Cancellation properties of nonunital and graded rings

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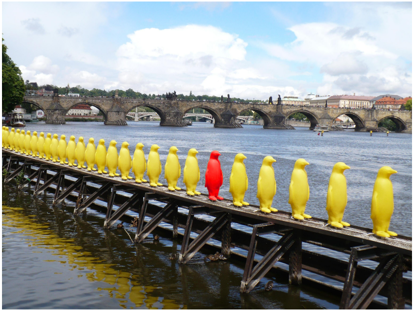


Unital ring

Nonunital ring

Let us start with a very general question...

If P is a property of unital rings, how does one define a generalized version of P suitable for nonunital rings?

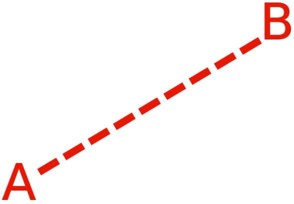
I.e., **what are suitable nonunital-ring generalizations** of the definitions which

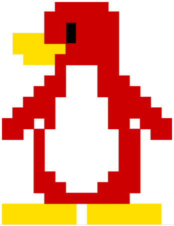
either

1. **refer to the identity,** e.g. R is *directly finite* if $(\forall x, y \in R)(xy = 1 \Rightarrow yx = 1)$,

or

2. **refer to an invertible element,** e.g. R is *unit-regular* if $(\forall x \in R)(\exists u \in U(R))\ x = xux$.

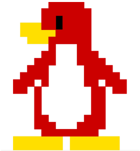


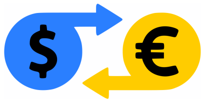


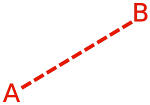
“Cancellation” properties

Module cancellation: $A \oplus C \cong B \oplus C \Rightarrow A \cong B$.

Module cancellation	**Corresponding ring property**
internal cancellation	**unit-regularity**
substitution	**stable range one**
(module-theoretic) exchange	(ring-theoretic) **exchange**
(module-theoretic) direct finiteness	(ring-theoretic) **direct finiteness**







“Sandwiched” properties

Sandwiched properties are also lumped together with the other cancellation properties.

unit-regular $\Rightarrow$ **clean** $\Rightarrow$ exchange

$$\textbf{clean} = (\forall x \in R)(\exists u \in U(R))$$
$$(\exists e \in I(R))x = u + e$$

The **module** cancellations can be considered for a **general** (i.e. possibly nonunital) ring. *What is the* **ring** *cancellation then?*

Unitizations

If R is a general ring, a unital ring S such that R embeds in S as a double-sided ideal of S is an **unitization** of R.

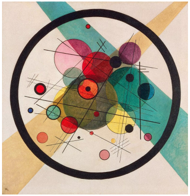
The **standard unitization** R^u of R is the $R \oplus \mathbb{Z}$ with coordinate-wise addition and the multiplication given by

$$(x, k)(y, l) = (xy + lx + ky, kl).$$

$(0, 1)$ is the identity of R^u.

There are also two operations $*$ and $\circ$

$$x * y = x + y + xy$$
$$x \circ y = x + y - xy$$



If R is unital, then $U(R, \cdot) \cong U(R, *) \cong U(R, \circ)$.

Known generalizations

Unital version	Generalization
unit-regular $(\forall x \in R)$ $(\exists u \in U(R))\ x = xux$	$(\forall x \in R)$ $(\exists u \in U(*))\ x = xux + x^2$.
stable range one $(\forall x, y \in R)$ $(xR + yR = R \Rightarrow$ $(\exists z \in R)(x + yz)R = R)$	$(\forall x \in R, y \in R^u)$ $(x, 1)R^u + yR^u = R^u \Rightarrow (\exists z$ $\in R^u)\ ((x, 1) + yz)R^u = R^u$.
exchange $(\forall x \in R)$ $(\exists e \in I(R) \cap xR)$ $1 - e \in (1 - x)R$	$(\forall x \in R)$ $(\exists e \in I(R) \cap xR)$ $e \in x \circ R$.
clean $(\forall x \in R)$ $(\exists u \in U(R))(\exists e \in I(R))$ $x = u + e$	$(\forall x \in R)$ $(\exists u \in U(*))\ (\exists e \in I(R))$ $x = u + e$.
directly finite $(\forall x, y \in R)$ $(xy = 1 \Rightarrow yx = 1)$	?

Mary-Patricio, Vaserstein, Ara, Nicholson-Zhou respectively.

Some questions...

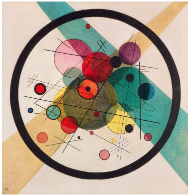
Why are these so different? Is there an unifying thread?

How can one define other properties (e.g. direct finiteness)?

Relating R^u, $*$ and $\circ$ helps.

Some facts.

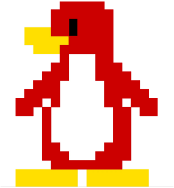
1. $x \to -x$ gives $(R, *) \cong (R, \circ)$.
2. $U(R^u) = \pm(U(*), 1) = \pm(U(\circ), -1)$
3. $(x, 1)R^u = R^u \Leftrightarrow 0 \in x * R \Leftrightarrow 0 \in -x \circ R$



Unit-regularity

If R is a general ring, the following conditions are equivalent.

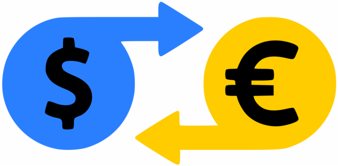
1. For any $x \in R$, $(x, 0)$ is a unit-regular element of R^u.
2. For any $x \in R$, there is $u \in U(*)$ such that $x = xux + x^2$.
3. For any $x \in R$, there is $u \in U(\circ)$ such that $x = xux - x^2$.
4. For any embedding $\phi : R \to S$ such that S is an unitization of R and for any $x \in R$, $\phi(x)$ is a unit-regular element of S.



Exchange

If R is a general ring, the following conditions are equivalent.

1. $(\forall x \in R)\ (\exists e \in I(R))\ e \in -xR$ and $e \in x * R$.
2. $(\forall x \in R)\ (\exists e \in I(R))\ e \in xR$ and $e \in x \circ R$.
3. $(\forall x \in R)\ (\exists e \in I(R))\ e \in xR$ and $(-e, 1) \in (-x, 1)R^u$.
4. For any embedding $\phi : R \to S$ such that S is an unitization of R, and $(\forall x \in R)\ (\exists e \in I(R))\ e \in xR$ and $1 - \phi(e) \in (1 - \phi(x))S$.

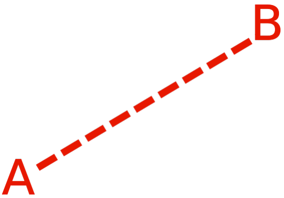


Direct finiteness

If R is a general ring, the following conditions are equivalent.

1. $(\forall x, y \in R)\ (x * y = 0 \Rightarrow y * x = 0)$.
2. $(\forall x, y \in R)\ (x \circ y = 0 \Rightarrow y \circ x = 0)$.
3. $(\forall u, v \in R^u)\ (uv = (0,1) \Rightarrow vu = (0,1))$.
4. $(\forall x, y \in R)\ ((x,1)(y,1) = (0,1) \Rightarrow (y,1)(x,1) = (0,1))$.
5. For any embedding $\phi : R \to S$ such that S is an unitization of R, and for any $x, y \in R$, $(\phi(x)+1)(\phi(y)+1) = 1$ implies $(\phi(y)+1)(\phi(x)+1) = 1$.

If R is unital, then the above conditions are equivalent with R being directly finite.



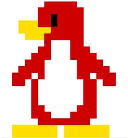
Relationship between the cancellation properties

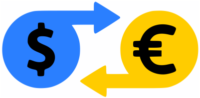
The unital case

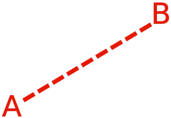
$$\begin{array}{ccccc} \text{UR} \Leftrightarrow & \text{Reg} + \text{sr}=1 & \Rightarrow \text{sr}=1 & \Rightarrow & \text{DF} \\ & \Downarrow & & & \\ & \text{Cln} & \Rightarrow \text{Exch} & & \end{array}$$

The general case

$$\begin{array}{ccccc} \text{UR} \Leftarrow & \text{Reg} + \text{sr}=1 & \Rightarrow \text{sr}=1 & \Rightarrow & \text{DF} \\ & \Downarrow & & & \\ & \text{Cln} & \Rightarrow \text{Exch} & & \end{array}$$







Further generalization

The algebras I've been working with a lot recently are also **graded**.

If Γ is a group, a ring R is Γ-**graded** if

$$R = \bigoplus_{\gamma \in \Gamma} R_\gamma \quad \text{such that} \quad R_\gamma R_\delta \subseteq R_{\gamma\delta}.$$

ring

graded ring

Homogeneous elements

Elements of each component R_γ are said to be **homogeneous**.

We use H to denote the set

$$\bigcup_{\gamma \in \Gamma} R_\gamma$$

of all homogeneous elements.

Example

If K is any field, let us grade $\mathbb{M}_2(K)$ by $\mathbb{Z}$ so that

$$\begin{bmatrix} K & K \\ K & K \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ K & 0 \end{bmatrix} + \begin{bmatrix} K & 0 \\ 0 & K \end{bmatrix} + \begin{bmatrix} 0 & K \\ 0 & 0 \end{bmatrix}$$

$$\mathbb{M}_2(K) = \ldots + 0 + \mathbb{M}_2(K)_{-1} + \mathbb{M}_2(K)_0 + \mathbb{M}_2(K)_1 + 0 \ldots$$

(thus $\mathbb{M}_2(K)_n = 0$ for $n \neq -1, 0, 1$).

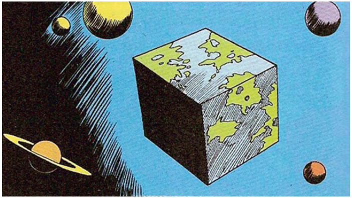
There are many elements which are not homogeneous.

The only homogeneous idempotents are $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, and $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

The only homogeneous invertible elements are $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ with $a, b \neq 0$.

The world of graded rings

To define the **graded version** P_{gr} of a property P, one replaces $\forall x$ and $\exists x$ with $\forall x \in H$ with $\exists x \in H$ in the definition.



field $= (\forall x)$	$\leftrightsquigarrow$	**graded field** $= (\forall x \in H)$
$x \neq 0 \;\Rightarrow\; \exists x^{-1}$	$\leftrightsquigarrow$	$x \neq 0 \;\Rightarrow\; \exists x^{-1}$
regular $= (\forall x)$	$\leftrightsquigarrow$	**graded regular** $= (\forall x \in H)$
$x \in xRx$	$\leftrightsquigarrow$	$x \in xRx$
dir. finite $= (\forall x, y)$	$\leftrightsquigarrow$	**graded dir. fin.** $= (\forall x, y \in H)$
$xy = 1 \Rightarrow yx = 1$	$\leftrightsquigarrow$	$xy = 1 \Rightarrow yx = 1$

Graded versions

Unital version	Graded unital generalization
unit-regular $(\forall x \in R)$ $(\exists u \in U(R))\ x = xux$	$(\forall x \in H)$ $(\exists u \in U(R) \cap H)\ x = xux.$
exchange $(\forall x \in H)$ $(\exists e \in I(R) \cap xR)$ $1 - e \in (1-x)R$	$(\forall x \in R)$ $(\exists e \in I(R) \cap xR \cap H)$ $1 - e \in (1-x)R$
clean $(\forall x \in R)$ $(\exists u \in U(R))(\exists e \in I(R))$ $x = u + e$	$(\forall x \in H)$ $(\exists u \in U(R) \cap H)\ (\exists e \in I(R) \cap H)$ $x = u + e.$

Positives:

$$\mathrm{UR}_{\mathrm{gr}} \Leftrightarrow \quad Reg_{\mathrm{gr}} + (\mathrm{sr}{=}1)_{\mathrm{gr}} \quad \Rightarrow (\mathrm{sr}{=}1)_{\mathrm{gr}} \quad \Rightarrow \mathrm{DF}_{\mathrm{gr}}$$

$$\mathrm{Cln}_{\mathrm{gr}} \qquad \Rightarrow \mathrm{Exch}_{\mathrm{gr}}$$

Downsides

$\mathrm{UR_{gr}}$ and $\mathrm{Cln_{gr}}$ are **independent.**

Both are very **restrictive**.

- $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is false for any $a, b \in K$.
- $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} + \begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix}$ is false for any $a, b, c, d \in K$.

So,

$\mathbb{M}_2(K)$ is neither graded unit-regular nor graded clean.

In fact, R is graded clean if and only if R_ε is clean and each nonzero element of R_γ is invertible for every $\gamma \neq \varepsilon$.

This makes us wonder:

Are the definitions of UR_{gr}, Cln_{gr} etc

meaningful?

Are they equivalent with the **graded versions** of module cancellations?

$$A \oplus C \cong_{\text{gr}} B \oplus C \Rightarrow A \cong_{\text{gr}} B$$

where $A \cong_{\text{gr}} B$ means that there is a module isomorphism f such that

$$f(A_\gamma) = B_\gamma.$$

In this case f is an element of $\text{HOM}_R(A, B)_\varepsilon$ where ε is the identity of Γ. So,

the component $\text{END}_R(R_R)_\varepsilon$ has a special significance.

The ε-cancellation properties

If P is a property of a ring, let us say that a Γ-graded ring R has P_ε if $\mathrm{END}_R(R_R)_\varepsilon$ has P.

If R is **unital** this boils down to R_ε has P.

With this definition,

Graded module cancellation	**ε-cancellation property**
graded internal cancellation	ε-unit-regularity
graded substitution	ε-stable range one
(module-theoretic) finite exchange	(ring-theoretic) ε-exchange
(module-theoretic) direct finiteness	(ring-theoretic) ε-direct finiteness

In addition, Cln_ε is closed under formation of matrix rings.

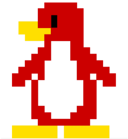
In addition,

The unital case

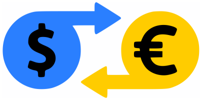
$$
\begin{array}{lllll}
\mathsf{UR}_\varepsilon \Leftrightarrow & \mathit{Reg}_{\mathrm{gr}} + (\mathrm{sr}{=}1)_\varepsilon & \Rightarrow (\mathrm{sr}{=}1)_\varepsilon & \Rightarrow \mathsf{DF}_\varepsilon \\
& \quad\Downarrow & & \\
& \quad\mathsf{Cln}_\varepsilon & \Rightarrow \mathsf{Exch}_\varepsilon &
\end{array}
$$

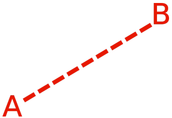
The general case

$$
\begin{array}{lllll}
\mathsf{UR}_\varepsilon \Leftarrow & \mathit{Reg}_{\mathrm{gr}} + (\mathrm{sr}{=}1)_\varepsilon & \Rightarrow (\mathrm{sr}{=}1)_\varepsilon & \Rightarrow \mathsf{DF}_\varepsilon \\
& \quad\Downarrow & & \\
& \quad\mathsf{Cln}_\varepsilon & \Rightarrow \mathsf{Exch}_\varepsilon &
\end{array}
$$









References: liavas.net