Three ways in which T. Y. Lam impacted my life

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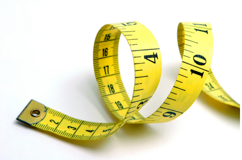


Graduate student in algebraic topology world

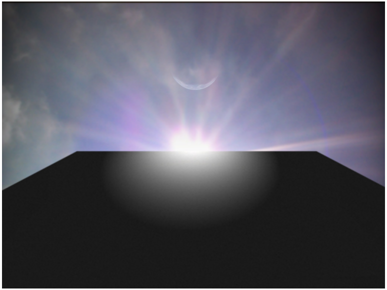
$$\nu^{-1}(H_*^{(2)}(\overline{X})) \cong \mathrm{P}H_*^G(\overline{X}, \mathcal{N}G)$$

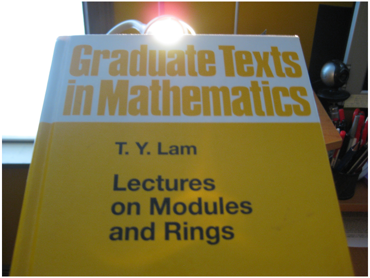
$$H_n^G(X, \mathcal{N}G) = H_n(\mathcal{N}G \otimes_{\mathbb{Z}G} C_*(X))$$

$$b_n^2(\overline{X}, \mathcal{N}G) = \dim_{\mathcal{N}G}\left(H_n^G(\overline{X}, \mathcal{N}G)\right)$$

$$H_n^{(2)}(X) = \ker c_n^{(2)} / \mathrm{cl}(\mathrm{im}\ c_{n+1}^{(2)})$$

2001 Odyssey





Lam on noncommutative localization

- **"The Good"** – some rings can be embedded in division rings;
- **"The Bad"** – not all rings can;
- **"The Ugly"** – even those that can might have nonisomorphic "division rings of fractions".

Symmetric perfect ring of quotients

The **total right ring of quotients** $Q^r_{\mathrm{tot}}(R)$ is not "bad" (exist for every ring) and not "ugly" (unique up to iso). Not exactly "good" but it's close enough.

$$Q^r_{\mathrm{cl}} \subseteq Q^r_{\mathrm{tot}} \subseteq Q^r_{\mathrm{max}}$$

$Q^r_{\mathrm{tot}}(R)$ seems to be **"just right"**.

Impact of the Good-bad-ugly idea

Direct.

1. Simplification of Morita's construction of $Q^r_{\mathrm{tot}}(R)$.
2. Symmetric version of perfect right rings of quotients.
3. Symmetric version $Q^{\sigma}_{\mathrm{tot}}(R)$ of Q^r_{tot}.

Indirect.

4. Different rings of quotients $\to$ Torsion theories.
5. Torsion theories $\to$ different closures of VNAs $\to$ dimension of VNA-like Baer rings.

- Perfect Symmetric Rings of Quotients, *J. of Algebra and its Appl.* 8(5) (2009).
- A Simplification of Morita's Construction of Q^r_{tot} for a Class of Rings, *J. of Algebra* 304(2) (2006).
- Dimension and Torsion Theories for a Class of Baer *-Rings, *J. of Algebra* 289(2) (2005).
- Torsion Theories for Finite von Neumann Algebras, *Comm. in Algebra* 33(3) (2005).

2005 – another important year

Philadelphia prior to 2005

Conference at Ohio Univ., Athens, OH, 2005

Lam asked a question...

Which von Neumann algebras are clean as rings?

Clean Rings

A ring R is **clean** if

every element = unit + idempotent

Additive version of unit-regular.

Examples: Unit-regular, local, semiperfect...

Non-examples: $\mathbb{Z}$, $R[x]$ for R commutative, not all regular ("Bergman example")...

Von Neumann algebras – "The Idea"

A finite VNA R has a **unit-regular** ring of quotients Q with the **same projections**.

Go up:
$a \in Q$

$\longrightarrow$

Have: $a = u + e$, e idempotent
Want: $a = u + p$, p projection

$\uparrow$ $\downarrow$

Start here:
$a \in R$

$p \in R$ and so $u = a - p \in R$
End with: $a = u + p$ in R

Two problems: (1) want idempotent, have projection; (2) u may not be unit in R.

Fix for (2) – Almost Clean Rings

A ring R is **almost clean** if

element = regular el. + idempotent

Additive version of (abelian) Rickart.

Examples: clean, abelian Rickart,...

$\mathbb{Z}$ is almost clean and not clean.

Non-examples: Couchot's paper.

Fix for (1) – Introducing stars

Von Neumann algebras are $*$-rings (have involution).

Involution $*$: is additive, $(xy)^* = y^*x^*$, and $(x^*)^* = x$.

For $*$-rings **projections** take over the role of **idempotents.**

- Baer becomes Baer *-ring,
- Rickart becomes Rickart *-ring,
- regular becomes *-regular.
- **So clean should become...**

*-clean

A *-ring R is ***-clean** if

element = unit + projection

A *-ring R is **almost *-clean** if

element = regular el. + projection

Using this concept, I could show that:

An AW^*-algebra (in particular von Neumann algebra) of type I_f is almost $*$-clean.

How to extend "The Idea"?

It works for **any** ring that has a
clean overing with the same idempotents.

Go up: $a \in Q$ $\longrightarrow$ Have: $a = u + e$, e idempotent, u unit

$\uparrow$ $\downarrow$

Start here: $a \in R$

$e \in R$ and so $u = a - e \in R$
End with: $a = u + e$, u regular.

Exploring "The Idea" with Evrim Akalan

- R r. quasi-continuous $\Rightarrow$

  $E(R)$ and R have same idempotents.

- R r. quasi-continuous + r. nonsingular $\Rightarrow$

  $E(R) = Q^r_{\max}(R)$ is clean, has same idempotents. So,

R is almost clean.

Stronger statements hold.

Module-case.

quasi-cont. + nonsingular $\Rightarrow$ almost clean.

Ring-case. (Recall CS = "complements are summands".)

right CS ~~r. quasi-continuous~~	right nonsingular $\longrightarrow$	almost clean
$\uparrow$		$\uparrow$
right continuous	$\longrightarrow$	clean

Corollary:

Finite AW^*-algebras are almost clean.

Camillo-Khurana Theorem and special clean

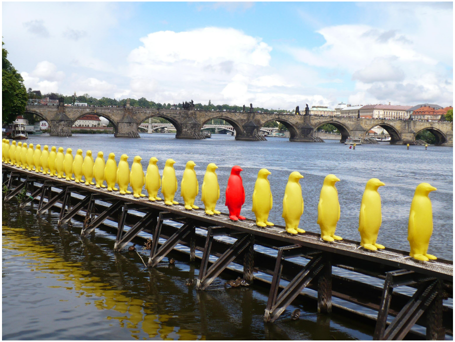
Camillo-Khurana:

unit-regular		special clean
$a = eu$	$\longleftrightarrow$	$a = e + u, \quad aR \cap eR = 0$

Almost clean – Rickart connection:

Rickart	abelian	special almost clean
$a = er$	$\longleftrightarrow$	$a = e + r, \quad aR \cap eR = 0$

Uniqueness



Rickart $a = er$	abelian r. quasi-cont. $\longleftrightarrow$	uniquely special almost clean $a = e + r$ unique, $aR \cap eR = 0$
$\uparrow$		$\uparrow$
unit-regular $a = eu$	abelian $\longleftrightarrow$	uniquely special clean $a = e + u$ unique, $aR \cap eR = 0$

Adapting the results to *-rings

Rickart *-ring	abelian $\longleftrightarrow$	special almost *-clean
$\uparrow$		$\uparrow$
*-regular	abelian $\longleftrightarrow$	special *-clean

Outcomes and some questions

- $*$-Clean Rings; Some Clean and Almost Clean Baer $*$-rings and von Neumann Algebras, *J. of Algebra*, 324 (12) (2010).
- **With E. Akalan**, Classes of almost clean rings, *Algebras and Representation Theory*, in print.

1. **Weaken abelian?** Cannot completely drop it.
2. **VNAs?** For AW^*-algebras:

$$\begin{array}{ccc} \text{finite, type } I & \longrightarrow & \text{finite} \\ \downarrow & & \downarrow \\ \text{almost } *\text{-clean} & \longrightarrow & \text{almost clean} \end{array}$$

Then 2010 came...

... and my sabbatical in Malaga, Spain.

Besides working on Leavitt path algebras,

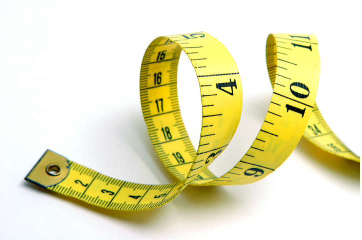
I was also reading on uniform dimension.

Rings with dimension

Inspiration: Uniform dimension, Goldie reduced rank.

Goal: Dimension similar to that on VNA-like Baer *-rings.

Strongly semihereditary rings with positive definite involution...



... a ring with any of the following:

- right nonsingular and every fin. gen. nonsingular module is projective.
- right nonsingular and R^n is CS for all n.

Positive definite: $\sum_{i=1}^{n} x_i^* x_i = 0 \Rightarrow x_i = 0$ for all i, for all n.

Rings with dimension

Examples:

1. Finite AW^*-algebras. Moreover: Von-Neumann-algebra-like-rings (Baer *-rings with some axioms).
2. Leavitt path algebras over finite no-exit graphs.

Not necessarily Baer *-rings.

- Strongly semihereditary rings and rings with dimension, *Algebras and Representation Theory*, in print.

References

Preprints of these papers are available on

http://www.usciences.edu/∼lvas and on arXiv.

Happy Birthday!