CANCELLATION PROPERTIES OF NONUNITAL RINGS. GRADED CLEAN AND GRADED EXCHANGE LEAVITT PATH ALGEBRAS

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ABSTRACT. Various authors have been generalizing some unital ring properties to nonunital rings in seemingly unrelated ways. We showcase the unifying elements of such generalizations for properties related to cancellation of modules (being unit-regular, having stable range one, being directly finite, exchange, or clean). We explore the relationships between these generalizations, the relationships between these generalizations and their "local" versions for rings with local units, and we extend the defined concepts to graded rings. With graded clean and graded exchange rings suitably defined, we study how these properties behave under the formation of graded matrix rings. We find properties of a graph E which are equivalent to the unital Leavitt path algebra $L_K(E)$ being graded clean. We also find some graph properties which are necessary and some which are sufficient for $L_K(E)$ to be graded exchange.

0. INTRODUCTION

We consider the question *"If P is a property of a unital ring, how does one define a generalized version of P suitable for nonunital rings?"* and address it for several different properties P. We say that a ring property is a *cancellation property* if it can be directly related to one of the cancellation properties of modules (internal cancellation, module-theoretic exchange, module-theoretic direct finiteness, and substitution) or if it is "sandwiched" between two properties directly relatable to module cancellation. For example, the properties that a ring is unit-regular, exchange, directly finite, or that it has stable range one can be directly related to internal cancellation, module-theoretic exchange, module-theoretic direct finiteness, and substitution respectively (see [18] or [28] for more details). We consider cleanness to also be a cancellation property because clean rings bridge the classes of unit-regular and exchange rings.

The cancellation properties mentioned above critically depend on the existence of the ring identity. Some of these properties have been adapted to nonunital rings using different approaches: [19] for unit-regularity, [30] for stable range one, [3] for exchange, and [22] for cleanness. In many cases, the approach from one of these papers do not seem readily relatable to the approach from the other. We review these approaches, introduce an equivalent approach for direct finiteness (Definition 2.7), and showcase the unifying elements so that a unital ring property can be generalized to nonunital rings in a more uniform way. Also, using the first two equivalent conditions from Propositions 2.2, 2.4, 2.6, 2.9, and 2.10, one can avoid a reference to a ring unitization in the generalization process. If UR stands for unit-regular, sr=1 for stable range one, DF for directly finite, Cln for clean, Exch for exchange, and Reg for von Neumann regular generalizations to possibly nonunital rings, the

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diagram below shows relations between these concepts.

$$\begin{array}{ccccccc} \text{UR} \Leftarrow & \text{Reg} + \text{sr=1} & \Rightarrow \text{sr=1} & \Rightarrow & \text{DF} \\ & \Downarrow & & & \\ & \text{Cln} & \Rightarrow \text{Exch} & & \end{array} \tag{D1}$$

In section 2.7, we turn to the "local" versions of cancellation properties. If P is a ring property, a ring R has *local* P if every finite subset of R is contained in a corner of R which has P. This definition is especially convenient for locally unital rings since finite subsets of such rings are contained in corners. We consider relations between P and local P when P is a cancellation property.

In section 3, we turn to rings which are graded. In [28], the following question was raised *"If P is a ring property, how does one define the graded version P_{gr} of the property P in a meaningful way?"* and considered it in the cases when P is unit-regularity, stable range one, or direct finiteness. In this paper, we focus on the remaining two cancellation properties: being an exchange ring and being a clean ring. We introduce the graded versions of being clean in the unital case and being exchange in the general case and study properties of these generalized concepts, in particular how they behave under the formation of graded matrix rings or algebras (Propositions 3.4 and 3.7).

In section 4, we consider Leavitt path algebras. It turns out to be useful if one can pair a ring property P with a graph property P' so that the Leavitt path algebra $L_K(E)$ of a graph E has P if and only if E has P'. Such pairings enable one to create rings with various predetermined properties by choosing suitable graphs (for example prime and not primitive rings, simple and not purely infinite simple rings, and so on). A number of ring-theoretic properties have been paired up in such a way: being regular, simple, purely infinite simple, hereditary, semisimple, Artinian, Noetherian, directly finite, Baer, to name some of them. In particular, such characterizations are known for properties of being locally unit-regular, locally directly finite, and exchange. Proposition 2.11 implies that the graph properties which characterize local unit-regularity and local direct finiteness also characterize the generalized versions of unit-regularity and direct finiteness.

Leavitt path algebras are naturally $\mathbb{Z}$-graded. Thus, pairing a *graded* ring property of $L_K(E)$ with a graph property of E is also of interest. This has previously been done for graded local direct finiteness for any E ([14]), and for graded unit-regularity and graded stable range one in the case when E is finite ([28]). By Proposition 4.4, we improve the result on graded unit-regularity by relaxing the assumption that E is finite to the requirement that E has finitely many vertices (no restriction on the number of edges). We present a necessary condition for a Leavitt path algebra to be graded exchange (Proposition 4.5) and a sufficient condition for the same property (Proposition 4.7) and we question if any of them is both necessary and sufficient. We also characterize which unital Leavitt path algebras are graded clean (Proposition 4.2). This latter result does not answer the long standing question *"Which Leavitt path algebras are clean?"*, but it settles it in the graded case.

The diagram below summarizes the relationship between the graded versions of the cancellation properties of Leavitt path algebras. If P stands for any of UR, sr=1, DF, Cln, or Exch, gr in the subscript indicates the graded version of these properties, and 0 in the subscript denotes the condition that the 0-component of the graded endomorphism ring of $L_K(E)$ has P. The number of vertices of the graph is assumed to be finite for the vertical arrows except for those related to Exch_{gr} and Reg_{gr} for which there are no restrictions on E.

Using the graph properties in the diagram, it is straightforward to produce examples showing that each horizontal implication is strict. This, in its own right, provides interesting examples illustrating differences between properties of rings and their graded generalizations. For example, $\mathrm{Reg}_{\mathrm{gr}} \not\Rightarrow \mathrm{Exch}_{\mathrm{gr}}$, contrasts Reg $\Rightarrow$ Exch, and $\mathrm{UR}_{\mathrm{gr}} \not\Rightarrow \mathrm{Cln}_{\mathrm{gr}}$, contrasts UR $\Rightarrow$ Cln.

$$\begin{array}{c}
\begin{array}{ccccccc}
\boxed{\mathrm{Cln}_{\mathrm{gr}}} & \Rightarrow & \boxed{\mathrm{UR}_{\mathrm{gr}},\ \mathrm{sr}{=}1_{\mathrm{gr}}} & \Rightarrow & \boxed{\mathrm{DF}_{\mathrm{gr}}} & \Rightarrow & \boxed{\mathrm{Reg}_{\mathrm{gr}},\ P_0 \text{ holds.}} \\
\Updownarrow & & \Updownarrow & & \Updownarrow & & \Updownarrow \\
\boxed{\begin{array}{c} E \text{ is finite } \bigsqcup \bullet \\ \text{or } E \text{ is } \bullet \end{array}} & \Rightarrow & \boxed{\begin{array}{c} E \text{ is as in part (3)} \\ \text{of Proposition 4.4.} \end{array}} & \Rightarrow & \boxed{\begin{array}{c} E \text{ is} \\ \text{no-exit.} \end{array}} & \Rightarrow & \boxed{E \text{ is any.}}
\end{array} \\
\\
\begin{array}{ccccc}
\boxed{\mathrm{Cln}_{\mathrm{gr}}} & \Rightarrow & \boxed{\mathrm{Exch}_{\mathrm{gr}}} & \Rightarrow & \boxed{\mathrm{Reg}_{\mathrm{gr}}} \\
\Updownarrow & & \Downarrow & & \Updownarrow \\
\boxed{\begin{array}{c} E \text{ is finite } \bigsqcup \bullet \\ \text{or } E \text{ is } \bullet \end{array}} & \Rightarrow & \boxed{E \text{ is no-exit.}} & \Rightarrow & \boxed{E \text{ is any.}}
\end{array}
\end{array} \tag{D2}$$

1. Review of the cancellation properties

Rings are assumed to be associative. If a ring has the identify, we call it a *unital ring*. We use the term a *general ring* to denote a possible lack of the identity. If R is a unital ring, $U(R)$ denotes the set of elements which are invertible in R.

We briefly review the definitions of the five cancellation properties we consider in the paper.

1.1. Unit-regular, stable range one, and directly finite rings.

A general ring is (von Neumann) *regular* if for every $x \in R$, $x \in xRx$. If R is a unital ring, R is *unit-regular* if every $x \in R$ is unit-regular meaning that there is an invertible $u \in R$ such that $x = xux$; R has *stable range one* if for every $x, y \in$ such that $xR + yR = R$, there is $z \in R$ such that $(x + yz)R = R$ (equivalently, $x + yz$ is invertible, see [30, Theorem 2.6]); R is *directly finite* if for every $x, y \in R$, $xy = 1$ implies $yx = 1$. If Reg, UR, sr=1, and DF denotes these properties for short, the following relations hold and all implications are strict. For more details, see [18] or [28].

$$\mathrm{Reg} \quad \Leftarrow \quad \mathrm{UR} \quad \Rightarrow \quad \mathrm{sr{=}1} \quad \Rightarrow \quad \mathrm{DF}$$

All of the above properties are closed when passing to a corner, i.e. a ring of the form eRe where $e \in R$ is an idempotent of a ring R ([18], [30] and [28] contain more details).

1.2. Exchange modules and rings.

Recall that a right R-module A has the *(finite) exchange property* if whenever (a copy of) A is a direct summand of a right module $B = \bigoplus_{i \in I} B_i$ where I is a (finite) index set, then there are submodules $A_i \leq B_i$ such that $\bigoplus_{i \in I} A_i$ is a complement of A in B. If I is infinite, we denote this property by $\mathrm{Exch}(A)$ and, if I is finite, by $\mathrm{FinExch}(A)$.

An element x of a unital ring R is an *exchange element* of R if there is an idempotent $e \in xR$ such that $1 - e \in (1 - x)R$ and R is an *exchange ring* if every element is exchange. We use Exch^r to shorten this last condition. The superscript r indicates the presence of right modules. Exch^l can be defined analogously. By [31, Theorem 2] and [21, Theorem 2.1], the following conditions are equivalent for a right R-module A (which we denote by writing A_R for A).

1. $\mathrm{FinExch}(A_R)$ holds.
2. Exch^r holds for $\mathrm{End}_R(A)$.
3. Exch^l holds for $\mathrm{End}_R(A)$.
4. $\mathrm{Exch}(\mathrm{End}_R(A)_{\mathrm{End}_R(A)})$ holds.

In the case when A is R_R and R is unital, we have that $\mathrm{End}_R(A) \cong R$ so that the properties $\mathrm{FinExch}(R_R)$, Exch^r for R, Exch^l for R, and $\mathrm{Exch}(R_R)$ are all equivalent. Because of this, we suppress writing l or r in the superscript of Exch.

By [21, Proposition 1.1], Exch is equivalent to the condition that for any $x \in R$, there is an idempotent $e \in R$ such that $e - x \in (x - x^2)R$. Since Exch is left-right symmetric, this condition is also left-right symmetric and we use Lift to denote it.

By [8, Lemma 3.10] and [21, Proposition 1.10], Exch is closed under the formation of matrix rings and corners of unital rings.

1.3. **Clean rings.** An element x of a unital ring R is a *clean element* of R if $x = u + e$ for some unit u and some idempotent e. In this case, the relation $x = u + e$ is a *clean decomposition* of x. A ring is *clean* if every element is clean and we use Cln to denote this requirement. Clean rings have been introduced in [21] where the second implication below was also shown.

$$\text{UR} \quad \Rightarrow \quad \text{Cln} \quad \Rightarrow \quad \text{Exch}$$

The first implication follows from the main result of [5]. We review the argument for the second implication. If $x = e + u$ is a clean decomposition of an element x of a clean ring R, then $f = u^{-1}(1-e)u$ is an idempotent such that

$$(x - x^2)u^{-1} = (e + u)u^{-1} - (eu^{-1} + ueu^{-1} + e + u) = f - e - u = f - x$$

and so the property Lift holds for R. Both implications are strict by examples from [6].

The property Cln is closed under the formation of matrix rings (see [11, Corollary 1]), it is not closed under the formation of corners (see [24]), and it is not Morita invariant (see [25]).

2. Cancellation properties of nonunital rings

2.1. **The standard unitization and operations $*$ and $\circ$.** If R is a general ring, a unital ring S such that R embeds in S as a double-sided ideal of S is an *unitization* of R. The *standard unitization* R^u of R is the ring $R^u = R \oplus \mathbb{Z}$ with the addition given coordinate-wise and with the multiplication given by

$$(x, k)(y, l) = (xy + lx + ky, kl).$$

The element $(0, 1)$ is the identity of R^u and R is a double-sided ideal of R^u. If R is a K-algebra for some field K, K can be used instead of $\mathbb{Z}$.

Consider the following two operations of R given below.

$$x * y = x + y + xy \quad \text{ and } \quad x \circ y = x + y - xy$$

Operation $*$ was used in [19] to adapt unit-regularity and in [22] to adapt cleanness to nonunital rings. Operation $\circ$ was used in [3] to adapt exchange property to nonunital rings. The property sr=1 was adapted in [30] using unitizations but without references to $*$ or $\circ$. The property DF was considered for locally unital rings in [28], but no definition appeared for rings which may not be locally unital. We relate all five cancellation properties both to $*$, $\circ$, to properties of R^u as well as of any other unitization of a general ring R.

The following lemma relates the operations $*$ and $\circ$ and the ring R^u. We use $U(*)$ and $U(\circ)$ to shorten $U(R, *)$ and $U(R, \circ)$ which denote the set of elements invertible under $*$ and $\circ$, respectively. The proofs of the properties in the lemma are short and direct, so we omit them.

Lemma 2.1. *The following properties hold for any general ring R and any $x, y \in R$.*

(1) $-(x * y) = (-x) \circ (-y)$. *Hence, the monoids $(R, *)$ and $(R, \circ)$ are isomorphic.*
(2) *An element (x, n) of R^u is invertible if and only if*

$$n = 1 \text{ and } x \in U(*) \quad \text{or} \quad n = -1 \text{ and } x \in U(\circ).$$

(3) $U(*) = -U(\circ)$ *and* $U(R^u) = \pm(U(*), 1) = \pm(U(\circ), -1)$.
(4) *The following conditions are equivalent for any* $x \in R$.

(i) $0 \in x * R$, *(ii)* $0 \in (-x) \circ R$, *(iii)* $(x, 1)R^u = R^u$.

(5) *If* $e \in R$ *is an idempotent, then* $x \in U(eRe)$ *if and only if* $x - e \in U(*)$.

2.2. **Nonunital unit-regularity.** In [19], the authors consider the property of a general ring R that for every $x \in R$, there is $u \in U(*)$ such that $x = xux + x^2$. By [19], this requirement is equivalent with conditions (3) and (4) in the proposition below. We note the equivalence with condition (2), given in terms of the operation $\circ$, also.

Proposition 2.2. *If* R *is a general ring, the following conditions are equivalent.*

(1) *For any* $x \in R$*, there is* $u \in U(*)$ *such that* $x = xux + x^2$.
(2) *For any* $x \in R$*, there is* $u \in U(\circ)$ *such that* $x = xux - x^2$.
(3) *For any* $x \in R$*,* $(x, 0)$ *is a unit-regular element of* R^u.
(4) *For any embedding* $\phi : R \to S$ *such that* S *is an unitization of* R *and for any* $x \in R$*,* $\phi(x)$ *is a unit-regular element of* S.

Proof. Assuming that (1) holds, let $x \in R$ and let $u \in U(*)$ be such that $-x = (-x)u(-x)+(-x)^2 = xux + x^2$. Then $x = x(-u)x - x^2$ and $-u \in U(\circ)$. The converse (2) $\Rightarrow$ (1) is similar. Conditions (1), (3), and (4) are equivalent by [19, Lemma 3.1, Theorem 4.1, and Corollary 4.5]. □

If R is a unital ring, any of the conditions of this proposition holds if and only if R is unit-regular (see [19, Corollary 2.5]). We continue to use the term *unit-regular* for a general ring which satisfies any of the conditions from Proposition 2.2. We also continue to use UR for this property.

The implication UR $\Rightarrow$ Reg continues to hold. Indeed, if R is unit-regular and if $x = xux + x^2$ for $x \in R$ and $u \in U(*)$, then $x^2 = xux^2 + x^3$ so that $x = xux + xux^2 + x^3 = x(u + ux + x)x$.

By [19, Example 4.3] which we review below, UR is not closed under the formation of corners.

Example 2.3. Let R_0 be a unital ring which is regular and not unit-regular. Let $R_n = \mathbb{M}_2(R_{n-1})$ for $n = 1, 2 \ldots$, embed R_n into R_{n+1} by $a \mapsto \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$, and let $R = \varinjlim_n R_n$. Then, R is unit-regular (more details are in [19, Example 4.3]) and the corner R_0 is not unit-regular.

2.3. **Nonunital stable range one.** In [30], a general ring R is said to have stable range one if for all $x \in R, \overline{y} \in R^u$, such that $(x, 1)R^u + \overline{y}R^u = R^u$, there is $\overline{z} \in R^u$ such that $((x, 1) + \overline{y}\,\overline{z})R^u = R^u$. We show the equivalence of this requirement and the conditions expressed in terms of $*$ and $\circ$ only and with no reference to R^u.

Proposition 2.4. *If* R *is a general ring, the following conditions are equivalent.*

(1) *For any* $x, y \in R$ *such that* $0 \in x * R + yR$*, there is* $z \in R$ *such that* $0 \in (x + yz) * R$.
(2) *For any* $x, y \in R$ *such that* $0 \in x \circ R + yR$*, there is* $z \in R$ *such that* $0 \in (x + yz) \circ R$.
(3) *For any* $x, y \in R$ *such that* $(x, 1)R^u + (y, 0)R^u = R^u$*, there is* $z \in R$ *such that* $(x+yz, 1)R^u = R^u$.
(4) *For any* $x \in R, \bar{y} \in R^u$ *such that* $\bar{x}R^u + \bar{y}R^u = R^u$ *where* $\bar{x} = (x, 1)$*, there is* $\bar{z} \in R^u$ *such that* $(\bar{x} + \bar{y}\bar{z})R^u = R^u$.
(5) *For any embedding* $\phi : R \to S$ *such that* S *is an unitization of* R*, and for any* $x, y \in S$ *such that* $x - 1 \in \phi(R)$ *and that* $xS + yS = S$*, there is* $z \in S$ *such that* $(x + yz)S = S$.

Proof. Assume that condition (1) holds and let $0 \in x \circ R + yR$ for some $x, y \in R$. Then $0 \in -(x \circ R + yR) = (-x) * R + (-y)R$. By (1), $0 \in (-x + (-y)z) * R = (-(x+yz)) * R = -((x+yz) \circ R)$ for some $z \in R$. Hence, $0 \in (x + yz) \circ R$. Similarly, (2) implies (1).

To show (1) $\Rightarrow$ (3), assume that (1) holds and let $x, y \in R$ be such that $(x, 1)R^u + (y, 0)R^u = R^u$. By [30, Lemma 3.5], this implies $(x, 1)(R, 1) + (y, 0)(R, 0) = (R, 1)$. Thus, $(0, 1) = (x, 1)(u, 1) + (y, 0)(v, 0) = (xu + x + u + yv, 1)$ for some $u, v \in R$, and so $0 = xu + x + u + yv \in x * R + yR$. By (1), there is $z \in R$ such $0 \in (x + yz) * R$ which implies that $(x + yz, 1)R^u = R^u$ by Lemma 2.1.

Conversely, if (3) holds and $x, y \in R$ are such that $0 = x + u + xu + yv$ for some $u, v \in R$ then $(0, 1) = (x, 1)(u, 1) + (y, 0)(v, 0)$ so $(x, 1)R^u + (y, 0)R^u = R^u$. By (3) there is $z \in R$ such that $(x + yz, 1)R^u = R^u$. Thus, $0 \in (x + yz) * R$ by Lemma 2.1.

The last three conditions are equivalent by [30, Theorems 3.6 and 3.10]. □

If the conditions from Proposition 2.4 hold, we say that R has *stable range one* and that sr=1 holds for R. This definition agrees with the unital case by [30, Theorem 3.4]. Next, we show that Reg + sr=1 $\Rightarrow$ UR. The converse does not hold by [19, Example 4.3] (also Example 2.3).

Proposition 2.5. *For general rings,* Reg + sr=1 $\Rightarrow$ UR.

Proof. Let R be a regular general ring with stable range one. By [9, Theorem 2], R has a regular unitization S. If we identify R with its image in S, R is contained in the directed union of the corners eSe for $e = e^2 \in R$ by [20, Lemma 1.1]. We also recall [20, Lemma 1.4] which states that a general ring I with a regular unitization T has sr=1 if and only if eTe is unit-regular for every $e = e^2 \in I$. By this result, any $x \in R$ is contained in a unit-regular corner eSe for some $e = e^2 \in R$. If $u \in U(eSe)$ is such that $x = xux$, then $1 - e + u \in U(S)$ is such that $x = x(1 - e + u)x$. Thus, x is unit-regular in S. By Proposition 2.2, R is unit-regular. □

2.4. **Nonunital direct finiteness.** Just as UR, the condition DF is given in terms of multiplication only, without any reference to addition. Thus, one can define a monoid $(M, \cdot, 1)$ to be directly finite if for every $x, y \in M$, $xy = 1$ implies $yx = 1$. Having this definition, we show the following.

Proposition 2.6. *If R is a general ring, the following conditions are equivalent.*

(1) *$(R, *)$ is directly finite.*
(2) *$(R, \circ)$ is directly finite.*
(3) *R^u is directly finite.*
(4) *For any embedding $\phi : R \to S$ such that S is an unitization of R, and for any $x, y \in R$, $(\phi(x) + 1)(\phi(y) + 1) = 1$ implies $(\phi(y) + 1)(\phi(x) + 1) = 1$.*

If R is unital, then the above conditions are equivalent with R being directly finite.

Proof. The equivalence of (1) and (2) directly follows from Lemma 2.1.

To show (1) $\Rightarrow$ (3), assume that $(x, k)(y, l) = (0, 1)$ holds in R^u. Then $k = l = 1$ or $k = l = -1$. In the first case, $x * y = x + y + xy = 0$ which implies that $y * x = 0$ by (1), and so $(y, 1)(x, 1) = (0, 1)$. In the second case, $-(x \circ y) = -x - y + xy = 0$ and so $x \circ y = x + y - xy = 0$ which implies that $y \circ x = 0$ by (2) and so $(y, -1)(x, -1) = (0, 1)$. Thus, (3) holds. The converse (3) $\Rightarrow$ (1) is direct since $x * y = 0$ readily implies $(x, 1)(y, 1) = (0, 1)$.

To show (1) $\Rightarrow$ (4), assume that (1) holds and let ϕ and S be as in (4). If $(\phi(x)+1)(\phi(y)+1) = 1$ holds in S, then $\phi(x + y + xy) = 0$ so $x * y = 0$ which implies that $y * x = 0$ by (1). Taking ϕ of both sides and adding 1 produces $(\phi(y) + 1)(\phi(x) + 1) = 1$. The converse (4) $\Rightarrow$ (1) is similar.

If R is unital, $xy = 1$ iff $(x-1)*(y-1) = 0$ and $x*y = 0$ iff $(x+1)(y+1) = 1$ for any $x, y \in R$. This implies the last sentence of the proposition. □

Proposition 2.6 justifies the validity of the following definition.

Definition 2.7. A general ring R is *directly finite* if any of the conditions from Proposition 2.6 holds.

We continue to use DF for the generalized DF property. It retains some favorable properties of the unital DF as we show next.

Proposition 2.8. *The property* DF *is closed under the formation of corners and* sr=1 $\Rightarrow$ DF*.*

Proof. To show the claim on corners, let R be a general ring, e an idempotent of R, and $x, y \in eRe$ be such that $xy = e$. Then $(x-e,1)(y-e,1) = (xy-x-y+e+x-e+y-e,1) = (xy-e,1) = (0,1)$ and so $(y-e,1)(x-e,1) = (0,1)$ which implies that $yx = e$.

To show sr=1 $\Rightarrow$ DF, assume that a general ring R has sr=1 and let $x, y \in R$ be such that $x*y = 0$. Then $(x,1)(y,1) = (0,1)$ in R^u and so $\bar{e} = (0,1)-(y,1)(x,1) = (-x-y-yx,0)$ is an idempotent in R^u such that $(0,1) = \bar{e}+(y,1)(x,1) \in \bar{e}R^u + (y,1)R^u$. By the assumption that sr=1, there is $\bar{z} \in R^u$ such that $((y,1)+\bar{e}\bar{z})R^u = R^u$. Since $(x,1)\bar{e} = (x,1)-(x,1)(y,1)(x,1) = (x,1)-(x,1) = (0,0)$, $(x,1)((y,1)+\bar{e}\bar{z}) = (x,1)(y,1) = (0,1)$. So, $(y,1)+\bar{e}\bar{z}$ has both a left and a right inverse and, thus, both this element and its left inverse $(x,1)$ are invertible. So, the right inverse $(y,1)$ of $(x,1)$ is also a left inverse of $(x,1)$. The relation $(y,1)(x,1) = (0,1)$ implies that $y*x = 0$. □

2.5. **Nonunital cleanness.** In [22], a general ring R is said to be *clean* if for any $x \in R$, there is an idempotent $e \in R$ and $u \in U(*)$ such that $x = e+u$. By [22, Proposition 7], conditions (1), (3), and (4) of the following proposition are equivalent. Assuming (1) and using it for $-x$ shows (2) by Lemma 2.1 and the converse is analogous. This shows the following.

Proposition 2.9. *If R is a general ring, the following conditions are equivalent.*

(1) *For any $x \in R$, there is an idempotent $e \in R$ and $u \in U(*)$ such that $x = e+u$.*
(2) *For any $x \in R$, there is an idempotent $e \in R$ and $u \in U(\circ)$ such that $x = -e+u$.*
(3) *For any $x \in R$, $(x,0)$ is a clean element of R^u.*
(4) *For any embedding $\phi : R \to S$ such that S is an unitization of R, and for any $x \in R$, $\phi(x)$ is a clean element of S.*

We say that Cln holds for a general ring if any condition from this proposition holds. This agrees with the cleanness of unital rings by [22, Lemma 1]. The property Cln does not transfer to corners even in the unital case by [24]. By [7, Theorem 2], Reg+sr=1 $\Rightarrow$ Cln.

2.6. **Nonunital exchange.** In [3], a general ring R is said to be an *exchange* ring if for any $x \in R$, there is an idempotent $e \in R$ and $y, z \in R$ such that $e = yx = x+z-zx$. Note that the existence of such y and z is equivalent with $e \in xR$ and $e \in x \circ R$. By [3, Theorem 1.2], this requirement is left-right symmetric (i.e. (1) $\Leftrightarrow$ (5) below).

Proposition 2.10. *If R is a general ring, the following conditions are equivalent.*

(1) *For any $x \in R$, there is an idempotent $e \in R$ such that $e \in -xR$ and $e \in x*R$.*
(2) *For any $x \in R$, there is an idempotent $e \in R$ such that $e \in xR$ and $e \in x \circ R$.*
(3) *For any $x \in R$, there is an idempotent $e \in R$ such that $e \in xR$ and $(-e,1) \in (-x,1)R^u$.*

(4) *For any embedding* $\phi : R \to S$ *such that* S *is an unitization of* R*, and for any* $x \in R$*, there is an idempotent* $e \in R$ *such that* $e \in xR$ *and* $1 - \phi(e) \in (1 - \phi(x))S$.
(5) *For any* $x \in R$*, there is an idempotent* $e \in R$ *such that* $e \in Rx$ *and* $e \in R \circ x$.

Proof. Assuming (1) and using it for $-x$ shows (2) by Lemma 2.1. Similarly, (2) implies (1).

If (2) holds, then $e = x \circ y$ for some $y \in R$. Thus, $(-e, 1) = (-x - y + xy, 1) = (-x, 1)(-y, 1) \in (-x, 1)R^u$ so (3) holds. Conversely, if (3) holds and $(-e, 1) = (-x, 1)(y, k)$ for some $y \in R$ and $k \in \mathbb{Z}$, then $k = 1$ and $-e = -x + y - xy$ so that $e = x + (-y) - x(-y) = x \circ (-y) \in x \circ R$.

The argument for (1) $\Leftrightarrow$ (4) is similar. (1) $\Leftrightarrow$ (5) holds by [3, Theorem 1.2]. $\square$

By [3, Lemma 1.1], using the term exchange ring and notation Exch for this generalized concept agrees with the unital case. The implication Cln $\Rightarrow$ Exch continues to hold by [22, Theorem 2]. Exch is closed under the formation of corners by [3, Proposition 1.3] and, using conditions (1) or (2) from Proposition 2.10, it is direct to show that Exch is closed under the formation of direct limits.

We note that all relations of diagram (D1) hold. Indeed, Proposition 2.5 shows that the relation $\Leftarrow$ of the diagram holds and [19, Example 4.3] (Example 2.3) shows that it is strict. Examples showing that three horizontal $\Rightarrow$ are strict are known to exist among unital rings. Example of a clean ring with a non-clean corner from [24] provides an example of a clean ring which does not have Reg+sr=1. Thus, all implications of diagram (D1) are strict.

2.7. **Local cancellation properties.** Recall that a general ring R is said be a *locally unital* ring and it is said to have *local units* if for every finite set F, there is an idempotent e such that $F \subseteq eRe$.

If P a property of (unital or general) rings, we say that a general ring R has *local* P if every finite set of elements of R is contained in a corner of R which has P. This "local" approach has been used for generalizing unital UR in [2] and unital DF in [26]. In [16], the properties of being Baer and Rickart were generalized using an approach equivalent to this one (since being Baer and being Rickart transfer to corners).

If a general ring R has local P, then it has local units. If P transfers to corners and R is a locally unital ring which has P, then R has local P (since any finite set is contained in a corner and every corner has P). We show the converse of ($P \Rightarrow$ local P) for all five cancellation properties.

Proposition 2.11. *If* P *is any of* UR, sr=1, DF, Cln, *or* Exch*, then (local* $P \Rightarrow P$*) holds.*

Proof. For UR, let $x \in R$ and let $e \in R$ be idempotent such that $x \in eRe$ and eRe is unit-regular. Let $u \in U(eRe)$ be such that $x = xux$. Thus, $u - e \in U(*)$ by Lemma 2.1 and $x(u - e)x + x^2 = xux - x^2 + x^2 = x$. Hence, x is a unit-regular element of R.

For sr=1, let $x, y \in R$ be such that $0 \in x * R + yR$ and let $e = e^2$ be such that $x, y \in eRe$ and that eRe has stable range one. If $0 = x * u + yv = x + u + xu + yv$ for some $u, v \in R$, then $x + ue + xue + yve = 0$. As $x, y, xue, yve \in eRe$, ue is in eRe and $yve = yeve$. Thus, $0 = x * ue + yve \in x * eRe + yeRe$. By the assumption that sr=1 holds for eRe, there is $z \in eRe \subseteq R$ such that $0 \in (x + yz) * eRe \subseteq (x + yz) * R$.

For DF, let $x, y \in R$ and let $e = e^2 \in R$ be such that $x, y \in eRe$ and eRe is DF. If $x * y = 0$, then $(x + e)(y + e) = e$ and so $(y + e)(x + e) = e$ which implies $y * x = 0$.

For Cln, let $x \in R$ and $e = e^2 \in R$ be such that $x \in eRe$ and eRe is clean. As $e + x \in eRe$, let $e + x = f + u$ for some $f = f^2 \in eRe$ and $u \in U(eRe)$. Then $x = f + (u - e)$ and $u - e \in U(*)$ by Lemma 2.1.

For Exch, if $x \in R$ and $e \in R$ is such that $x \in eRe$ and eRe is exchange, then there is $f = f^2 \in eRe$ such that $f \in xeRe \subseteq xR$ and $f \in x \circ eRe \subseteq x \circ R$. Thus, x is exchange in R. □

3. Cancellation properties of graded rings

We turn to graded rings now. Unless stated otherwise, Γ denotes an arbitrary group and ε denotes its identity element.

3.1. **Graded rings prerequisites.** A general ring R is *graded* by a group Γ if $R = \bigoplus_{\gamma\in\Gamma} R_\gamma$ for additive subgroups R_γ and $R_\gamma R_\delta \subseteq R_{\gamma\delta}$ for all $\gamma, \delta \in \Gamma$. The elements of the set $H = \bigcup_{\gamma\in\Gamma} R_\gamma$ are said to be *homogeneous*. The grading is *trivial* if $R_\gamma = 0$ for every $\varepsilon \neq \gamma \in \Gamma$. We adopt the standard definitions of graded ring homomorphisms, graded left and right R-modules, graded module homomorphisms, graded algebras, graded left and right ideals, graded left and right free and projective modules as defined in [23] and [13]. A Γ-graded unital ring R is a *graded division ring* if every $0 \neq x \in H$ is invertible. In this case, R is a *graded field* if R is also commutative.

If M is a graded right R-module and $\gamma \in \Gamma$, the *γ-shifted or γ-suspended* graded right R-module $(\gamma)M$ is defined as the module M with the Γ-grading given by $(\gamma)M_\delta = M_{\gamma\delta}$ for any $\delta \in \Gamma$. If M and N are graded right R-modules and $\gamma \in \Gamma$, then $\operatorname{Hom}_R(M,N)_\gamma$ denotes the following

$$\operatorname{Hom}_R(M,N)_\gamma = \{f \in \operatorname{Hom}_R(M,N) \mid f(M_\delta) \subseteq N_{\gamma\delta} \text{ for any } \delta \in \Gamma\},$$

then the subgroups $\operatorname{Hom}_R(M,N)_\gamma$ of $\operatorname{Hom}_R(M,N)$ intersect trivially and $\operatorname{HOM}_R(M,N)$ denotes their direct sum $\bigoplus_{\gamma\in\Gamma} \operatorname{Hom}_R(M,N)_\gamma$. The notation $\operatorname{END}_R(M)$ is used in the case if $M = N$. If M is finitely generated (which is the case we often consider), then $\operatorname{Hom}_R(M,N) = \operatorname{HOM}_R(M,N)$ for any N (both [23] and [13] contain details) and $\operatorname{End}_R(M) = \operatorname{END}_R(M)$ is a Γ-graded unital ring.

In [13], for a Γ-graded unital ring R and $\gamma_1, \dots, \gamma_n \in \Gamma$, $\mathbb{M}_n(R)(\gamma_1, \dots, \gamma_n)$ denotes the ring of matrices $\mathbb{M}_n(R)$ with the Γ-grading given by

$$(r_{ij}) \in \mathbb{M}_n(R)(\gamma_1, \dots, \gamma_n)_\delta \quad \text{if} \quad r_{ij} \in R_{\gamma_i^{-1}\delta\gamma_j} \text{ for } i, j = 1, \dots, n.$$

In [23], $\mathbb{M}_n(R)(\gamma_1, \dots, \gamma_n)$ is defined so that it is $\mathbb{M}_n(R)(\gamma_1^{-1}, \dots, \gamma_n^{-1})$ using the definition from [13] (more details on the relations between two definitions can be found in [29, Section 1]). We opt to use the definition from [13]. With this definition, if F is the graded free right module $(\gamma_1^{-1})R \oplus \cdots \oplus (\gamma_n^{-1})R$ (and any finitely generated graded free right R-module is of that form, see [23] or [13]), then $\operatorname{Hom}_R(F,F) \cong_{\mathrm{gr}} \mathbb{M}_n(R)(\gamma_1, \dots, \gamma_n)$ as graded unital rings.

We also recall [13, Theorem 1.3.3] stating the following lemma for Γ abelian only, but the proof generalizes to arbitrary Γ. [23, Remark 2.10.6] also states the first two parts for arbitrary Γ.

Lemma 3.1. [13, Theorem 1.3.3]. *Let R be a Γ-graded unital ring and $\gamma_1, \dots, \gamma_n \in \Gamma$.*

(1) *If π a permutation of the set $\{1, \dots, n\}$, then*
$$\mathbb{M}_n(R)(\gamma_1, \gamma_2, \dots, \gamma_n) \cong_{\mathrm{gr}} \mathbb{M}_n(R)(\gamma_{\pi(1)}, \gamma_{\pi(2)} \dots, \gamma_{\pi(n)}).$$
(2) *If δ in the center of Γ, $\mathbb{M}_n(R)(\gamma_1, \gamma_2, \dots, \gamma_n) = \mathbb{M}_n(R)(\gamma_1\delta, \gamma_2\delta, \dots, \gamma_n\delta)$.*
(3) *If $\delta \in \Gamma$ is such that there is an invertible element u_δ in R_δ, then*
$$\mathbb{M}_n(R)(\gamma_1, \gamma_2, \dots, \gamma_n) \cong_{\mathrm{gr}} \mathbb{M}_n(R)(\gamma_1\delta, \gamma_2 \dots, \gamma_n).$$

We recall some examples we often consider in the rest of the paper. Let K be a field and consider it to be trivially graded by $\mathbb{Z}$. By Lemma 3.1, every graded matrix algebra over K is graded isomorphic to $\mathbb{M}_n(K)(0, l_1, \dots, l_{n-1})$ where l_i are integers such that $0 \leq l_i \leq l_{i+1}$ for $i = 1, \dots, n-2$.

If m is a positive integer, let $K[x^m, x^{-m}]$ denote the ring of Laurent polynomials over x^m. We consider this ring $\mathbb{Z}$-graded by $K[x^m, x^{-m}]_{km} = Kx^{km}$ and $K[x^m, x^{-m}]_{km+l} = 0$ for any integer k and $l \in \{1, \ldots, m-1\}$. By Lemma 3.1, every graded matrix ring over $K[x^m, x^{-m}]$ is graded isomorphic to $\mathbb{M}_n(K[x^m, x^{-m}])(0, l_1, \ldots, l_{n-1})$ where $l_i \in \{0, 1, \ldots, m-1\}$ and $l_i \leq l_{i+1}$ for $i = 1, \ldots, n-2$.

3.2. **Cancellation properties of graded rings.** If P is a (general or unital) ring property, the term *graded property* P has been used for the property P_{gr} obtained by replacing every $\forall x$ and $\exists x$ appearing in P by the restricted versions $\forall x \in H$ and $\exists x \in H$ where H is the set of homogeneous elements. For example, since the property $(\forall x)(\exists y)(xyx = x)$ defines a regular ring, a graded ring R is *graded regular* if $(\forall x \in H)(\exists y \in H)(xyx = x)$ and we denote this condition by Reg_{gr}.

Similarly, if R is a graded unital ring, then R is *graded unit-regular* if for every $x \in H$, there is $u \in U(R) \cap H$ such that $xux = x$; R has *graded stable range one* if the condition $(\gamma^{-1})xR + (\delta^{-1})yR = R$ for some $x \in R_\gamma, y \in R_\delta$, implies that there is $z \in H$ such that $(\gamma^{-1})(x + yz)R = R$; R is *graded directly finite* if $xy = 1$ implies $yx = 1$ for all $x, y \in H$. By [28], $\text{UR}_{\text{gr}} \Rightarrow \text{sr=1}_{\text{gr}} \Rightarrow \text{DF}_{\text{gr}}$.

We define the graded versions of Cln and Exch similarly. We note that the graded nil clean property, analogously defined, was considered in [17].

Definition 3.2. Let R be a graded unital ring. We say that R is *graded clean* if every homogeneous $x \in R$ is a sum of a homogeneous unit and a homogeneous idempotent and we call such sum a *graded clean decomposition* of x. If R is graded clean, we say that Cln_{gr} holds for R.

We say that R is *graded right exchange* and that $\text{Exch}^r_{\text{gr}}$ holds for R if for every homogeneous $x \in R$, there is a homogeneous idempotent $e \in R$ such that $e \in xR$ and $1 - e \in (1 - x)R$. The condition $\text{Exch}^l_{\text{gr}}$ is defined analogously. The ring R is graded exchange if it is both graded left and graded right exchange and we say that Exch_{gr} holds in this case.

These definitions coincide with those of Cln and Exch respectively if the grading is trivial. We examine the graded versions of Cln and Exch in more details next.

3.3. **Graded cleanness.** The condition Cln_{gr} is very restrictive as the following lemma shows. This lemma can be compared with [28, Lemma 2.5] which states that every nonzero component of a graded unit-regular (unital) graded ring contains an invertible element.

Lemma 3.3. *Let R be a Γ-graded unital ring. Then R is graded clean if and only if R_ε is clean and each nonzero element of R_γ is invertible for every $\gamma \neq \varepsilon$.*

Proof. If Cln_{gr} holds for R then R_ε is clean. If $0 \neq x \in R_\gamma$ for some $\gamma \neq \varepsilon$ and $x = u + e$ for some $e = e^2 \in R_\varepsilon$ and $u \in R_\delta \cap U(R)$, then $\gamma = \delta$ and $e = 0$. Thus, $x = u$ is invertible.

Conversely, let $x \in R_\gamma$. If $\gamma = \varepsilon$, then x is in R_ε so a clean decomposition of x in R_ε is a graded clean decomposition of x in R. If $\gamma \neq \varepsilon$ and $x = 0$, $x = 1 + (-1)$ is a graded clean decomposition. If $\gamma \neq \varepsilon$ and $x \neq 0$, then $x \in U(R)$ and so $x = 0 + x$ is a graded clean decomposition. □

By this lemma, a graded division ring is graded clean. The converse does not hold (take a clean ring which is not a division ring and grade it trivially).

We use Lemma 3.3 to characterize when a graded matrix ring is graded clean next.

Proposition 3.4. *Let n be a positive integer, $\gamma_1, \gamma_2, \ldots, \gamma_n \in \Gamma$, and R be a Γ-graded unital ring.*

(1) *If R is trivially graded, then $\mathbb{M}_n(R)(\gamma_1, \gamma_2 \ldots, \gamma_n)$ is graded clean if and only if $\gamma_1 = \gamma_2 = \ldots = \gamma_n$ and $\mathbb{M}_n(R)$ is clean.*

(2) *If R is not trivially graded, then $\mathbb{M}_n(R)(\gamma_1, \gamma_2 \ldots, \gamma_n)$ is graded clean if and only if $n = 1$ and R is graded clean.*

Proof. Let $S = \mathbb{M}_n(R)(\gamma_1, \gamma_2 \ldots, \gamma_n)$. The standard matrix unit e_{ij} is in $S_{\gamma_i\gamma_j^{-1}}$ so it is homogeneous.

(1) If S is graded clean and $\gamma_i \neq \gamma_j$, then e_{ij} is not invertible and in $S_{\gamma_i\gamma_j^{-1}} \neq S_\varepsilon$. By Lemma 3.3 this cannot happen and so $\gamma_i = \gamma_j$. Thus, $S_\varepsilon =$ and $\mathbb{M}_n(R_\varepsilon) = \mathbb{M}_n(R)$ since R is trivially graded. Thus, the assumption that S is graded clean implies that $S_\varepsilon = \mathbb{M}_n(R)$ is clean. To show the converse, assume that $\gamma_1 = \gamma_2 = \ldots = \gamma_n$ and that $\mathbb{M}_n(R) = S_\varepsilon$ is clean. Then $S_\gamma = 0$ for all $\gamma \neq \varepsilon$. Thus, S is graded clean by Lemma 3.3.

(2) As R is not trivially graded, there is $\delta \neq \varepsilon$ so that $R_{\gamma_1^{-1}\delta\gamma_1} \neq 0$. Let $0 \neq x \in R_{\gamma_1^{-1}\delta\gamma_1}$. If $n > 1$, then $xe_{11} \in S_\delta$ is noninvertible. If S is graded clean, this cannot happen, so $n = 1$. Since $S = \mathbb{M}_1(R)(\gamma_1) \cong_{gr} (\gamma_1^{-1})R(\gamma_1)$ is graded clean, so is R. Conversely, if $n = 1$, then $S \cong_{\mathrm{gr}} (\gamma_1^{-1})R(\gamma_1)$. The graded cleanness of R implies the graded cleanness of $(\gamma_1^{-1})R(\gamma_1)$. □

Proposition 3.4 shows that $\mathrm{Cln}_{\mathrm{gr}}$ is not closed under the formation of graded matrix rings. The property $\mathrm{Cln}_{\mathrm{gr}}$ is also not closed under the formation of direct sums. Indeed, $K[x, x^{-1}]$ is graded clean but $(x, 0)$ is a homogeneous element of $R = K[x, x^{-1}] \oplus K[y, y^{-1}]$ which is not invertible, so R is not graded clean by Lemma 3.3.

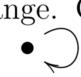
The next example shows that the conditions $\mathrm{UR}_{\mathrm{gr}}$ and $\mathrm{Cln}_{\mathrm{gr}}$ are independent properties.

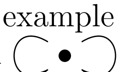
Example 3.5. Let K be a field trivially graded by $\mathbb{Z}$, let $K[x^2, x^{-2}]$ be naturally $\mathbb{Z}$-graded (see section 3.1), and let $R = \mathbb{M}_2(K[x^2, x^{-2}])(0, 1)$. By [28, Proposition 5.1], R is graded unit-regular. By Proposition 3.4, R is not graded clean. Thus, $\mathrm{UR}_{\mathrm{gr}} \not\Rightarrow \mathrm{Cln}_{\mathrm{gr}}$.

If a unital ring which is clean but not unit-regular (for example, the ring from [25, Section 3]) is graded trivially by any group, we obtain an example showing that $\mathrm{Cln}_{\mathrm{gr}} \not\Rightarrow \mathrm{UR}_{\mathrm{gr}}$.

The pairs (Cln, $\mathrm{Cln}_{\mathrm{gr}}$) and (Exch, $\mathrm{Exch}_{\mathrm{gr}}$) are pairs of mutually independent properties.

Example 3.6. The $\mathbb{Z}$-graded ring $\mathbb{M}_2(K)(0, 1)$ is clean (because $\mathbb{M}_2(K)$ is clean) but not graded clean by Proposition 3.4.

The ring $K[x, x^{-1}]$ is a graded field so it is graded clean and graded exchange. However, $K[x, x^{-1}]$ is neither clean nor exchange. One can see this last fact by representing this algebra as a Leavitt path algebra of the graph  and using [4, Theorem 4.5].

The Leavitt algebra $L_K(1, 2)$ is a universal example of a K-algebra R such that $R^2 \cong R$, and, also, it is the Leavitt path algebra of the graph . This algebra is exchange (by [4, Theorem 4.5]) and neither left nor graded right exchange by Proposition 4.5 (Example 4.6 has more details).

3.4. **Unital graded exchange.** If the grading is trivial, $\mathrm{Exch}^r_{\mathrm{gr}}$ and $\mathrm{Exch}^l_{\mathrm{gr}}$ are equivalent. Recall that this holds by the proof of

$$\mathrm{Exch}^r \text{ holds for } \mathrm{End}_R(R_R) \;\Rightarrow\; \mathrm{FinExch}(R_R) \text{ holds } \;\Rightarrow\; \mathrm{Exch}^l \text{ holds for } \mathrm{End}_R(R_R).$$

This proof does not transfer to the graded case. Also, the condition "$\mathrm{FinExch}(R_R)$ holds" is equivalent with $\mathrm{End}_R(R_R)_\varepsilon$ being exchange which is strictly weaker than $\mathrm{Exch}^r_{\mathrm{gr}}$ or $\mathrm{Exch}^l_{\mathrm{gr}}$ holding on $\mathrm{End}_R(R_R)$ (Example 4.6 shows this). However, if R and its opposite ring are graded isomorphic, then the conditions $\mathrm{Exch}^r_{\mathrm{gr}}$ and $\mathrm{Exch}^l_{\mathrm{gr}}$ are equivalent. This is the case when R is a graded involutive ring (i.e. if R has an involution $*$ such that $(R_\gamma)^* \subseteq R_{\gamma^{-1}}$ for any $\gamma \in \Gamma$). The rings we focus on (graded matrix algebras over graded fields and Leavitt path algebras) are graded involutive.

If R is a graded general ring, let $\text{Lift}^r_{\text{gr}}$ denote the condition that for every homogeneous $x \in R$, there is a homogeneous idempotent $e \in R$ such that $e - x \in (x - x^2)R$. If R is unital, $\text{Exch}^r_{\text{gr}}$ and $\text{Lift}^r_{\text{gr}}$ are equivalent. Indeed, following the proof of Exch $\Leftrightarrow$ Lift from [21, Proposition 1.1], one can take e from $\text{Exch}^r_{\text{gr}}$ condition and use it in $\text{Lift}^r_{\text{gr}}$ condition since if $e = xr$ and $1 - e = (1 - x)s$, then $e - x = e - xe - x + xe = (1 - x)e - x(1 - e) = (1 - x)xr - x(1 - x)s = (x - x^2)(r - s)$. Conversely, if $\text{Lift}^r_{\text{gr}}$ holds for $x \in H$ with $e = e^2 \in H$, and $e - x = (x - x^2)r$, then $e = x + (x - x^2)r \in xR$ and $1 - e = 1 - x - (x - x^2)r = (1 - x)(1 - xr) \in (1 - x)R$ showing $\text{Exch}^r_{\text{gr}}$.

Since $\text{Exch}^r_{\text{gr}} \Leftrightarrow \text{Lift}^r_{\text{gr}}$, the same argument showing Cln $\Rightarrow$ Exch (see the end of section 1) can be used for $\text{Cln}_{\text{gr}} \Rightarrow \text{Exch}^r_{\text{gr}}$. As Cln_{gr} is left-right symmetric, we have that $\text{Cln}_{\text{gr}} \Rightarrow \text{Exch}_{\text{gr}}$.

We explore some conditions under which certain graded matrix algebras are graded exchange.

Proposition 3.7. *Let K be a trivially $\mathbb{Z}$-graded field, m, n positive integers, and $\gamma_1, \ldots, \gamma_n \in \mathbb{Z}$. Then, the following hold.*

(1) *$\mathbb{M}_n(K)(\gamma_1, \ldots, \gamma_n)$ is graded exchange.*
(2) *Let k_i denote the cardinality of the set $\{\gamma \in \{\gamma_1, \ldots, \gamma_n\} \mid \gamma \equiv i \text{ modulo } m\}$ for $i = 0, \ldots, m-1$. If k_i is either 0 or 1 for each $i = 0, \ldots, m-1$, then $\mathbb{M}_n(K[x^m, x^{-m}])(\gamma_1, \ldots, \gamma_n)$ is graded exchange.*

Proof. Graded rings in both parts are graded involutive ([15] has more details), so showing that $\text{Exch}^r_{\text{gr}}$ holds ensures that $\text{Exch}^l_{\text{gr}}$ also holds.

To show (1), let $S = \mathbb{M}_n(K)(\gamma_1, \ldots, \gamma_n)$. By Lemma 3.1, we can assume that $\gamma_1 = 0$ and $\gamma_i \le \gamma_{i+1}$ for all $i = 1, \ldots n-1$. Then S_l consists of matrices with zeros on and above the main diagonal if $l > 0$, and, S_l consists of matrices with zeros on and below the main diagonal if $l < 0$. Thus, $1_S - a$ is invertible for any $a \in S_l$ if $l \neq 0$. So, $e = 0$ is such that $e \in aS$ and $1_S = 1_S - e \in (1_S - a)S = S$. Since S_0 is a direct sum of matrix algebras over K, S_0 is exchange.

To show (2), let $S = \mathbb{M}_n(K[x^m, x^{-m}])(\gamma_1, \ldots, \gamma_n)$. By Lemma 3.1, we can assume that $\gamma_j \in \{0, 1, \ldots, m-1\}$ and that $\gamma_j \le \gamma_{j+1}$ for all $j = 1, \ldots n$. Assume that $k_i = 0$ or $k_i = 1$ for all $i = 0, \ldots, m-1$. For an arbitrary homogeneous element a of S, we claim that either a is right invertible, $1_S - a$ is right invertible, or a is in the K-linear span of some of the standard matrix units $e_{ii}, i = 1, \ldots, n$. By taking $e = 1_S$ in the first case, $e = 0$ in the second case, and $e = \sum_{a_{ii} \neq 0} e_{ii}$ in the third case, we obtain a homogeneous idempotent e such that $e \in aS$ and $(1_S - e) \in (1_S - a)S$.

Let a be in $S_{l'}$ and let $l' = km + l$, for some $l = 0, \ldots, m-1$. Consider the cases $l = 0$ and $l \neq 0$.

If $l = 0$, then a is diagonal with the diagonal entries in Kx^{km}. Thus, $a = \sum_{a_{ii} \neq 0} a_{ii} e_{ii}$.

If $l \neq 0$, we consider two cases: (1) $k_i = 1$ for all i and, (2) $k_i = 0$ for some i.

If all k_i are 1, then $n = m$, $S = \mathbb{M}_m(K(x^m, x^{-m}))(0, 1, \ldots, m-1)$, and a has the following form for some $a_1, \ldots, a_m \in K$.

$$a = \left[\begin{array}{cccc|cccc} 0 & 0 & \ldots & 0 & a_1x^{km} & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 & 0 & a_2x^{km} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & a_lx^{km} \\ \hline a_{l+1}x^{(k+1)m} & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\ 0 & a_{l+1}x^{(k+1)m} & \ldots & 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & a_mx^{(k+1)m} & 0 & 0 & \ldots & 0 \end{array}\right]$$

We explore the condition that $1_S - a$ is right invertible. If $(1_S - a)b = 1_S$ for some $b = [b_{ij}]$, then

$$b_{ij} - a_i x^{km} b_{(m-l+i)j} = \delta_{ij} \text{ for } i = 1, \dots, l \text{ and } j = 1, \dots, m \text{ and}$$

$$b_{ij} - a_i x^{(k+1)m} b_{(i-l)j} = \delta_{ij} \text{ for } i = l+1, \dots, m \text{ and } j = 1, \dots, m$$

where δ_{ij} stands for $\delta_{ij} = 1$ if $i = j$ and 0 otherwise. If we fix $j = 1, \dots, m$ and consider these equations as equations in m unknowns $b_{ij}, i = 1, \dots, m$, then one can eliminate the variables b_{ij} for $i \neq j$ using the relations above and reduce the system to the following single equation in b_{jj}.

$$b_{jj} \left(1 - \left(\prod_{i=1}^{m} a_i \right) x^{m((k+1)m-l)} \right) = 1$$

Thus, if any of $a_1, \dots, a_m$ is zero, this equation has a unique solution $b_{jj} = 1$ which determines the values of all other b_{ij} for $i \neq j$ so $1_S - a$ is right invertible.

If none of $a_1, \dots, a_m$ is zero, then a is an invertible matrix with the inverse

$$a^{-1} = \left[\begin{array}{cccc|cccc} 0 & 0 & \dots & 0 & a_{l+1}^{-1} x^{-(k+1)m} & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & a_{l+2}^{-1} x^{-(k+1)m} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & a_m^{-1} x^{-(k+1)m} \\ \hline a_1^{-1} x^{-km} & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & a_2^{-1} x^{-km} & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_l^{-1} x^{-km} & 0 & 0 & \dots & 0 \end{array} \right].$$

It remains to consider the case when some k_i is zero in which case $n < m$. Adding $m - 1 - i$ to all shifts, we can assume that $i = m - 1$. We use induction on $m - n$ to show that $1_S - a$ is right invertible. If $m - n = 1$, then $k_0 = \dots = k_{m-2} = 1$ and S embeds in $S' = \mathbb{M}_m(0, 1, \dots, m-1)$ by $\phi : s \mapsto \begin{bmatrix} s & 0 \\ 0 & 0 \end{bmatrix}$ for $s \in S$. Since ϕ is a graded map, $\phi(a)$ is homogeneous in S'. As $\phi(a)$ is not right invertible in S', $1_{S'} - \phi(a)$ is right invertible by case (1). If a right inverse of $1_{S'} - \phi(a)$ has b in the upper-left block, then b is a right inverse of $1_S - a$.

Assuming the induction hypothesis, let us show the claim for $m - n > 1$. In this case, there is a graded embedding ϕ of S in $S' = \mathbb{M}_{n+1}(k_0(0), k_1(1), \dots, k_{m-2}(m-2), m-1)$ where $k_i(i)$ is the empty list if $k_i = 0$ and $k_i(i) = i$ if $k_i = 1$. By the induction hypothesis, $1_{S'} - \phi(a)$ is right invertible. If b is the upper-left block, then b is a right inverse of $1_S - a$. □

Using Proposition 3.7, we show that the implication $\mathrm{Cln}_{\mathrm{gr}} \Rightarrow \mathrm{Exch}_{\mathrm{gr}}$ is strict.

Example 3.8. Let $R = \mathbb{M}_2([x^2, x^{-2}])(0,1)$ (note that R can also be realized as the Leavitt path algebra of the graph $\bullet \rightleftarrows \bullet$). By Proposition 3.7, R is graded exchange but, by Proposition 3.4, R is not graded clean. Thus, $\mathrm{Exch}_{\mathrm{gr}} \not\Rightarrow \mathrm{Cln}_{\mathrm{gr}}$.

$\mathrm{Exch}^r_{\mathrm{gr}}$ is closed under the formation of graded corners (i.e. the corner formed by a homogeneous idempotent) and graded direct limits (the proof is completely analogous to the proof of the non-graded case). Besides these favorable properties, $\mathrm{Exch}^r_{\mathrm{gr}}$ is still a rather restrictive condition. For instance, Example 4.6 shows that $\mathrm{Reg}_{\mathrm{gr}} \not\Rightarrow \mathrm{Exch}^r_{\mathrm{gr}}$.

3.5. **Nonunital graded exchange.** The equivalence of the following conditions enables us to consider $\mathrm{Exch}^r_{\mathrm{gr}}$ in the nonunital case also. The proof of Proposition 2.10 transfers directly to the proof of the following.

Proposition 3.9. *If R is a graded general ring and $\gamma \in \Gamma$, the following conditions are equivalent.*

(1) *For any $x \in R_\gamma$, there is $e = e^2 \in R_\varepsilon$ such that $e \in -xR$ and $e \in x * R$.*
(2) *For any $x \in R_\gamma$, there is $e = e^2 \in R_\varepsilon$ such that $e \in xR$ and $e \in x \circ R$.*
(3) *For any $x \in R_\gamma$, there is $e = e^2 \in R_\varepsilon$ such that $e \in xR$ and $(-e, 1) \in (-x, 1)R^u$.*
(4) *For any graded embedding $\phi : R \to S$ such that S is a graded unitization of R, and for any $x \in R_\gamma$, there is $e = e^2 \in R_\varepsilon$ such that $e \in xR$ and $1 - \phi(e) \in (1 - \phi(x))S$.*

If R is graded unital, then R is graded right exchange if and only if the above conditions hold.

We continue to say that a ring is *graded right exchange* and that it has $\mathrm{Exch}^r_{\mathrm{gr}}$ if the equivalent conditions above hold. $\mathrm{Exch}^l_{\mathrm{gr}}$ and $\mathrm{Exch}_{\mathrm{gr}}$ are similarly generalized for graded general rings.

Proposition 3.10. *The property $\mathrm{Exch}^r_{\mathrm{gr}}$ is closed under the formation of graded corners and graded direct limits of graded* general *rings.*

Proof. The proof of [3, Proposition 1.3] adapts to the graded case to show the claim on graded corners. Parts (1) and (2) of Proposition 3.9 can be used for the claim on graded direct limits. □

3.6. **Graded local cancellation properties.** Recall that a graded general ring R is said to have graded local units if every finite set of homogeneous elements of R is contained in a graded corner. In this case, one can meaningfully consider the graded (and nonunital) versions of ring properties by considering their graded local versions as follows. If R is a graded general ring and P is a ring property, we say that R has *graded local* P (and that local P_{gr} holds for R) if every finite set of homogeneous elements of R is contained in a graded corner which has the property P_{gr}.

If R is a graded unital ring, then $P_{\mathrm{gr}} \Leftrightarrow$ local P_{gr} for any property P. For nonunital rings, considering the trivial grading, and examples from section 2.7, one can show that $P_{\mathrm{gr}} \not\Rightarrow$ local P_{gr} in general. Also, the proof of (local UR $\Rightarrow$ UR) does not generalize to the graded case since $u \in U(*) \cap H$ such that $x = xux + x^2$ for some $x \in H$ can be found only if $x \in R_\varepsilon$. This enables us to create an example of a locally $\mathrm{UR}_{\mathrm{gr}}$ ring such that the graded version of condition (1) from Proposition 2.2 fails.

Example 3.11. Consider $K[x^2, x^{-2}]$ with the standard $\mathbb{Z}$-grading (see section 3.1), let $R_{2n} = \mathbb{M}_{2n}(K[x^2, x^{-2}])(0, 1, 0, 1, \ldots, 1)$ and $R_{2n+1} = \mathbb{M}_{2n+1}(K[x^2, x^{-2}])(0, 1, 0, 1, \ldots, 0)$. The number of shifts which are zero is equal to the number of shifts which are one for R_{2n} so $\mathrm{UR}_{\mathrm{gr}}$ holds on R_{2n} by [28, Proposition 5.1]. By the same result, $\mathrm{UR}_{\mathrm{gr}}$ does not hold on R_{2n+1}. Embed R_n into R_{n+1} using the same embedding as in Example 2.3 and let $R = \varinjlim_n R_n$. Every homogeneous matrix of R_{2n+1} embeds in R_{2n+2} which is graded isomorphic to a graded unit-regular corner of R. Hence, R is locally $\mathrm{UR}_{\mathrm{gr}}$. However, no homogeneous $u \in U(*) \cap H$ such that $x = xux + x^2$ can be found for any $x \in H$ when x is not in the 0-component.

This example also shows that having graded corners which are not graded unit-regular is not an obstruction for a graded locally unital ring to be graded locally unit-regular. This contrasts the non-graded case in which a locally UR ring cannot have non-UR corners.

For graded unital rings, it is direct to see that local $\mathrm{DF}_{\mathrm{gr}}$ is equivalent to $\mathrm{DF}_{\mathrm{gr}}$. For Exch, $\mathrm{Exch}^r_{\mathrm{gr}} \Leftrightarrow$ local $\mathrm{Exch}^r_{\mathrm{gr}}$ holds for all graded *locally* unital rings. Indeed, the direction $\Rightarrow$ holds since $\mathrm{Exch}^r_{\mathrm{gr}}$ transfers to graded corners and the direction $\Leftarrow$ holds by the proof analogous to the non-graded case (see Proposition 2.11).

3.7. ε **cancellation properties.** Besides the graded versions of ring properties, one can define the graded versions of module properties as follows. If $P(A)$ is a property of a module A, we let $P_{\mathrm{gr}}(A)$ denote the statement on a graded module A obtained by replacing every instance of "module" by "graded module" and every instance of "homomorphism" by "graded homomorphism" in $P(A)$.

Since a graded homomorphism between right modules A and B is an element of $\mathrm{HOM}_R(A,B)_\varepsilon$, the ε-component of $\mathrm{END}_R(R_R)$ has a special significance. If P is a property of graded (unital or general) rings, let P_ε denote the requirement of a graded general ring R that $\mathrm{END}_R(R_R)_\varepsilon$ has P.

With this definition, if $\mathrm{END}_R(R_R)$ is graded regular, UR_ε holds for R if and only if R_R has graded internal cancellation. [28, Proposition 3.4] shows this in the unital case using [10, Theorem 4.1]. This last results holds in the nonunital case and the proof adapts to the grading case also. The condition that R has sr$=1_\varepsilon$ is equivalent with the requirement that the graded module R_R has graded substitution (the proof of [28, Theorem 4.4] generalizes to the nonunital case also). We also have that DF_ε holds for R if and only if R_R is graded directly finite as a graded module (the proof from [28, Section 4.3] in the unital case carries to the nonunital case).

With Cln_ε and $\mathrm{Exch}_\varepsilon$ defined analogously, the implications below are direct since the cancellation ε-properties reduce to the non-graded case.

$$\mathrm{UR}_\varepsilon \;\Rightarrow\; \mathrm{sr}{=}1_\varepsilon \;\Rightarrow\; \mathrm{DF}_\varepsilon \quad \text{and} \quad \mathrm{UR}_\varepsilon \;\Rightarrow\; \mathrm{Cln}_\varepsilon \;\Rightarrow\; \mathrm{Exch}_\varepsilon$$

If R is unital, $\mathrm{End}_R(R_R)=\mathrm{END}_R(R_R)\cong_{\mathrm{gr}} R$, so $P_{\mathrm{gr}} \;\Rightarrow\; P_\varepsilon$ for any of the cancellation properties P. When P is UR, sr=1, or DF, this implication is strict by [28]. The first ring from Example 3.5 shows that $\mathrm{Cln}_{\mathrm{gr}} \not\Leftarrow \mathrm{Cln}_\varepsilon$. The 0-component of a unital Leavitt path algebra $L_K(E)$ is a matricial algebra over K so $\mathrm{Exch}_\varepsilon$ holds. If E has exits, $L_K(E)$ does not have $\mathrm{Exch}_{\mathrm{gr}}$ by Proposition 4.5.

The ε-properties are much less restrictive than the gr-properties and sufficient to consider when one is interested in graded versions of module properties. In [28], it is shown that UR_ε, sr$=1_\varepsilon$, and DF_ε retain many properties of their non-graded counterparts. The same holds for Cln_ε and $\mathrm{Exch}_\varepsilon$. For example, Cln_ε is closed under the formation of graded matrix rings while $\mathrm{Cln}_{\mathrm{gr}}$ is not. Indeed, if $\gamma\in\Gamma$, then $\mathbb{M}_1(R)(\gamma)=\mathrm{End}_R((\gamma^{-1})R)\cong_{\mathrm{gr}}(\gamma^{-1})R(\gamma)$. The ε-component of this last ring is $R_{\gamma^{-1}\varepsilon\gamma}=R_\varepsilon$, so R has Cln_ε if and only if $\mathbb{M}_1(R)(\gamma)$ has Cln_ε. This, combined with the proof that Cln is closed under the formation of matrix rings ([11, Lemma, page 2598]) shows the claim.

$\mathrm{Exch}_\varepsilon$ is left-right symmetric since Exch is such. Also, requiring R_ε to be exchange is sufficient for the graded version of another important property of Exch to hold. We say that homogeneous idempotents *lift modulo a graded right ideal* I if, for a homogeneous element x such that $x-x^2\in I$, there is $e=e^2\in R_\varepsilon$ such that $e-x\in I$.

Proposition 3.12. *Let R be a graded unital ring. If R_ε is exchange, then homogeneous idempotents lift modulo every graded right ideal.*

Proof. Let I be a graded right ideal, $x\in R_\gamma$ and $x-x^2\in I$. Consider the cases $\gamma=\varepsilon$ and $\gamma\neq\varepsilon$. In the first case, $x-x^2$ is in I_ε which is a right ideal of R_ε. Since R_ε is exchange, Lift holds so there is $e=e^2\in R_\varepsilon$ such that $e-x\in I_\varepsilon\subseteq I$. If $\gamma\neq\varepsilon$, then $x\in I_\gamma\subseteq I$ (and $x^2\in I_{\gamma^2}$) since I is graded. So, one can take $e=0$ and have $e-x=-x\in I$. □

If R is graded right exchange, then R_ε is exchange, so the conclusion of Proposition 3.12 holds. This conclusion is not sufficient for R to be graded exchange since the right ideal $(x-x^2)R$ is not graded if $x \in R_\gamma$ and $\gamma \neq \varepsilon$. In particular, Example 4.6 exhibits a $\mathbb{Z}$-graded ring R such that R_0 is exchange (so the conclusion of Proposition 3.12 holds) and R is not graded exchange.

If R is a graded unital ring and A is a graded right R-module, let $\mathrm{FinExch}_{\mathrm{gr}}(A)$ stand for the graded version of $\mathrm{FinExch}(A)$. The proof of [21, Theorem 2.1] directly translates to the proof of the equivalence

$$\mathrm{FinExch}_{\mathrm{gr}}(A_R) \text{ holds } \quad \Leftrightarrow \quad \text{Exch holds for } \mathrm{END}_R(A)_\varepsilon. \tag{*}$$

Thus, $\mathrm{FinExch}_{\mathrm{gr}}(R_R)$ holds if and only if R has $\mathrm{Exch}_\varepsilon$. If Morita equivalence is suitably defined for graded unital rings (as in [13, Section 2.3]), the following holds.

Proposition 3.13. *The condition* $\mathrm{Exch}_\varepsilon$ *is graded Morita invariant for graded unital rings.*

Proof. If A and B are graded right R-modules, $\mathrm{FinExch}_{\mathrm{gr}}(A \oplus B)$ holds if and only if $\mathrm{FinExch}_{\mathrm{gr}}(A)$ and $\mathrm{FinExch}_{\mathrm{gr}}(B)$ hold (the proof of [8, Lemma 3.10] adapts to the graded case). We also claim that $\mathrm{FinExch}_{\mathrm{gr}}(A)$ holds if and only if $\mathrm{FinExch}_{\mathrm{gr}}((\gamma)A)$ holds for any graded right module A and any $\gamma \in \Gamma$. Indeed, if $(\gamma)A$ is a direct summand of some module $B = B_1 \oplus B_2$, and $\mathrm{FinExch}_{\mathrm{gr}}(A)$ holds, then A is a direct summand of $(\gamma^{-1})B = (\gamma^{-1})B_1 \oplus (\gamma^{-1})B_2$. Thus, there are graded submodules A_i of $(\gamma^{-1})B_i$ such that $A_1 \oplus A_2$ is a complement of A in $(\gamma^{-1})B$. The modules $(\gamma)A_i$ are graded submodules of B_i such that $(\gamma)A_1 \oplus (\gamma)A_2$ is a complement of $(\gamma)A$ in B.

Because of the equivalence (*), this shows that the property $\mathrm{Exch}_\varepsilon$ is closed under the formation of graded matrix algebras. If e is a homogeneous idempotent in R, then $(eRe)_\varepsilon = eR_\varepsilon e$. Thus, the fact that Exch is closed under the formation of corners directly implies that $\mathrm{Exch}_\varepsilon$ is closed under the formation of graded corners. Hence, $\mathrm{Exch}_\varepsilon$ is graded Morita invariant. □

4. Graded cancellation properties of Leavitt path algebras

4.1. **Leavitt path algebras review.** Let E be a directed graph, let E^0 denote the set of vertices, E^1 the set of edges, and $\mathbf{s}$ and $\mathbf{r}$ denote the source and range maps of E. If both E^0 and E^1 are finite, E is a *finite* graph. A *sink* of E is a vertex which does not emit edges. A vertex of E is *regular* if it is not a sink and if it emits finitely many edges and E is said to be *row-finite* if every vertex is either a sink or it is regular. A *path* is a sequence of edges $e_1e_2\ldots e_n$ such that $\mathbf{r}(e_i) = \mathbf{s}(e_{i+1})$ for $i = 1, \ldots, n-1$ or a single vertex. A *cycle* is a closed path such that different edges in the path have different sources. The graph is *acyclic* if there are no cycles. A cycle has *an exit* if a vertex on the cycle emits an edge outside of the cycle. The graph is *no-exit* if no cycle has an exit and it has *Condition (K)* if no vertex is on exactly one cycle.

Extend a graph E to the graph with the same vertices and with edges $E^1 \cup \{e^* \mid e \in E^1\}$ where the range and source relations are the same as in E for $e \in E^1$ and $\mathbf{s}(e^*) = \mathbf{r}(e)$ and $\mathbf{r}(e^*) = \mathbf{s}(e)$ for the added edges. If K is any field, the *Leavitt path algebra* $L_K(E)$ of E over K is a free K-algebra generated by the set $E^0 \cup E^1 \cup \{e^* \mid e \in E^1\}$ such that for all vertices v, w and edges e, f,

(V) $vw = 0$ if $v \neq w$ and $vv = v$, (E1) $\mathbf{s}(e)e = e\mathbf{r}(e) = e$,
(E2) $\mathbf{r}(e)e^* = e^*\mathbf{s}(e) = e^*$, (CK1) $e^*f = 0$ if $e \neq f$ and $e^*e = \mathbf{r}(e)$,
(CK2) $v = \sum_{e \in \mathbf{s}^{-1}(v)} ee^*$ for each regular vertex v.

By the first four axioms, every element of $L_K(E)$ can be represented as a sum of the form $\sum_{i=1}^n k_i p_i q_i^*$ for some n, paths p_i and q_i, and elements $k_i \in K$, for $i = 1, \ldots, n$ where $v^* = v$ for $v \in E^0$ and $p^* = e_n^* \ldots e_1^*$ for a path $p = e_1 \ldots e_n$. Using this representation, one can make $L_K(E)$ into an involutive ring by $(\sum_{i=1}^n k_i p_i q_i^*)^* = \sum_{i=1}^n k_i^* q_i p_i^*$ where $k_i \mapsto k_i^*$ is any involution on K. It is direct to see that $L_K(E)$ is locally unital (with the set of the finite sums of vertices as the set of local units) and that $L_K(E)$ is unital if and only if E^0 is finite in which case the sum of all vertices is the identity. For more details on these basic properties, see [1].

We note a short lemma we need later on.

Lemma 4.1. *If $p = e_1 \dots e_n$ is a path, then $pp^* = \mathbf{s}(p)$ if and only if $\mathbf{s}(e_i)$ emits only e_i for every $i = 1, \dots, n$.*

Proof. We use induction on n. If $n = 1$, $p = e \in E^1$, and $ee^* = \mathbf{s}(e)$, assume that $\mathbf{s}(e)$ emits more than one edge so that $S = \{f \in \mathbf{s}^{-1}(\mathbf{s}(e)) \mid f \neq e\} \neq \emptyset$. If $\leq$ stands for the order on the set of projections (selfadjoint idempotents) given by $p \leq q$ iff $pq = qp = p$, then $ee^* \leq ee^* + ff^*$ and $ee^* = \mathbf{s}(e) \geq ee^* + ff^*$ for any $f \in S$. Thus, $ee^* + ff^* = ee^*$ which implies that $ff^* = 0$ and so $f = f\mathbf{r}(f) = ff^*f = 0f = 0$. This is a contradiction since f is a basis element of $L_K(E)$.

Assuming the induction hypothesis, let $p = eq$ for $e \in E^1$ and a path q with $|q| \geq 1$. If $pp^* = \mathbf{s}(p)$, multiply this relation by e^* on the left and by e on the right to have that $qq^* = \mathbf{r}(e) = \mathbf{s}(q)$. The induction hypothesis applies to q, so it remains to show that $\mathbf{s}(e)$ emits no other edges but e. This holds also by the induction hypothesis since $ee^* = e\mathbf{s}(q)e^* = eqq^*e^* = pp^* = \mathbf{s}(p) = \mathbf{s}(e)$.

The converse is direct by the (CK2) axiom. □

If we consider K to be trivially graded by $\mathbb{Z}$, $L_K(E)$ is naturally graded by $\mathbb{Z}$ so that the n-component $L_K(E)_n$ is the K-linear span of the elements pq^* for paths p, q with $|p| - |q| = n$ where $|p|$ denotes the length of a path p. While one can grade a Leavitt path algebra by any group Γ (see [13, Section 1.6.1]), we always consider the natural grading by $\mathbb{Z}$.

4.2. **Cancellation and graded cancellation properties of Leavitt path algebras.** By [2, Theorems 1 and 2], the conditions that $L_K(E)$ is locally unit-regular, that $L_K(E)$ is regular, and that E is acyclic are equivalent. By Proposition 2.11, local unit-regularity implies unit-regularity so if E is acyclic, $L_K(E)$ is unit-regular. The converse holds since unit-regularity implies regularity which, for a Leavitt path algebra, implies that the underlying graph is acyclic.

By [26, Theorem 4.12], $L_K(E)$ is locally directly finite if and only if E is no-exit. This is equivalent with $L_K(E)$ being directly finite since DF $\Rightarrow$ local DF as DF transfers to corners and since local DF $\Rightarrow$ DF by Proposition 2.11.

We claim that $L_K(E)$ has sr=1 if and only if E is acyclic. If E is acyclic, then $L_K(E)$ is a direct limit of matricial algebras over K. Such direct limit has local sr=1 and so $L_K(E)$ has sr=1 by Proposition 2.11. Conversely, if $L_K(E)$ has sr=1, then it has DF by Proposition 2.8, so E is no-exit. Assuming that there is a cycle in E, let v be any vertex of this cycle. By [1, Lemma 2.2.7], the corner $vL_K(E)v$ is isomorphic to $K[x, x^{-1}]$. This algebra does not have sr=1, so $L_K(E)$ has a non-sr=1 corner making it not having sr=1. Thus, E is acyclic.

By [4, Theorem 4.5], $L_K(E)$ is exchange if and only if E has Condition (K). Thus, if $L_K(E)$ is clean, E has Condition (K). If E is acyclic, $L_K(E)$ has Reg + sr=1, so Cln as well. It is not known exactly which class of graphs sandwiched between the acyclic graphs and graphs with Condition (K) correspond to clean Leavitt path algebras. The diagram below summarizes the relations.

$$\begin{array}{ccc} \boxed{\text{UR, sr=1, Reg}} & \Rightarrow & \boxed{\text{DF}} \\ \Updownarrow & & \Updownarrow \\ \boxed{E\text{ is acyclic.}} & \Rightarrow & \boxed{E\text{ is no-exit.}} \end{array} \qquad \begin{array}{ccccc} \boxed{\text{UR, sr=1, Reg}} & \Rightarrow & \boxed{\text{Cln}} & \Rightarrow & \boxed{\text{Exch}} \\ \Updownarrow & & \Updownarrow & & \Updownarrow \\ \boxed{E\text{ is acyclic.}} & \Rightarrow & \boxed{\textit{To be determined.}} & \Rightarrow & \boxed{E\text{ has Condition (K).}} \end{array}$$

We turn to the graded cancellation properties of Leavitt path algebras now. The 0-component $L_K(E)_0$ is an ultramatricial algebra over K so it is locally unit-regular. Thus, the conditions UR_ε, sr$=1_\varepsilon$, DF_ε, Cln_ε, and $\mathrm{Exch}_\varepsilon$ hold for $L_K(E)$ of any graph E.

By [14, Theorem 3.7], $L_K(E)$ is graded locally directly finite if and only if E is no-exit. If E^0 is finite, these conditions are equivalent with $L_K(E)$ being graded directly finite (see section 3.6). We consider the graded versions of other cancellation properties for Leavitt path algebras next.

4.3. **Graded clean Leavitt path algebras.** For Leavitt path algebras, the condition $\mathrm{Cln}_{\mathrm{gr}}$ is the most restrictive of all gr-cancellation properties.

Proposition 4.2. *If K is a field and E is a graph with finitely many vertices, the following conditions are equivalent.*

(1) *$L_K(E)$ is graded clean.*
(2) *E is either a collection of disjoint vertices with no edges between them or E is $\bullet \circlearrowright$.*

Proof. If $L_K(E)$ is graded clean and E has edges, they are invertible by Lemma 3.3. Assuming that E has more than one vertex and some edges, let u be the inverse of an edge e and let v be a vertex different than $\mathbf{s}(e)$. Then $ve = v\mathbf{s}(e)e = 0$ and so $0 = veu = v \neq 0$ which is a contradiction. Thus, $\mathbf{s}(e) = 1$ is the only vertex in E. Since $e^*e = \mathbf{s}(e) = 1 = ue$, $e^* = u$. This implies that $ee^* = 1 = \mathbf{s}(e)$ and Lemma 4.1 ensures that e is the only edge $\mathbf{s}(e)$ emits.

Conversely, if E is a collection of disjoint vertices, then $L_K(E)$ is graded isomorphic to a sum of finitely many copies of K, graded trivially, and this ring is graded clean. If E is $\bullet \circlearrowright$, then $L_K(E)$ is graded isomorphic to $K[x, x^{-1}]$ which is a graded field and, hence, graded clean. □

4.4. **Graded unit-regular Leavitt path algebras.** We move on to the condition $\mathrm{UR}_{\mathrm{gr}}$. Let (EDL) be the graph property below.

(EDL) For every cycle of length m, the lengths, considered modulo m, of all paths which do not contain the cycle and which end in an arbitrary but fixed vertex of the cycle, are

$$0, 0, \ldots, 0,\ 1, 1, \ldots, 1,\ \ldots\ldots\ldots,\ m-1, m-1 \ldots m-1$$

where each i is repeated the same number of times in the above list for $i = 0, \ldots, m-1$.

The notation EDL shortens "equally distributed lengths". By [28, Theorem 5.3], if E is a finite graph, $L_K(E)$ has $\mathrm{UR}_{\mathrm{gr}}$ if and only if E is a no-exit graph such that every sink is isolated and that Condition (EDL) holds. We relax the assumption for this result by requiring only E^0 to be finite.

Lemma 4.3. *If $L_K(E)$ is unital and graded unit-regular, then E is row-finite.*

Proof. Assume that $v \in E^0$ emits infinitely many edges. If $e \in \mathbf{s}^{-1}(v)$, let u be an invertible element in R_{-1} such that $e = eue$. Since $u^{-1} \in R_1$, u^{-1} can be written as a K-linear combination of monomials pq^* where path p is one edge longer than path q. Let $f \in \mathbf{s}^{-1}(v)$ be an edge different than any edge appearing in any paths p, q of such representation of u^{-1} so that $f^*u^{-1} = 0$ by (CK1). Multiplying the relation $u^{-1}u = 1$ by f^* on the left we obtain that $0 = f^*$ which is a contradiction. □

Proposition 4.4. *If $L_K(E)$ is unital, the following conditions are equivalent.*

(1) *$L_K(E)$ is graded unit-regular.*
(2) *$L_K(E)$ has graded stable range one.*

(3) *E is finite, no-exit graph such that every sink is isolated and Condition (EDL) holds.*

Proof. Since every Leavitt path algebra is graded regular (by [12, Theorem 9]), the first two conditions are equivalent (see [28, Proposition 4.2]). If (1) holds, then E is a finite graph by Lemma 4.3. Thus, [28, Theorem 5.3] ensures that (1) $\Leftrightarrow$ (3). $\square$

Although $\mathrm{UR}_{\mathrm{gr}}$ and $\mathrm{Cln}_{\mathrm{gr}}$ are independent in general, $\mathrm{Cln}_{\mathrm{gr}} \Rightarrow \mathrm{UR}_{\mathrm{gr}}$ for unital Leavitt path algebras by Propositions 4.2 and 4.4.

4.5. **Graded exchange Leavitt path algebras.** Note that a Leavitt path is a graded involutive ring, so the conditions $\mathrm{Exch}^r_{\mathrm{gr}}$ and $\mathrm{Exch}^l_{\mathrm{gr}}$ are equivalent. In the next result, we formulate a necessary condition for a Leavitt path algebra to be graded exchange.

Proposition 4.5. *For an arbitrary graph E, if $L_K(E)$ is graded exchange, then E is no-exit.*

Proof. Let $R = L_K(E)$ be graded exchange. Assume that there is a cycle c with an edge exiting c. If v is the source of that exit, consider c so that $v = \mathbf{s}(c) = \mathbf{r}(c)$.

Let $|c| = m > 0$. Since $c \in R_m$, there is $u = u^2 \in R_0$ such that $u \in cR$ and $u \in c \circ R$. The first relation implies that $u = cr$ for some $r \in R$. Since $c = cv$, we can replace r by vr so that $r = vr$.

As $u \in c \circ R$, $u = c + s - cs$ for some $s \in R$. Let s_n denote the n-component of s. If there is $k > 0$ such that $s_{-k} \neq 0$, then we can take k to be the largest such so that $s_{-l} = 0$ for all $l > k$. By considering the $(-k)$-component of the relation $u = c+s-cs$, we have that $0 = s_{-k} - cs_{-m-k} = s_{-k}$ which is a contradiction with the choice of k. Hence, $s_{-k} = 0$ for all $k > 0$.

Considering the 0-component of $u = c+s-cs$, we have that $u = s_0$. Considering the i-component of the same relation for $i = 1, \ldots, m-1$ implies that $s_i = 0$ and this ensures that $s_i = 0$ for each i which is not a multiple of m.

Considering the m-component of $u = c+s-cs$, we have that $0 = c + s_m - cs_0 \Rightarrow s_m = cs_0 - c = cu - c$. Considering the km-component for $k = 2, 3, \ldots$, we obtain that $s_{km} = c^{k-1}(cu-c)$. Since only finitely many components of s are nonzero, there is a positive integer k such that $c^{k-1}(cu-c) = 0$. Multiplying this relation by $(c^*)^{k-1}$, we obtain that $cu - c = 0$ and so $c = cu$.

Combining the relations $u = cr$ and $c = cu$, we have that $c = c^2 r$. Multiplying by c^* on the left produces $c^*c = c^*c^2 r \Rightarrow v = cr = u$. On the other hand, multiplying $c = c^2 r$ with $(c^*)^2$ on the left produces $c^* = vr = r$ so that $u = cr = cc^*$. Thus, $cc^* = u = v = c^*c$. By Lemma 4.1, c has no exits which is a contradiction with the choice of c. $\square$

Using Proposition 4.5, we can obtain examples showing $\mathrm{Reg}_{\mathrm{gr}} \not\Rightarrow \mathrm{Exch}^r_{\mathrm{gr}}$ and $\mathrm{Exch} \not\Rightarrow \mathrm{Exch}^r_{\mathrm{gr}}$.

Example 4.6. Let E be the graph . Since E has a cycle with an exit, $L_K(E)$ is not graded exchange by Proposition 4.5. On the other hand, E does have Condition (K) so $L_K(E)$ is exchange by [4, Theorem 4.5] and $L_K(E)$ is graded regular by [12, Theorem 9] (stating that every Leavitt path algebra is graded regular).

Together with the previous results, this shows that the implications and the equivalences of the diagram (D2) in the introduction hold and that each horizontal implication in that diagram is strict.

If E is a no-exit graph, $L_K(E)$ is graded isomorphic to a direct limit of algebras which are a finite direct sum of $\mathbb{M}_n(K)(\gamma_1, \ldots, \gamma_n)$ and $\mathbb{M}_n(K[x^m, x^{-m}])(\gamma_1, \ldots, \gamma_n)$ for various n and m by [27, Proposition 3.3]. The algebras of the first type are graded exchange by Proposition 3.7. Using the definition of k_i from Proposition 3.7, the algebras of the second type are graded exchange if $k_i < 2$, but at this point, we make no claim on what happens if $k_i \geq 2$. We also note that $k_i > 0$. Indeed, recall that n corresponds to the number of paths which end in an arbitrarily selected vertex of a cycle of length m but without considering the cycle itself. The shifts $\gamma_1, \ldots, \gamma_n$ correspond to the lengths of these paths. There is at least one set of paths of lengths $0, 1, \ldots m-1$ which correspond to the paths of the cycle ending at that vertex. Thus, $k_i > 0$.

The above arguments also show that the converse of Proposition 4.5 would hold if the question whether $\mathbb{M}_n(K[x^m, x^{-m}])(\gamma_1, \ldots, \gamma_n)$ is graded exchange for every $k_i > 0$ had an affirmative answer. With the question still unanswered currently, we only have one direction both in Proposition 4.5 and in the next proposition, containing a sufficient condition for $L_K(E)$ to be graded exchange.

Proposition 4.7. *If E is a disjoint union of acyclic graphs and graphs consisting of a single cycle each, then $L_K(E)$ is graded exchange.*

Proof. If E is as specified, then $L_K(E)$ is graded isomorphic to a direct sum of algebras of two types: first, direct limits of finite sums of algebras of the form $\mathbb{M}_n(K)(\gamma_1, \ldots, \gamma_n)$ and, second, algebras $\mathbb{M}_m(K[x^m, x^{-m}])(0, 1, \ldots, m-1)$ for various positive integers m. By Propositions 3.10 and 3.7, $L_K(E)$ is graded exchange. □

By Propositions 4.5 and 4.7, the class of graphs characterizing graded exchange Leavitt path algebras strictly contains acyclic graphs and it is contained in the class of no-exit graphs.

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