My mathematical journey

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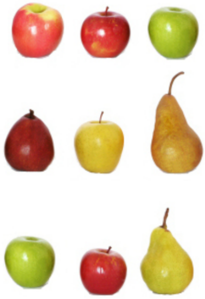
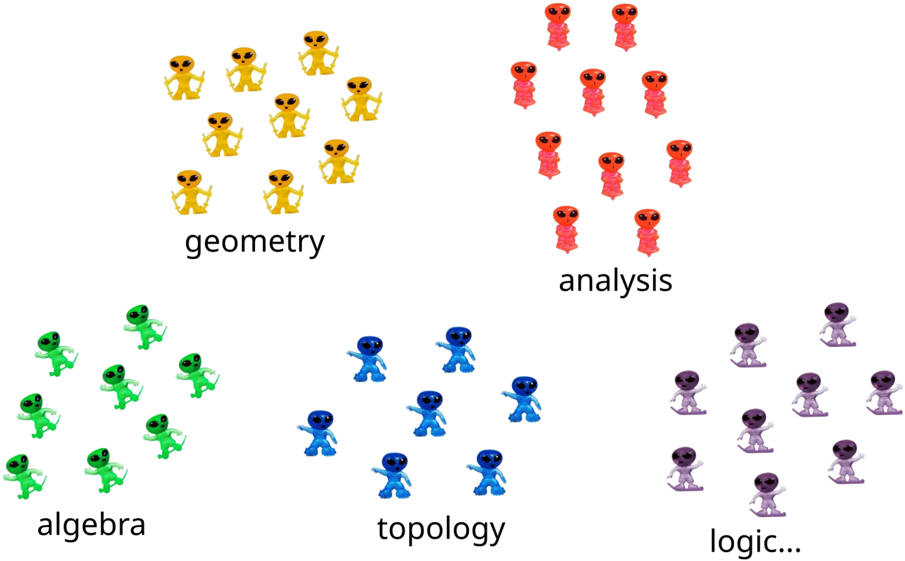
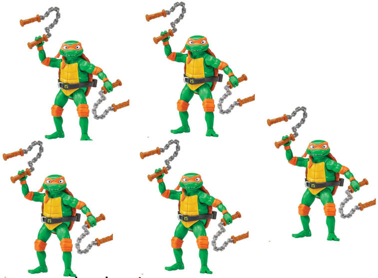


classification

Modern mathematics...

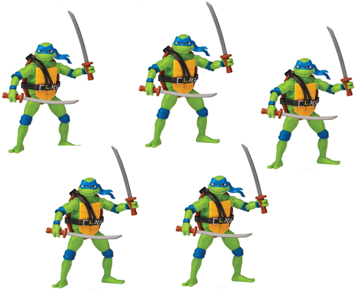


And then there are also mutants...



algebraic geometry

"continuous" algebra



algebraic topology

universal algebra

Some bridges were built...

solving polynomials	**Galois group** $\leftrightsquigarrow$	groups
(some) differential equations	**Laplace transform** $\leftrightsquigarrow$	algebraic equations
topological structures	**homology** $\leftrightsquigarrow$	algebraic structures



So, building bridges seemed important...

... and I ended up working on one of them.

Operator theory

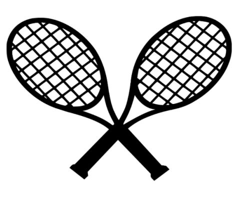
Algebra

(or at least on one lane of the multi-lane highway of this bridge)

But, let me start from my roots...

Serbia

šljivovica



Novak Đoković

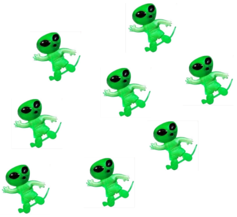
Nikola Jokić

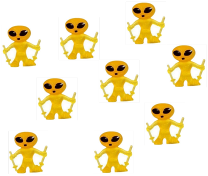
Serbian mathematician and a recognizable name ...

Mileva Marić with Albert Einstein in 1912

She was born near my home town, **Novi Sad**, and lived in it for a while. Their first daughter was born there.

As a student in Novi Sad (early to mid 90s)

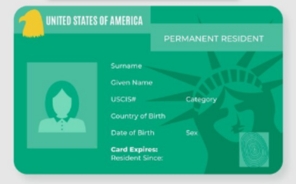




logic

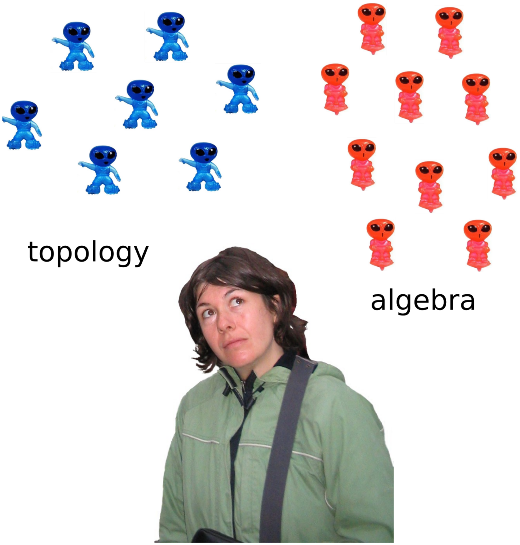
universal algebra

In 1996, I got a green card (on the lottery)...

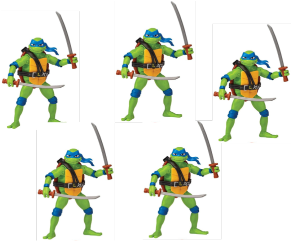


... and was accepted to the PhD program at the University of Maryland

And the quest continued...



In 2002, I graduated...

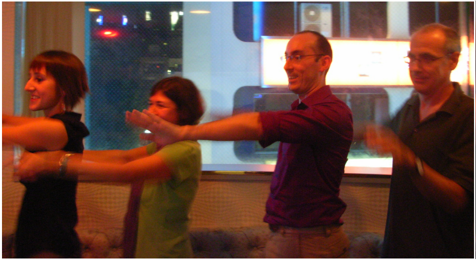
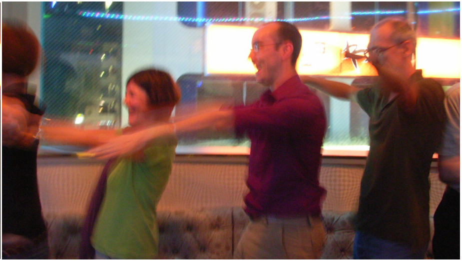


with a thesis technically in **algebraic topology**
but motivated by the **operator-theory-to-algebra bridge**...

... and moved to **Philadelphia.**

My current (research) story starts in late 2000s...

When I met a group of mathematicians with similar interests of crossing bridges....

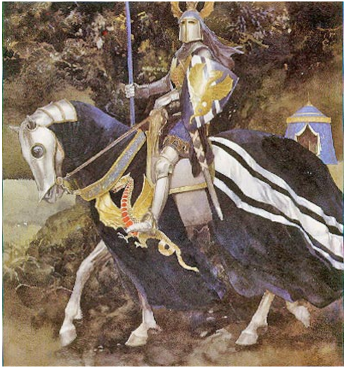
... and travelling.

Gonzalo Aranda Pino

got me closer to algebras defined over graphs called

Leavitt path algebras

Graph algebras



Damsel in distress: a **graph** C^***-algebra**

Knight in shining armour: a **Leavitt path algebra.**

Why "damsel in distress"?...

The examples of C^*-algebras were so vast and so diverse, that a need for their **classification** became evident.

This initiative is known as the

Elliott Classification Program.

Elliott completely classified one type of C^*-algebras (the "best behaved" type).

Graphs and paths

Start with a directed **graph**: vertices, edges, and source and range map, s and r.

$$\mathsf{s}(e)\bullet \xrightarrow{\;e\;} \bullet\mathsf{r}(e)$$

For example, the paths of

$$\bullet^{u} \xrightarrow{\;e\;} \bullet^{v} \xrightarrow{\;f\;} \bullet^{w}$$

are
u, v, w (length 0),
e, f (length 1) and
ef (length 2).

Adding and multiplying paths – path algebra

Addition: a new element $p + q$.

Multiplication: **concatenation.**

pq is $\bullet \overset{p}{\dashrightarrow} \bullet \overset{q}{\dashrightarrow} \bullet$ if range(p) =source(q) and 0 otherwise.

For example, for $\bullet^u \xrightarrow{e} \bullet^v \xrightarrow{f} \bullet^w$, $e \cdot f = ef$ and $f \cdot e = 0$.

Form a vector space over your favorite coefficient field with the paths as the *basis*. This is the **path algebra**.

For example, some elements of the path algebra over our example graph: $3e + \sqrt{5}\, ef$, and $2v - \frac{3}{4} f$.

Example $\bullet^u \xrightarrow{e} \bullet^v \xrightarrow{f} \bullet^w$ continued

The **standard matrix units**: $e_{11} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$,

$e_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ etc.

The map

$$u \mapsto e_{11}, \quad e \mapsto e_{12}, \quad ef \mapsto e_{13}, \quad v \mapsto e_{22}, \quad f \mapsto e_{23}, \quad \text{and} \quad w \mapsto e_{33}$$

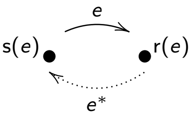
extends to an isomorphism of the path algebra and the algebra of the **upper triangular matrices**. We represent this iso by

$$\begin{bmatrix} u & e & ef \\ 0 & v & f \\ 0 & 0 & w \end{bmatrix}$$

Ghost paths

To get to a C^*-algebra we need to have an involution (glorified transpose).

Add the **ghost edges** – elements of the form e^* for $e \in E^1$.



The vertices are **selfadjoint:** $v^* = v$ for $v \in E^0$.

Example 1 – Matrices

Generalizes to *n*-line.

$$\bullet \xrightarrow{e_1} \bullet \xrightarrow{e_2} \bullet \cdots\cdots\cdots \bullet \xrightarrow{e_{n-1}} \bullet$$

Path algebra: upper triangular *n*x*n* matrices
Leavitt path algebra: all *n*x*n* matrices

Graph C*-**algebra:** all *n*x*n* matrices but considered with the norm.

Example 2 – Loop

Paths: $v = 1 = e^0, e = e^1, e^2, e^3, \ldots$ Representation: $e = x$

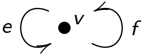
Path algebra: Polynomials with coefficients in K. Think that $3v + 5e^2 \leftrightsquigarrow 3 + 5x^2$.

Ghost edge e^*. Representation: $e^* = x^{-1}$

Leavitt path algebra: Laurent polynomials (like regular polynomials but with negative powers of x also). For example, $2e^* + 3v + 5e^2 \leftrightsquigarrow 2x^{-1} + 3 + 5x^2$.

Graph C^***-algebra:** continuous functions on a circle $C(S^1)$.

Example 3 – roses

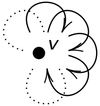


two-petal rose

Leavitt path algebra is known as the Leavitt algebra $L(1,2)$.
It is a universal example of a ring R with $R^2 \cong R$.

Graph C^***-algebra:**
Cuntz algebra $\mathcal{O}_2$.

Generalizes to n-**rose.**



Some research trends

1. **Characterizations.** For a given algebra property P, find a graph property Q so that $L_K(E)$ has P iff E has Q.

2. **Generalizations.** For example, generalize the class of graphs.

3. **Classifications.** For example, are $L_K(E) \cong L_K(F)$ as algebras $\Leftrightarrow$ the Grothendieck groups are isomorphic?

Grothendieck groups

Dimensions of vector spaces: $1, 2, 3 \dots$

One would want these dimensions to form a group, the

group of dimensions.

So, we "add" zero and inverses

$$\dots, -3, -2, -1, 0, 1, 2, 3, \dots$$

And this is it: the zero Grothendieck group (of the underlying coefficient field) is the group of integers $\mathbb{Z}$.

Graded structure and GCS

Grothendieck groups of graph algebras are

graded

(additional structure coming from the lengths of the paths).

Graded Classification Conjecture.

$L_K(E) \cong L_K(F)$ are isomorphic as graded algebras $\Leftrightarrow$ their graded Grothendieck groups are (pointed) isomorphic.

Asked by **Roozbeh Hazrat** in 2011.

2015 – 2023

disjoint cycles

classification

porcupine

graded rings

porcupine quotient

Sabbatical and Syracusa, Feb 2024

Oberwolfach, March 2024

Current state of things