how does f "know" that E and F are not the "same"?

The tails can also be made canonical...

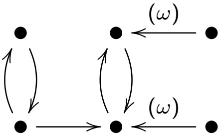




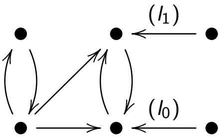
Connecting paths of the “ending” part...



Sometimes they produce the same thing.



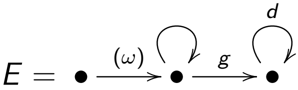
Reduced graphs

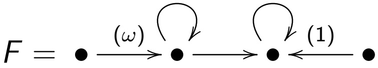


Reduced iff $l_0, l_1 \in \{0, \omega\}$.

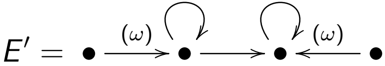
Recurring theme:
make something smallest or
make it change invariant.

The cut map





Moving (the only) exit of both creates



Then E is the **cut form** of both E' and F.

The cut map has

$$f_1 \mapsto g_0 g d^*, \quad g_n \mapsto g_n - g_n g g^* + g_{n+1} g g^*,$$

and its inverse has

$$g_0 \mapsto g_0 - g_0 g g^* + f_1 d g^* \text{ and } g_n \mapsto g_n - g_n g g^* + g_{n-1} g g^* \text{ for } n > 0$$

Idea of the proof

3 steps.

1. **The Quotient Proposition** (deals with I^+).
2. **The Cut Lemma** (deals with the cuts).
3. **Realization** (deals with O).

Induction base $n = 2$. E and F with c, c' emitting cycles, d, d' terminal cycles.

1. **Initial realization**: if $E = E_{\text{can}}$ there is F_{can} such that the "connecting part" of E_{can} and F_{can} is the same. There are exit moves ϕ_E and ϕ_F such that

$$g([\mathbf{s}(c)]) = [\mathbf{s}(c')] \text{ and } g([\mathbf{s}(d)]) = [\mathbf{s}(d')]$$

for $g = \overline{\phi_F} f \overline{\phi_E}^{-1}$.

The Quotient Proposition and the Cut Lemma

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & J^\Gamma(H_1) & \longrightarrow & M_E^\Gamma & \longrightarrow & M_{E/H}^\Gamma & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_{2/1}} & & \\
0 & \longrightarrow & J^\Gamma(G_1) & \longrightarrow & M_F^\Gamma & \longrightarrow & M_{F/G_1}^\Gamma & \longrightarrow & 0
\end{array}
$$

2. **The Quotient Proposition**. There is an isomorphism $\iota : E/H \cong F/G$ such that $f_{2/1} = g_{2/1} = \bar{\iota}$.
3. **The Cut Lemma**. If E and F are cut and I-plus-reduced graphs such that that there is $\iota : E/H \cong F/G$ which maps $\mathbf{s}(c)$ onto $\mathbf{s}(c')$. If f is such that $f_{2/1} = \bar{\iota}$, then ι can be extended to a graph isomorphism $\iota : E \cong F$ such that $f = \bar{\iota}$.

$n > 2$

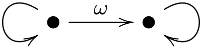
$$(\emptyset,\emptyset) \subsetneq (H_1, S_1) \subsetneq (H_2, S_2) \subsetneq \dots$$

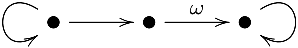
$$\subsetneq (H_{n-1}, S_{n-1}) \subsetneq (H_n, \emptyset) = (E^0, \emptyset)$$

can be made into

$$(\emptyset,\emptyset) \subsetneq (G_1,\emptyset) \subsetneq (G_2,\emptyset) \subsetneq \dots \subsetneq (G_{n-1},\emptyset) \subsetneq (G_n,\emptyset) = (E^0,\emptyset)$$

by out-splits. For example,





The General Quotient Proposition

Assume that the General Cut Lemma and GCC hold for the $(n-1)$-S-NE graphs. Assume that the $(n-1)$-quotients of E and F are canonical.

(i) $f([\mathbf{s}(c_2)]) = [\mathbf{s}(c_2')]$ and $f([\mathbf{s}(c_1)]) = [\mathbf{s}(c_1')]$.

(ii) The 21-connecting matrices are equal.

(iii) There is an isomorphism $\iota_{n/2} : E/H_2 \cong F/G_2$ such that $f_{n/2} = \bar{\iota}_{n/2}$.

Under these assumptions, there is $\iota_{n/1} : E/H_1 \cong F/G_2$ such that $f_{n/1} = \bar{\iota}_{n/1}$.

The General Cut Lemma

Let E and F be direct-exit n-S-NE graphs with canonical $(n-1)$-quotients such that that there is $\iota_{n/1} : E/H_1 \cong F/G_1$ such that $\iota_{n/1}(\mathbf{s}(c_j)) = \mathbf{s}(c'_j)$ for $j = 2, \ldots, n$.
If there is a pointed isomorphism f of the Γ-monoids such that

(i) $f_{n/1} = \bar{\iota}_{n/1}$,

(ii) $f([\mathbf{s}(c_1)]) = [\mathbf{s}(c'_1)]$

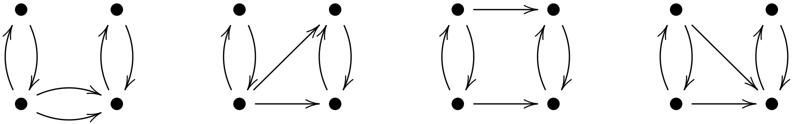
(iii) the $j1$-connecting matrices are equal,

(iv) $\iota_{n-1}(\mathbf{s}(c_j)) = \mathbf{s}(c'_j)$ for $j = 2, \ldots, n$,

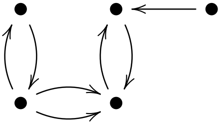
then $\iota_{n/1}$ can be extended to $\iota : E \cong F$ such that $f = \bar{\iota}$.

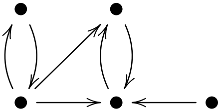
Counting tails example

E_1, E_2, E_3, and E_4 :



Canonical for E_3 and E_4 :





- $E_1 \not\approx E_2$ – connecting parts different
- $E_1 \not\approx E_3$ and $E_2 \not\approx E_4$ – different number of tails.

The end argument

- Use induction for E/H_1 and F/G_1 and obtain canonical forms with $E/H_1 \cong F/G_1$.
- Messing with H_2-to-H_1 part does not impact the quotient any more, only H_j-to-H_1 part. We make the H_2-to-H_1 part canonical and then ...
- ... proceed with the H_3-to-H_1 part and so on until we reach H_n-to-H_1.

Corollaries

- The GCC holds for graphs with disjoint cycles and finitely many vertices.
- All the results hold for the **graph C^*-algebras.**
- The graph operations **preserve the diagonal** (so $\approx$ leads to 111 relation in Eilers-Ruiz 3-bit codification).
- The **Graded Isomorphism Conjecture** holds:

1, 2, 3 are equivalent with any of these.

4, 5, 6. There is a (diagonal-preserving) graded algebra (*-)isomorphism $L_K(E) \to L_K(F)$.

7, 8, 9. There is a (diagonal-preserving) graded ring (*-)isomorphism $L_K(E) \to L_K(F)$.

10, 11. There is an equivariant (equiv. graded) isomorphism $C^*(E) \to C^*(F)$.