EVERY GRADED IDEAL OF A LEAVITT PATH ALGEBRA IS *GRADED* ISOMORPHIC TO A LEAVITT PATH ALGEBRA

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ABSTRACT. We show that every graded ideal of a Leavitt path algebra is *graded* isomorphic to a Leavitt path algebra. It is known that a graded ideal I of a Leavitt path algebra is isomorphic to the Leavitt path algebra of a graph, known as the generalized hedgehog graph, which is defined based on certain sets of vertices uniquely determined by I. However, this isomorphism may not be graded. We show that replacing the short "spines" of the generalized hedgehog graph with possibly fewer, but then necessarily longer spines, we obtain a graph (which we call the porcupine graph) such that its Leavitt path algebra is *graded* isomorphic to I.

0. INTRODUCTION

If E is a graph and $L_K(E)$ is its Leavitt path algebra, it is known that every graded ideal I of $L_K(E)$ is uniquely determined by a pair (H, S) of certain subsets of vertices, known as an admissible pair (we review the definition in section 1.2). For every such admissible pair, one can define a graph, referred to as the generalized hedgehog graph in [1], such that I is isomorphic to the Leavitt path algebra of this graph. The name "hedgehog" comes from the construction in which one attaches new edges (the "spines") to $H \cup S$ (the "body"). We modify this construction by replacing the added edges with possibly fewer paths but then necessarily of length larger than one. Because of the longer spines, we call the resulting graph the porcupine graph of (H, S). In the main result of the paper, Theorem 2.3, we construct a *graded* isomorphism of I and the Leavitt path algebra of the porcupine graph of the admissible pair corresponding to I.

1. PREREQUISITES

1.1. Graded rings prerequisites. A ring R (not necessarily unital) is *graded* by a group Γ if $R = \bigoplus_{\gamma \in \Gamma} R_\gamma$ for additive subgroups R_γ and $R_\gamma R_\delta \subseteq R_{\gamma\delta}$ for all $\gamma, \delta \in \Gamma$. The elements of the set $\bigcup_{\gamma \in \Gamma} R_\gamma$ are said to be *homogeneous* and γ is the *degree* of any nonzero element of R_γ. The grading is *trivial* if $R_\gamma = 0$ for every $\gamma \in \Gamma$ which is not the group identity. We adopt the standard definitions of graded ring homomorphisms, graded algebras, and graded ideals as defined in [2].

1.2. Leavitt path algebras prerequisites. Let E be a directed graph, let E^0 denote the set of vertices, E^1 the set of edges, and $\mathbf{s}$ and $\mathbf{r}$ denote the source and range maps of E. A *sink* of E is a vertex which emits no edges and an *infinite emitter* is a vertex which emits infinitely many edges. A vertex of E is *regular* if it is not a sink or an infinite emitter. A *path* is a single vertex or a sequence of edges $e_1 e_2 \dots e_n$ for some positive integer n such that $\mathbf{r}(e_i) = \mathbf{s}(e_{i+1})$ for $i = 1, \dots, n-1$.

Extend a graph E to the graph with the same vertices and with edges $E^1 \cup \{e^* \mid e \in E^1\}$ where the range and source relations are the same as in E for $e \in E^1$ and $\mathbf{s}(e^*) = \mathbf{r}(e)$ and $\mathbf{r}(e^*) = \mathbf{s}(e)$ for

the added edges. If K is any field, the *Leavitt path algebra* $L_K(E)$ of E over K is a free K-algebra generated by the set $E^0 \cup E^1 \cup \{e^* \mid e \in E^1\}$ such that for all vertices v, w and edges e, f,

(V) $vw = 0$ if $v \neq w$ and $vv = v$, (E1) $\mathbf{s}(e)e = e\mathbf{r}(e) = e$,
(E2) $\mathbf{r}(e)e^* = e^*\mathbf{s}(e) = e^*$, (CK1) $e^*f = 0$ if $e \neq f$ and $e^*e = \mathbf{r}(e)$,
(CK2) $v = \sum_{e \in \mathbf{s}^{-1}(v)} ee^*$ for each regular vertex v.

By the first four axioms, every element of $L_K(E)$ can be represented as a sum of the form $\sum_{i=1}^n k_i p_i q_i^*$ for some n, paths p_i and q_i, and elements $k_i \in K$, for $i = 1, \ldots, n$ where $v^* = v$ for $v \in E^0$ and $p^* = e_n^* \ldots e_1^*$ for a path $p = e_1 \ldots e_n$. Using this representation, one can make $L_K(E)$ into an involutive ring by $(\sum_{i=1}^n k_i p_i q_i^*)^* = \sum_{i=1}^n k_i^* q_i p_i^*$ where $k_i \mapsto k_i^*$ is any involution on K. For more details on these basic properties, see [1].

If we consider K to be trivially graded by $\mathbb{Z}$, $L_K(E)$ is naturally graded by $\mathbb{Z}$ so that the n-component $L_K(E)_n$ is the K-linear span of the elements pq^* for paths p, q with $|p| - |q| = n$ where $|p|$ denotes the length of a path p. While one can grade a Leavitt path algebra by any group (see [2, Section 1.6.1]), we consider only the natural grading by $\mathbb{Z}$.

If a K-algebra R contains elements a_v, b_e, and c_{e^*} which satisfy the axioms (V), (E1), (E2), (CK1), and (CK2) where $v \in E^0$ and $e \in E^1$ for some graph E, the Universal Property of $L_K(E)$ ensures that the map $\phi : v \mapsto a_v, e \mapsto b_e, e^* \mapsto c_{e^*}$ has a unique K-algebra homomorphism extension $\phi : L_K(E) \to R$ (see [1, Remark 1.2.5]). If R is $\mathbb{Z}$-graded and the elements $\phi(v)$, $\phi(e)$, and $\phi(e^*)$ have degrees 0, 1, and -1 respectively, such extension is graded and the Graded Uniqueness Theorem ([1, Theorem 2.2.15]) states that ϕ is injective if and only if $\phi(v) \neq 0$ for every $v \in E^0$. If R is involutive and if $a_v^* = a_v$ for every $v \in E^0$ and $(b_e)^* = c_{e^*}$ for every $e \in E^1$, then ϕ is a $*$-homomorphism (see [5, Lemma 4.7]).

1.3. Graded ideals of a Leavitt path algebra.
A subset H of E^0 is said to be *hereditary* if $\mathbf{r}(p) \in H$ for any path p such that $\mathbf{s}(p) \in H$. The set H is *saturated* if $v \in H$ for any regular vertex v such that $\mathbf{r}(\mathbf{s}^{-1}(v)) \subseteq H$. We recall a construction from [4]. If H is hereditary and saturated, let

$$B_H = \{v \in E^0 - H \mid v \text{ infinite emitter and } \mathbf{s}^{-1}(v) \cap \mathbf{r}^{-1}(E^0 - H) \text{ is nonempty and finite}\} \text{ and}$$

$$v^H = v - \sum_{e \in \mathbf{s}^{-1}(v) \cap \mathbf{r}^{-1}(E^0 - H)} ee^* \quad \text{for } v \in B_H.$$

An *admissible pair* is an ordered pair (H, S) where $H \subseteq E^0$ is hereditary and saturated and $S \subseteq B_H$. For such a pair, let $I(H, S)$ denote the graded ideal generated by homogeneous elements $H \cup \{v^H \mid v \in S\}$. The ideal $I(H, S)$ is the K-linear span of the elements pq^* for paths p, q with $\mathbf{r}(p) = \mathbf{r}(q) \in H$ and the elements $pv^H q^*$ for paths p, q with $\mathbf{r}(p) = \mathbf{r}(q) = v \in S$ (see [4, Lemma 5.6]). The converse holds as well: for every graded ideal I, the vertices in I form a hereditary and saturated set H and the set of infinite emitters such that $v^H \in I$ is a subset of B_H ([4, Theorem 5.7], also [1, Theorem 2.5.8]). For an admissible pair (H, S), let $E_{(H,S)}$ be the graph defined by

$F_1(H, S) = \{e_1 \ldots e_n \text{ is a path of } E \mid \mathbf{r}(e_n) \in H, \mathbf{s}(e_n) \notin H \cup S\}$,
$F_2(H, S) = \{p \text{ is a path of } E \mid \mathbf{r}(p) \in S, |p| > 0\}$, $\overline{F_i}(H, S)$ is a copy of $F_i(H, S), i = 1, 2$,
$E_{(H,S)}^0 = H \cup S \cup F_1(H, S) \cup F_2(H, S)$, and
$E_{(H,S)}^1 = \{e \in E^1 \mid \mathbf{s}(e) \in H\} \cup \{e \in E^1 \mid \mathbf{s}(e) \in S, \mathbf{r}(e) \in H\} \cup \overline{F_1}(H, S) \cup \overline{F_2}(H, S)$ with $\mathbf{s}$ and $\mathbf{r}$ the same as on E^1 for $e \in E^1 \cap E_{(H,S)}^1$ and $\mathbf{s}(\overline{p}) = p, \mathbf{r}(\overline{p}) = \mathbf{r}(p)$ for $\overline{p} \in \overline{F_1}(H, S) \cup \overline{F_2}(H, S)$.

By [3, Theorem 6.1] (also [1, Theorem 2.5.22]), the ideal $I(H, S)$ is isomorphic to $L_K(E_{(H,S)})$.

In [1], the graph $E_{(H,\emptyset)}$ is referred to as the *hedgehog graph* and the graph $E_{(H,S)}$ is referred to as the *generalized hedgehog graph*. The term "hedgehog" is chosen since the added edges ending at $H \cup S$ resemble spines of the hedgehog body $H \cup S$. The isomorphism $L_K(E_{(H,S)}) \to I(H, S)$ is defined so that the edges of $E_{(H,S)}$ are mapped to certain homogeneous elements of $I(H, S)$ of *positive* degree, not necessarily degree one. Thus, the degrees are not necessarily preserved under this map and so this isomorphism may not be graded.

2. The main result

We modify the construction of the generalized hedgehog graph of an admissible pair (H, S) by making the spines added to the body $H \cup S$ possibly fewer in number but then necessarily longer in length. Because of the longer spines, we call the resulting graph the porcupine graph and denote it by $P_{(H,S)}$. The modified construction enables us to create a degree preserving isomorphism $L_K(P_{(H,S)}) \to I(H, S)$.

Definition 2.1. For an admissible pair (H, S), we keep the definitions of $F_1(H, S)$ and $F_2(H, S)$. For each $e \in (F_1(H, S) \cup F_2(H, S)) \cap E^1$, let w^e be a new vertex and f^e a new edge such that $\mathbf{s}(f^e) = w^e$ and $\mathbf{r}(f^e) = \mathbf{r}(e)$. Continue this process inductively as follows. For each path $p = eq$ where $q \in F_1(H, S) \cup F_2(H, S)$ and $|q| \geq 1$, add a new vertex w^p and a new edge f^p such that $\mathbf{s}(f^p) = w^p$ and $\mathbf{r}(f^p) = w^q$.

We define the *porcupine graph* $P_{(H,S)}$ as follows. The set of vertices of $P_{(H,S)}$ is

$$H \cup S \cup \{w^p \mid p \in F_1(H, S) \cup F_2(H, S)\}.$$

The set of edges of $P_{(H,S)}$ is

$$\{e \in E^1 \mid \mathbf{s}(e) \in H\} \cup \{e \in E^1 \mid \mathbf{s}(e) \in S, \mathbf{r}(e) \in H\} \cup \{f^p \mid p \in F_1(H, S) \cup F_2(H, S)\}$$

The $\mathbf{s}$ and $\mathbf{r}$ maps of $P_{(H,S)}$ are the same as in E for the common edges and they are defined as above for the new edges.

Before formulating and proving the main result, we present examples comparing and contrasting the generalized hedgehog and the porcupine graphs of an admissible pair.

Example 2.2. Let E be the Toeplitz graph $e \, \circlearrowleft \, \bullet^w \xrightarrow{\;g\;} \bullet^v$ and let $H = \{v\}$. The hedgehog graph of $(H, \emptyset)$ is listed first and the porcupine graph second.
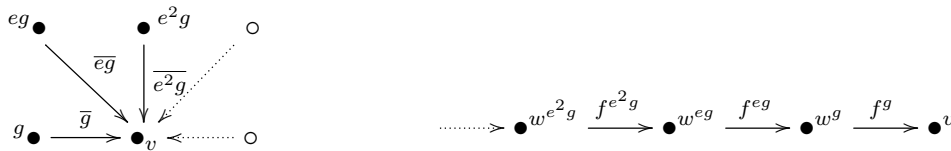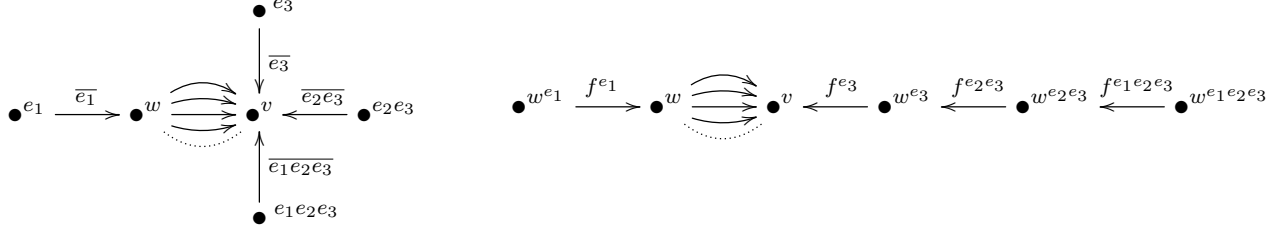

The graded isomorphism of Theorem 2.3 is such that the path $e^{n-1}g$ of length n corresponds to the path $f^{e^{n-1}g} f^{e^{n-2}g} \ldots f^g$ of length n also.

We consider another example with an infinite emitter. Let E be the graph

and consider $H = \{v\}, S = \{w\}$. In this case, $F_1(H, S) = \{e_3, e_2 e_3, e_1 e_2 e_3\}$ and $F_2(H, S) = \{e_1\}$.

The generalized hedgehog graph of (H, S) is listed first and the porcupine graph second.

Theorem 2.3. *For an admissible pair (H, S) of a graph E,*

$$I(H, S) \text{ and } L_K(P_{(H,S)}) \text{ are graded isomorphic.}$$

Thus, every graded ideal of a Leavitt path algebra is graded isomorphic to a Leavitt path algebra.

Proof. We define a function ϕ which maps the vertices and edges of $P_{(H,S)}$ to elements of $I(H, S)$ as follows. For a vertex v of $P_{(H,S)}$, we let

$$\phi(v) = \begin{cases} v & \text{if } v \in H \\ v^H & \text{if } v \in S \\ pp^* & \text{if } v = w^p \text{ and } p \in F_1(H, S) \\ p\mathbf{r}(p)^H p^* & \text{if } v = w^p \text{ and } p \in F_2(H, S) \end{cases}$$

For an edge g of $P_{(H,S)}$, we let

$$\phi(g) = \begin{cases} e & \text{if } g = e \in E^1 \text{ or if } g = f^e \text{ for some } e \in F_1(H, S) \cap E^1 \\ e\mathbf{r}(e)^H & \text{if } g = f^e \text{ for some } e \in F_2(H, S) \cap E^1 \\ epp^* & \text{if } g = f^{ep} \text{ for some } e \in E^1 \text{ and } p \in F_1(H, S) \\ ep\mathbf{r}(p)^H p^* & \text{if } g = f^{ep} \text{ for some } e \in E^1 \text{ and } p \in F_2(H, S) \end{cases}$$

Extend ϕ to g^* for $g \in P^1_{(H,S)}$ by $\phi(g^*) = \phi(g)^*$ and use definitions to check that (V), (E1), (E2), and (CK1) hold for $\phi(v), \phi(g), \phi(g^*), v \in P^0_{(H,S)}, g \in P^1_{(H,S)}$. We present more details for checking that (CK2) holds. If v is a regular vertex of $P_{(H,S)}$, then v cannot be in S, so $v \in H$ or $v = w^p$ for some $p \in F_1(H, S) \cup F_2(H, S)$. In the first case, the edges v emits are in E since H is hereditary and the relation $v = \sum_{e \in \mathbf{s}^{-1}(v)} ee^*$ holds in $L_K(P_{(H,S)})$ since it holds in $L_K(E)$. In the second case, consider the four possible cases: $p = e \in F_1(H, S) \cap E^1$, $p = e \in F_2(H, S) \cap E^1$, $p = eq$ for $e \in E^1, q \in F_1(H, S)$, and $p = eq$ for $e \in E^1, q \in F_2(H, S)$. Note that in each case, v emits only one edge f^p so it is sufficient to check that $\phi(f^p)\phi((f^p)^*) = \phi(v)$ in each case.

If $p = e \in F_1(H, S) \cap E^1$, then $\phi(f^e)\phi((f^e)^*) = ee^* = \phi(v)$.

If $p = e \in F_2(H, S) \cap E^1$, then $\phi(f^e)\phi((f^e)^*) = e\mathbf{r}(e)^H \mathbf{r}(e)^H e^* = e\mathbf{r}(e)^H e^* = \phi(v)$.

If $p = eq$ for $e \in E^1, q \in F_1(H, S)$, then $\phi(f^p)\phi((f^p)^*) = eqq^* qq^* e^* = eqq^* e^* = pp^* = \phi(v)$.

If $p = eq$ for $e \in E^1, q \in F_2(H, S)$, then $\phi(f^p)\phi((f^p)^*) = eq\mathbf{r}(p)^H q^* q\mathbf{r}(p)^H q^* e^* = eq\mathbf{r}(p)^H q^* e^* = p\mathbf{r}(p)^H p^* = \phi(v)$. This shows that (CK2) holds.

By the Universal Property, ϕ has a unique extension to a K-algebra homomorphism $L_K(P_{(H,S)}) \to I(H, S)$. Since ϕ preserves degrees of $v \in P^0_{(H,S)}$ and of g and g^* for $g \in P^1_{(H,S)}$, this extension, which we denote also by ϕ, is a *graded* homomorphism. As $\phi(v)^* = \phi(v)$ for $v \in P^0_{(H,S)}$ and $\phi(g^*) = \phi(g)^*$ for $g \in P^1_{(H,S)}$, ϕ is a $*$-homomorphism. The map ϕ is nonzero on every vertex of $P_{(H,S)}$, so ϕ is injective by the Graded Uniqueness Theorem.

Since ϕ is a $*$-homomorphism, to show surjectivity of ϕ, it is sufficient to show (1) and (2) where (1) denotes the condition that p is in the image of ϕ for every path such that $\mathbf{r}(p) \in H$, and (2) denotes the condition that $p\mathbf{r}(p)^H$ is in the image of ϕ for every path p such that $\mathbf{r}(p) \in S$. Both claims hold for paths of zero length since $\phi(v) = v$ if $v \in H$ and $\phi(v) = v^H$ if $v \in S$. Thus, consider a path $p = e_1 \ldots e_n$ for a positive integer n.

To show (1) assume that $\mathbf{r}(p) \in H$. If $\mathbf{s}(p) \in H$, then $\phi(p) = p$ so the claim holds. If $\mathbf{s}(p) \notin H$, let $i \in \{1, 2, \ldots, n\}$ be the largest such that $\mathbf{s}(e_i) \notin H$. We consider three cases: (i) $\mathbf{s}(e_i) \notin S$, (ii) $\mathbf{s}(e_i) \in S$ and $i = 1$, and (iii) $\mathbf{s}(e_i) \in S$ and $i > 1$.

In case (i), $e_j \ldots e_i \in F_1(H, S)$ for all $j = 1, \ldots i$ and we have that

$$\phi(f^{e_1 \ldots e_i} f^{e_2 \ldots e_i} \ldots f^{e_i} e_{i+1} \ldots e_n) = e_1 \ldots e_i (e_2 \ldots e_i)^* (e_2 \ldots e_i)(e_3 \ldots e_i)^* \ldots e_{i-1} e_i e_i^* e_i \ e_{i+1} \ldots e_n =$$

$$= e_1 \ldots e_i \ e_{i+1} \ldots e_n = e_1 \ldots e_n = p$$

if $i < n$ and the analogous argument applies to the case $i = n$.

In case (ii), $\phi(e_j) = e_j$ for every $j = 1, \ldots, n$ and $\phi(p) = p$.

In case (iii), $e_j \ldots e_{i-1} \in F_2(H, S)$ for every $j = 1, \ldots i - 1$ and $\mathbf{r}(e_{i-1})^H e_i = e_i$ so that

$$\phi(f^{e_1 \ldots e_{i-1}} f^{e_2 \ldots e_{i-1}} \ldots f^{e_{i-1}} e_i \ldots e_n) = e_1 \ldots e_{i-1} \mathbf{r}(e_{i-1})^H (e_2 \ldots e_{i-1})^* e_2 \ldots e_{i-1} \mathbf{r}(e_{i-1})^H \ldots$$

$$\ldots e_{i-1}^* e_{i-1} \mathbf{r}(e_{i-1})^H e_i e_{i+1} \ldots e_n = e_1 \ldots e_{i-1} \mathbf{r}(e_{i-1})^H e_i e_{i+1} \ldots e_n = e_1 \ldots e_{i-1} e_i e_{i+1} \ldots e_n = p.$$

This shows that (1) holds.

To show (2), assume that $\mathbf{r}(p) \in S$. Then $e_i \ldots e_n$ is in $F_2(H, S)$ for every $i = 1, \ldots, n$ and

$$\phi(f^{e_1 \ldots e_n} f^{e_2 \ldots e_n} \ldots f^{e_n}) = e_1 \ldots e_n \mathbf{r}(p)^H (e_2 \ldots e_n)^* (e_2 \ldots e_n) \mathbf{r}(p)^H (e_3 \ldots e_n)^* \ldots$$

$$\ldots e_{n-1} e_n \mathbf{r}(p)^H e_n^* e_n \mathbf{r}(p)^H = e_1 \ldots e_n \mathbf{r}(p)^H = p\mathbf{r}(p)^H.$$

This shows that ϕ is surjective.

The second sentence of the theorem is a direct corollary of the first sentence and [4, Theorem 5.7] (also [1, Theorem 2.5.8]). □

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