Porcupine-quotient, the fourth primary color...

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hedgehog

porcupine

porcupine-quotient

three primary colors the fourth one

.... and some updates about the Graded Classification Conjecture.

Composition series

A very general question: given a **composition series**

$$0 = I_0 \lneq I_1 \lneq \dots \lneq I_n = R$$

(so I_{i+1}/I_i is simple).

how to patch the information from its components to recover the information on R?

Let $R =$ your favorite graph (or groupoid) algebra and I_i their appropriate substructures.

For me, $R = L_K(E)$ is a **Leavitt path algebra** with its grading and $I_i = I(H_i, S_i)$ for an admissible pair (H_i, S_i).

Everything I say also holds for **graph C^*-algebras**.

And now imagine....

1. ... that we know such a composition series exists.
2. ... that each composition quotient $I(H_{i+1}, S_{i+1})/I(H_i, S_i)$ is again a LPA, say of E_i.
3. and that we have a method of proving that

   if the **Graded Classification Conjecture (GCC)** holds for each E_i,

   then it holds for E.

And if your thoughts and dreams ...

about 1. to 3. become real, then that plus

4. The GCC holds for cofinal graphs with a **sink** or a **cycle without exits**.

implies that

The Graded Classification Conjecture holds for graphs with disjoint cycles

Making 2. happen

quotient

porcupine

porcupine-quotient

The quotient graph construction is "old" (2006-2008).

Examples. Let E be $\circlearrowleft \bullet^{v} \overset{\infty}{\rightrightarrows} \bullet^{w}$ and $H = \{w\}$. Then

$B_H = \{v\}$ and

$E/(H, B_H)$ is $\circlearrowleft \bullet^{v}$ and $E/(H, \emptyset)$ is $\circlearrowleft \bullet^{v} \longrightarrow \bullet^{v'}$.

Hedgehog graph

$$
\begin{aligned}
F_1(H,S) &= \{e_1\dots e_n \text{ is a path of } E \mid \mathrm{r}(e_n)\in H, \mathrm{s}(e_n)\notin H\cup S\},\\
F_2(H,S) &= \{p \text{ is a path of } E \mid \mathrm{r}(p)\in S,\ |p|>0\},\\
\overline{F_i}(H,S) &\text{ is a copy of } F_i(H,S), i=1,2,
\end{aligned}
$$

Then

$$
\begin{aligned}
E^0_{(H,S)} &= H\cup S\cup F_1(H,S)\cup F_2(H,S), \text{ and}\\
E^1_{(H,S)} &= \{e\in E^1 \mid \mathrm{s}(e)\in H\}\cup\{e\in E^1 \mid \mathrm{s}(e)\in S, \mathrm{r}(e)\in H\}\cup\\
&\quad \overline{F_1}(H,S)\cup\overline{F_2}(H,S) \text{ with}\\
\mathrm{s}(\overline{p}) &= p, \mathrm{r}(\overline{p}) = \mathrm{r}(p) \text{ for } \overline{p}\in\overline{F_1}(H,S)\cup\overline{F_2}(H,S).
\end{aligned}
$$

Examples. Let E be $\begin{tikzcd} \bullet^v \arrow[loop left, "e"] \arrow[r, "g"] & \bullet^w \end{tikzcd}$ and $H=\{w\}$. Then

$F_1=\{e^n g \mid n=0,1,\dots\}$ and the hedgehog is

$$
\begin{tikzcd}
\bullet \arrow[dr] & \bullet \arrow[d] & \circ \arrow[dl, dotted] \\
\bullet \arrow[r] & \bullet & \circ \arrow[l, dotted]
\end{tikzcd}
$$

Some positives and some negatives

Positives

$$L_K(\text{quotient graph}) \cong_{\text{gr}} L_K(E)/I(H,S)$$

$$L_K(\text{hedgehog}) \cong I(H,S)$$

Negatives

$$L_K(\text{hedgehog}) \ncong_{\text{gr}} I(H,S)$$

Indeed, if E is $\begin{tikzcd} \bullet^v \arrow[loop left, "e"] \arrow[r, "g"] & \bullet^w \end{tikzcd}$ and $H = \{w\}$,

the path $eeeg$ (of **length 4**) of $I(H)$ corresponds to an edge $\overline{eeeg}$ so it has **length 1** in the LPA of the hedgehog.

How to fix this?
Make the "spines" longer and get...

Porcupine graph $P_{(H,S)}$ (2021)

Keep the definitions of F_1 and F_2.

For each $e \in (F_1 \cup F_2) \cap E^1$, let w^e be a **new vertex** and f^e a **new edge** such that $\mathrm{s}(f^e) = w^e$ and $\mathrm{r}(f^e) = \mathrm{r}(e)$.

For each path $p = eq$ where $q \in F_1 \cup F_2$ and $|q| \geq 1$, add a **new vertex** w^p and a **new edge** f^p such that $\mathrm{s}(f^p) = w^p$ and $\mathrm{r}(f^p) = w^q$. Then let

$$P^0_{(H,S)} = H \cup S \cup \{w^p \mid p \in F_1(H,S) \cup F_2(H,S)\} \text{ and}$$

$$P^1_{(H,S)} = \{e \in E^1 \mid \mathrm{s}(e) \in H\} \cup \{e \in E^1 \mid \mathrm{s}(e) \in S, \mathrm{r}(e) \in H\} \cup \{f^p \mid p \in F_1(H,S) \cup F_2(H,S)\}$$

We get a graded iso by

$$w^p \leftrightsquigarrow pp^*, p \in F_1, \quad w^p \leftrightsquigarrow p\mathrm{r}(p)^H p^*, p \in F_2,$$

$$f^{ep} \leftrightsquigarrow epp^*, p \in F_1, \quad f^{ep} \leftrightsquigarrow ep\mathrm{r}(p)^H p^*, p \in F_2,$$

Example

Let E be $\begin{tikzcd} \bullet^{v} \arrow[loop left, "e"] \arrow[r, "g"] & \bullet^{w} \end{tikzcd}$ and $H = \{w\}$. Then

$$F_1(H) = \{g, eg, eeg, eeeg, \ldots\}$$

and the porcupine is

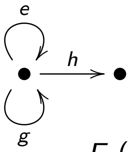
$$\begin{tikzcd} \cdots \arrow[r, dotted] & \bullet^{w^{e^2g}} \arrow[r, "f^{e^2g}"] & \bullet^{w^{eg}} \arrow[r, "f^{eg}"] & \bullet^{w^{g}} \arrow[r, "f^{g}"] & \bullet^{w} \end{tikzcd}$$

We unroll the loop and make it into a single spine.

The graded iso is

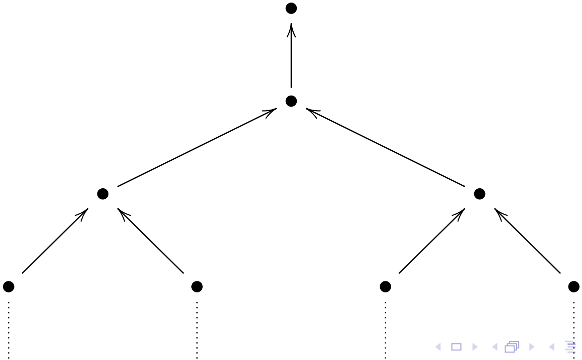
$$eeeg \leftrightsquigarrow f^{e^3g} f^{e^2g} f^{eg} f^{g}.$$

Another example


and $H = \{w\}$. Then

$$F_1(H) = \{h, eh, gh, eeh, egh, geh, ggh, eeeh, \ldots\}$$

and the **hedgehog** graph is the **same** as in the previous example. The, **porcupine**, on the other hand, is



Porcupine-quotient (2023)

Given $(H, S) \leq (G, T)$ (this means $H \subseteq G$ and $S \subseteq G \cup T$)

we want to do the **quotient** construction with (H, S) but **relative** to the **porcupine** graph of (G, T).

and we want to get

$$L_K((G, T)/(H, S)) \cong_{\text{gr}} I(G, T)/I(H, S).$$

The definition of $(G, T)/(H, S)$

$$B_H^G = \{v \in E^0 - H \text{ inf. emitter s.t. } 0 < |\mathrm{s}^{-1}(v) \cap \mathrm{r}^{-1}(G-H)| < \infty\}.$$

$$F_1(G-H, T-S) = \{e_1 e_2 \dots e_n \text{ is a path of } E \mid \mathrm{r}(e_n) \in G-H, \\ \mathrm{s}(e_n) \notin (G-H) \cup (T-S)\}$$

$$F_2(G-H, T-S) = \{p \text{ is a path of } E \mid \mathrm{r}(p) \in T-S, |p| > 0\}$$

The set of **vertices** is

$$(G-H) \cup (T-S) \cup \{w^p \mid p \in F_1(G-H, T-S) \cup F_2(G-H, T-S)\} \cup \\ \{v' \mid v \in ((G \cup T) - S) \cap B_H^G\}.$$

The set of **edges** is

$$\{e \in E^1 \mid \mathrm{r}(e) \in G-H \text{ and either } \mathrm{s}(e) \in G-H \text{ or } \mathrm{s}(e) \in T-S\} \cup \\ \{f^p \mid p \in F_1(G-H, T-S) \cup F_2(G-H, T-S)\} \cup \\ \{e' \mid \mathrm{r}(e) \in ((G \cup T) - S) \cap B_H^G\}.$$

Example 1

Let E be

$$\begin{tikzcd} \bullet_{u_0} & \bullet_{v_0} & \bullet_{w_0} \\ \bullet_{u_1} \arrow[u] \arrow[r, "e"] & \bullet_{v_1} \arrow[u] \arrow[r, "g"] & \bullet_{w_1} \arrow[u] \end{tikzcd}$$

and let $H = \{w_0, w_1\}$ and $G = H \cup \{v_0, v_1\}$. Then, G/H is

$$\begin{tikzcd} & \bullet_{v_0} \\ \bullet \arrow[r, "f^e"] & \bullet_{v_1} \arrow[u] \end{tikzcd}.$$

The porcupine graph of H is

$$\begin{tikzcd} & & \bullet_{w_0} \\ \bullet \arrow[r, "f^{eg}"] & \bullet \arrow[r, "f^g"] & \bullet_{w_1} \arrow[u] \end{tikzcd}$$

and the quotient graph E/G is

$$\begin{tikzcd} \bullet_{u_0} \\ \bullet_{u_1} \arrow[u] \end{tikzcd}.$$

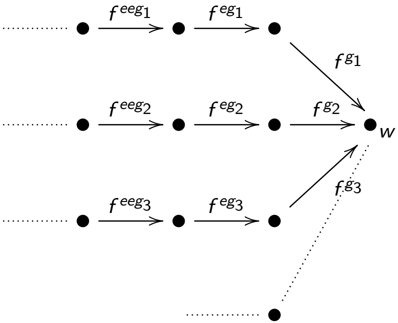
The chain $\emptyset \leq H \leq G \leq E^0$ has cofinal porcupine-quotients.

Example 2

Let E be $\begin{tikzcd} \bullet_v \arrow[loop left, "e"] \arrow[r, bend left=40] \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \bullet_w \end{tikzcd}$ If $H = \{w\}$, then $B_H = \{v\}$ and we have

$$(\emptyset, \emptyset) \leq (H, \emptyset) \leq (H, \{v\}) \leq (E^0, \emptyset)$$

$(H, \{v\})/(H, \emptyset)$ is $\cdots\cdots \bullet \xrightarrow{f^{eee}} \bullet \xrightarrow{f^{ee}} \bullet \xrightarrow{f^{e}} \bullet_v$. The porcupine graph of $(H, \emptyset)$ is



.

The quotient $E/(H, \{v\})$ is $\begin{tikzcd} \bullet_v \arrow[loop left, "e"] \end{tikzcd}$.

Everything agrees

If $(H, S) = (\emptyset, \emptyset)$,

porcupine-quotient is porcupine

If $G = E^0$ (so $T = \emptyset$),

porcupine-quotient is quotient

The best part

For graph monoids and pre-ordered monoid maps:

$$\boxed{M_{(G,T)/(H,S)} \cong J(G,T)/J(H,S)}$$

For talented monoids and pre-ordered Γ-monoid maps:

$$\boxed{M^{\Gamma}_{(G,T)/(H,S)} \cong J^{\Gamma}(G,T)/J^{\Gamma}(H,S).}$$

So, the requirements that a **composition series** of any of these exists are equivalent:

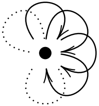
- admissible pairs of E,
- graded ideals of $L_K(E)$,
- order-ideals of M_E,
- Γ-order-ideals of M_E^{Γ}.

Graded simple LPAs

$L_K(E)$ is graded simple iff
no nontrivial and proper admissible pairs

i.e. composition series for E (equiv. $L_K(E)$) have **length 1**.

Three basic examples $\leftrightsquigarrow$ three primary colors.

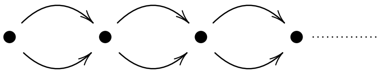


Three types of "terminal" vertices

1. A **sink** connects to no other vertex in the graph except, trivially, to itself.
2. The vertices on a **cycle without exits** do not connect to any vertices outside of the cycle.
3. An **extreme cycle** is a cycle such that the range of every exit from the cycle connects back to a vertex in the cycle. The vertices in such a cycle c connect only to the vertices on cycles in the same "cluster" as c.

The fourth type

The algebra of



is graded simple but E has no sinks, no-exit nor extreme c-s.

$T(V) =$ **tree** of V, vertices to which $v \in V$ emits paths,
$R(V) =$ **root** of V, vertices from which $v \in V$ receives paths.

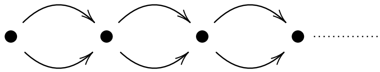
An infinite path α of E is **terminal** if

- no element of $T(\alpha^0)$ is an infinite emitter
- or on a cycle and
- for every infinite path β with $s(\beta) \in \alpha^0$, $T(\beta^0) \subseteq R(\beta^0)$.

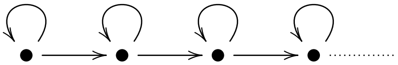
Everything that originates in α comes back to α.

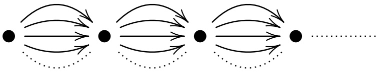
Examples

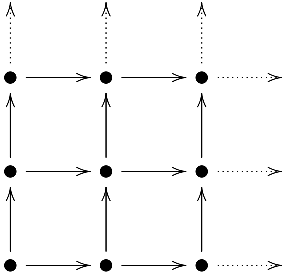
Each infinite path is terminal:



None of the infinite paths is terminal:







Four color characterization of graded simplicity

$L_K(E)$ is **graded simple** (E is cofinal) iff exactly one of the following holds.

1. $E^0 = \overline{\{v\}}$ for v a sink. If so, E is row-finite and acyclic.
2. $E^0 = \overline{c^0}$ for a cycle c without exits. If so, E is row-finite and c is the only cycle.
3. $E^0 = \overline{c^0}$ for an extreme cycle c. If so, every cycle is extreme and every inf. emitter is on a cycle.
4. $E^0 = \overline{\alpha^0}$ for a terminal path α. If so, E is row-finite and acyclic.

Corollaries. $L_K(E)$ is **simple** iff 1, 3, or 4 hold.

$L_K(E)$ is **(graded) purely infinite simple** iff 3 holds.

Comparison with Trichotomy Principle

Trichotomy Principle. $L_K(E)$ is **graded simple** iff exactly one of the following holds.

1. $L_K(E)$ is locally matricial.
2. $L_K(E) \cong \mathbb{M}_\kappa(K[x, x^{-1}])$ for a cardinal κ.
3. $L_K(E)$ is purely-infinite simple.

The four-color characterization does not contradicts this. It

- refines 1 and
- gives a graph, not algebra, characterization.

Necessary cond. for having a composition series

Let

$$\mathrm{Ter}(E) = \overline{\text{terminal vertices}} =$$

$$\overline{\text{sinks}} \cup \overline{\text{no-exits}} \cup \overline{\text{extremes}} \cup \overline{\text{terminal paths}}.$$

If E^0 is finite, $I(\mathrm{Ter}(E))$ has been known as I_{lce}.

If a graph E has a composition series, the following hold.

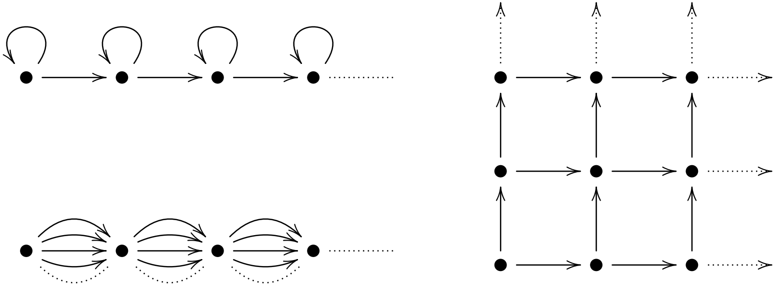
(a) $\mathrm{Ter}(E)$ is **nonempty**.

(b) The set of terminal vertices of E contains **finitely many clusters.**

(c) The set of breaking vertices of $\mathrm{Ter}(E)$ is **finite**.

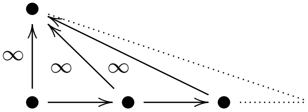
3 types of graphs not having a composition series

Ter(E) is empty:



Infinitely many clusters: 

Infinitely many breaking vertices of Ter(E) :



Characterization of having a composition series

One more type of obstruction

$$\begin{tikzcd} {} \arrow[r, dotted, no head] & \bullet \arrow[loop above] \arrow[r] & \bullet \arrow[loop above] \arrow[r] & \bullet \arrow[loop above] \end{tikzcd}$$

Define the **composition quotients** F_n of E. Let $F_0 = E$.
If $\mathrm{Ter}(F_n) \subsetneq F_n^0$, let

$$F_{n+1} = F_n/(\mathrm{Ter}(F_n), B_{\mathrm{Ter}(F_n)}).$$

If $\mathrm{Ter}(F_n) = F_n^0$, let $F_{n+1} = F_{n+2} = \ldots = \emptyset$.

The graph E **has a composition series** iff

1. Conditions (a), (b), and (c) hold for F_n for each n for which $F_n \neq \emptyset$.
2. There is $n \geq 0$ such that $F_{n+1} = \emptyset$ and $F_n \neq \emptyset$.

The proof is constructive. A corollary.

If $C_1, \ldots, C_n$ are clusters in $\mathrm{Ter}(E)$, start by

$$(\emptyset, \emptyset) \leq (\overline{C_1}, \emptyset) \leq (\overline{C_1} \cup \overline{C_2}, \emptyset) \leq \ldots (\mathrm{Ter}(E), \emptyset).$$

If $v_1, \ldots, v_m$ are breaking vertices of $\mathrm{Ter}(E)$, continue with

$$(\mathrm{Ter}(E), \emptyset) \leq (\mathrm{Ter}(E), \{v_1\}) \leq (\mathrm{Ter}(E), \{v_1, v_2\}) \leq \ldots$$

$$(\mathrm{Ter}(E), B_{\mathrm{Ter}(E)}).$$

"Extend" a series for F_1 by $(\mathrm{Ter}(E), B_{\mathrm{Ter}(E)})$ and append it to this.

Corollary.

Every **unital** $L_K(E)$ has a graded composition series.

Surprising? **Yes** – LPAs appear to be "wilder" than this. **No** – there are fin many vertices and we are cutting nonzero many in each step.

We have 4.

GCC holds for cofinal graphs with sinks and no-exit cycles.

Moreover, TFAE for a cofinal graph E with a sink or a no-exit cycle and any graph F.

1. The talented monoid are "pointed" isomorphic.
2. F is cofinal with the same type of the terminal cluster is the same ($m = 0$ if sink $m > 0$ if cycle of length m) and the number and the lengths of paths ending in a terminal vertex are the same modulo m.

Say $E \approx F$ in this case.

3. The algebras are graded isomorphic.

Graph condition for GCC for disjoint cycles

If E has a finite composition series and disjoint cycles, and F is any graph.

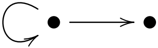
1. The talented monoid are "pointed" isomorphic.
2. $E \approx F$
3. The algebras are graded isomorphic.

If $n = 2$, $\emptyset \leq H \leq E^0$ and $\emptyset \leq G \leq F^0$, then $E \approx F$ if

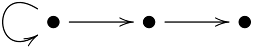
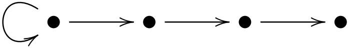
- The **porcupine** parts match $P_H \approx P_G$
- The **quotient** parts match $E/H \approx F/G$, and
- The **numbers and lengths** of paths from the terminal cluster of E/H to the terminal cluster of P_H match the corresponding numbers and lengths for F/G and P_G.

Canonical form and deflated graphs

Let E be



Then each of the graphs below is an out-splits of E.

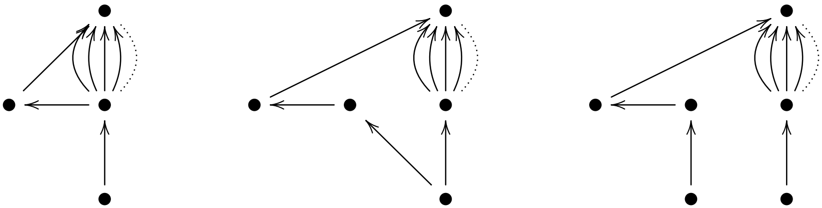
  $\cdots$

So, before you count the paths for the last requirement, you have to ensure your graph is the “smallest” in the same sense as E is the smaller than the graphs above.

A graph is **deflated** if it no out-amalgamations making it “smaller” are possible.

Canonical form and neat graphs

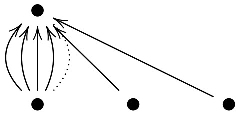
For an infinite emitter v emitting paths to a terminal cluster, you want to count only the paths such that there is infinitely many of those of the same length.



The last graph is the **neat** form of the first.

Canonical form and combed-out graphs

You also want to comb out all the loose "hair" sticking out.

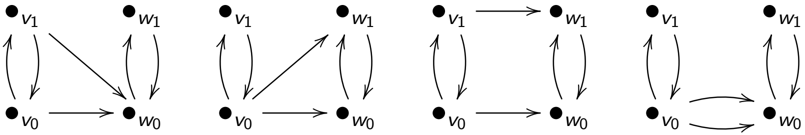


The second graph is the combed-out form of the first.

When you have your graph in a **canonical form** (deflated, neat, and combed-out), then you can start counting the paths.

Example with four deflated graphs

Let E_1, E_2, E_3, and E_4 be the four graphs below.

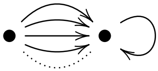


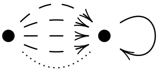
For E_1, $\begin{pmatrix} (1) & (0) \\ (1) & (0) \end{pmatrix}$. For E_2, $\begin{pmatrix} (1) & (1) \\ (0) & (0) \end{pmatrix}$.

For E_3, $\begin{pmatrix} (1) & (0) \\ (0) & (1) \end{pmatrix}$. For E_4, $\begin{pmatrix} (2) & (0) \\ (0) & (0) \end{pmatrix}$.

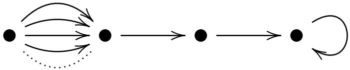
Row and column permutations leave all these different.

You also have to count the in-between vertices





These two show that you do not want to consider the lengths modulo the length of the terminal cycle.



This graph and the second above show that the number of in-between vertices matters.

A hope for settling GCC for all graphs

- Show the GCC for cofinal graphs with extreme cycles.

  The methods of the "step 3" would work to show that GCC holds for all E with a composition series.

  And more:
- Define $\approx$ so that TFAE
  1. The talented monoid are pointed isomorphic.
  2. $E \approx F$
  3. The algebras are graded isomorphic.