Porcupine-quotient and the fourth primary color

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hedgehog

porcupine

porcupine-quotient

three primary colors the fourth one

Substructures and quotients

A very general question: given a short exact sequence

$$0 \to I \to R \to R/I \to 0$$

how to patch the information from I and R/I to recover the information on R?

More specifically, given g and h, how to get f?

$$\begin{tikzcd}
0 \arrow[r] & I \arrow[r] \arrow[d, "g"] & R \arrow[r] \arrow[d, dotted, "f"] & R/I \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & J \arrow[r] & S \arrow[r] & S/J \arrow[r] & 0
\end{tikzcd}$$

Where do these questions come from?

Let R, S be your favourite graph (or groupoid) algebras and I, J their appropriate substructures. Let $\overline{R}, \overline{S}$, etc, denote some exact functor (e.g. pointed K_0^{gr}).

Having

$$\begin{tikzcd}
0 \arrow[r] & \overline{I} \arrow[r] \arrow[d] & \overline{R} \arrow[r] \arrow[d] & \overline{R/I} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \overline{J} \arrow[r] & \overline{S} \arrow[r] & \overline{S/J} \arrow[r] & 0
\end{tikzcd}$$

and getting isomorphisms g and h using some inductive process, we would like to have an iso f as below.

$$\begin{tikzcd}
0 \arrow[r] & I \arrow[r] \arrow[d, "g"] & R \arrow[r] \arrow[d, dotted, "f"] & R/I \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & J \arrow[r] & S \arrow[r] & S/J \arrow[r] & 0
\end{tikzcd}$$

This brings us to the following ...

1. Substructures and quotients can also be represented as algebras of graphs (or groupoids).

quotient

porcupine

porcupine-quotient

2. Finite sequences of such substructures lead us to **composition series** $0 = I_0 \lneq I_1 \lneq \ldots \lneq I_n = R.$

3. The requirement that I_{k+1}/I_k is **simple** leads us to consideration of exactly four types of algebras.

My algebra of choice today is...

Leavitt path algebra $L_K(E)$ of a graph E over a field K.

It is naturally **graded** by $\mathbb{Z}$ so that a path of length n is in the n-th component.

The "substructure" is a **graded ideal.**

Any such ideal I is generated by the set $H \cup S^H$ where $H = I \cap E^0$,

$$B_H = \{v \in E^0 - H \text{ inf. emitter s.t. } 0 < |\mathrm{s}^{-1}(v) \cap \mathrm{r}^{-1}(E^0 - H)| < \infty\},$$

$$\text{for } v \in B_H, \quad \text{let} \quad v^H = v - \sum_{e \in \mathrm{s}^{-1}(v) \cap \mathrm{r}^{-1}(E^0 - H)} ee^*,$$

$$S = \{v \in B_H \mid v^H \in I\} \text{ and } S^H = \{v^H \mid v \in S\}.$$

The vertices in B_H are **breaking vertices**.

Admissible pair $\leftrightsquigarrow$ graded ideal

Such H is **hereditary** ($v \in H$ implies that the tree of v is in H) and **saturated** (a regular v with $\mathrm{r}(\mathrm{s}^{-1}(v))$ in H is itself in H).

Conversely, if H is any hereditary and saturated set of vertices and $S \subseteq B_H$, then the ideal generated by $H \cup S^H$ is **graded**. The pair

$$(H, S)$$

is called an **admissible pair** and we write $I(H, S)$ for the graded ideal generated by $H \cup S^H$.

We want $I(H, S)$ and $L_K(E)/I(H, S)$ to be

Leavitt path algebras.

Quotient graph

This is an "old" construction (2006-2008).

$$(E/(H,S))^0 = E^0 - H \cup \{v' \mid v \in B_H - S\},$$
$$\text{(think } v \leftrightsquigarrow v - v^H \text{ and } v' \leftrightsquigarrow v^H\text{)}$$
$$(E/(H,S))^1 = \{e \in E^1 \mid \mathrm{r}(e) \notin H\} \cup$$
$$\{e' \mid e \in E^1 \text{ and } \mathrm{r}(e) \in B_H - S\},$$

with $\mathrm{s}(e') = \mathrm{s}(e)$, $\mathrm{r}(e') = \mathrm{r}(e)'$.
This ensures that CK2 holds in $E/(H,S)$ for $v \in B_H - S$.

Examples. Let E be $\begin{tikzcd} \bullet^v \arrow[loop left] \arrow[r, bend left] \arrow[r, bend left=10] \arrow[r] \arrow[r, bend right=10] \arrow[r, dotted, no head, bend right=50] & \bullet^w \end{tikzcd}$ and $H = \{w\}$. Then

$B_H = \{v\}$ and

$E/(H,B_H)$ is $\begin{tikzcd} \bullet^v \arrow[loop left] \end{tikzcd}$ and $E/(H,\emptyset)$ is $\begin{tikzcd} \bullet^v \arrow[loop left] \arrow[r] & \bullet^{v'} \end{tikzcd}$.

Hedgehog graph

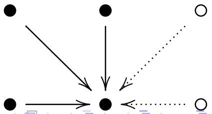
$F_1(H,S) = \{e_1 \dots e_n \text{ is a path of } E \mid \mathrm{r}(e_n) \in H, \mathrm{s}(e_n) \notin H \cup S\}$,
$F_2(H,S) = \{p \text{ is a path of } E \mid \mathrm{r}(p) \in S,\ |p| > 0\}$,
$\overline{F_i}(H,S)$ is a copy of $F_i(H,S), i = 1,2$,

Then

$$E^0_{(H,S)} = H \cup S \cup F_1(H,S) \cup F_2(H,S), \text{ and}$$
$$E^1_{(H,S)} = \{e \in E^1 \,|\, \mathrm{s}(e) \in H\} \cup \{e \in E^1 \,|\, \mathrm{s}(e) \in S, \mathrm{r}(e) \in H\} \cup \overline{F_1}(H,S) \cup \overline{F_2}(H,S) \text{ with}$$
$$\mathrm{s}(\overline{p}) = p, \mathrm{r}(\overline{p}) = \mathrm{r}(p) \text{ for } \overline{p} \in \overline{F_1}(H,S) \cup \overline{F_2}(H,S).$$

Examples. Let E be $e \circlearrowleft \bullet^v \xrightarrow{g} \bullet^w$ and $H = \{w\}$. Then

$F_1 = \{e^n g \,|\, n = 0,1,\dots\}$ and the hedgehog is



Some good and some bad news

Good

$$L_K(\text{quotient graph}) \cong_{\text{gr}} L_K(E)/I(H,S)$$

$$L_K(\text{hedgehog}) \cong I(H,S)$$

Bad

$$L_K(\text{hedgehog}) \ncong_{\text{gr}} I(H,S)$$

Indeed, if E is $\begin{tikzcd} \bullet^v \arrow[loop left, "e"] \arrow[r, "g"] & \bullet^w \end{tikzcd}$ and $H = \{w\}$,

the path $eeeg$ (of **length 4**) of $I(H)$ corresponds to an edge $\overline{eeeg}$ so it has **length 1** in the LPA of the hedgehog.

How to fix this?
Make the "spines" longer and get...

Porcupine graph $P_{(H,S)}$ (2021)

Keep the definitions of F_1 and F_2.

For each $e \in (F_1 \cup F_2) \cap E^1$, let w^e be a **new vertex** and f^e a **new edge** such that $s(f^e) = w^e$ and $r(f^e) = r(e)$.

For each path $p = eq$ where $q \in F_1 \cup F_2$ and $|q| \geq 1$, add a **new vertex** w^p and a **new edge** f^p such that $s(f^p) = w^p$ and $r(f^p) = w^q$. Then let

$$P^0_{(H,S)} = H \cup S \cup \{w^p \mid p \in F_1(H,S) \cup F_2(H,S)\} \text{ and}$$

$$P^1_{(H,S)} = \{e \in E^1 \mid s(e) \in H\} \cup \{e \in E^1 \mid s(e) \in S, r(e) \in H\} \cup \{f^p \mid p \in F_1(H,S) \cup F_2(H,S)\}$$

We get a graded iso by

$$w^p \leftrightsquigarrow pp^*, p \in F_1, \quad w^p \leftrightsquigarrow pr(p)^H p^*, p \in F_2,$$

$$f^{ep} \leftrightsquigarrow epp^*, p \in F_1, \quad f^{ep} \leftrightsquigarrow epr(p)^H p^*, p \in F_2,$$

Example

Let E be $\begin{tikzcd} \bullet^{v} \arrow[loop left, "e"] \arrow[r, "g"] & \bullet^{w} \end{tikzcd}$ and $H = \{w\}$. Then

$$F_1(H) = \{g, eg, eeg, eeeg, \ldots\}$$

and the porcupine is

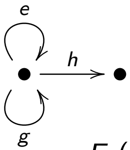
$$\begin{tikzcd} \cdots \arrow[r, dotted] & \bullet^{w^{e^2 g}} \arrow[r, "f^{e^2 g}"] & \bullet^{w^{eg}} \arrow[r, "f^{eg}"] & \bullet^{w^{g}} \arrow[r, "f^{g}"] & \bullet^{w} \end{tikzcd}$$

We unroll the loop and make it into a single spine.

The graded iso is

$$eeeg \leftrightsquigarrow f^{e^3 g} f^{e^2 g} f^{eg} f^{g}.$$

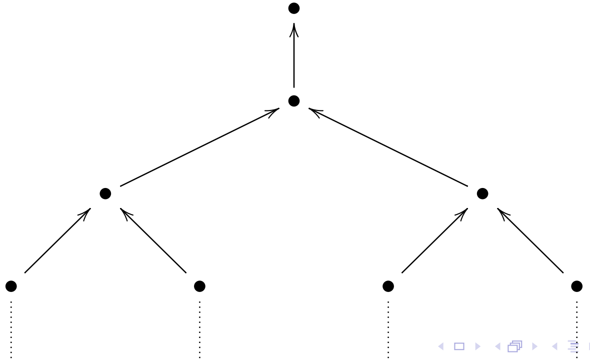
Another example



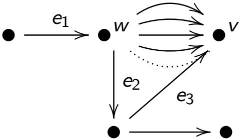
and $H = \{w\}$. Then

$$F_1(H) = \{h, eh, gh, eeh, egh, geh, ggh, eeeh, \ldots\}$$

and the **hedgehog** graph is the **same** as in the previous example. The, **porcupine**, on the other hand, is

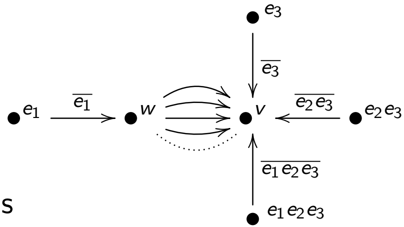


Yet another example

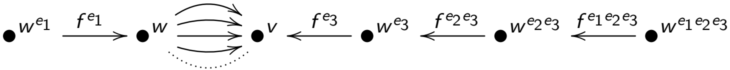
Let E be 

and $H = \{v\}, S = \{w\}$. Here

$F_1(H,S) = \{e_3, e_2e_3, e_1e_2e_3\}$ and $F_2(H,S) = \{e_1\}$.
The hedgehog is



The porcupine is



Porcupine-quotient (2023)

Given $(H, S) \leq (G, T)$ (this means $H \subseteq G$ and $S \subseteq G \cup T$)

we want to do the **quotient** construction with (H, S) but **relative** to the **porcupine** graph of (G, T).

and we want to get

$$L_K((G, T)/(H, S)) \cong_{\mathrm{gr}} I(G, T)/I(H, S).$$

The definition of $(G,T)/(H,S)$

$$B_H^G = \{v \in E^0 - H \text{ inf. emitter s.t. } 0 < |\mathrm{s}^{-1}(v) \cap \mathrm{r}^{-1}(G-H)| < \infty\}.$$

$$F_1(G-H, T-S) = \{e_1 e_2 \dots e_n \text{ is a path of } E \mid \mathrm{r}(e_n) \in G-H, \\ \mathrm{s}(e_n) \notin (G-H) \cup (T-S)\}$$

$$F_2(G-H, T-S) = \{p \text{ is a path of } E \mid \mathrm{r}(p) \in T-S, |p| > 0\}$$

The set of **vertices** is

$$(G-H) \cup (T-S) \cup \{w^p \mid p \in F_1(G-H, T-S) \cup F_2(G-H, T-S)\} \cup \\ \{v' \mid v \in ((G \cup T) - S) \cap B_H^G\}.$$

The set of **edges** is

$$\{e \in E^1 \mid \mathrm{r}(e) \in G-H \text{ and either } \mathrm{s}(e) \in G-H \text{ or } \mathrm{s}(e) \in T-S\} \cup \\ \{f^p \mid p \in F_1(G-H, T-S) \cup F_2(G-H, T-S)\} \cup \\ \{e' \mid \mathrm{r}(e) \in ((G \cup T) - S) \cap B_H^G\}.$$

Example 1

Let E be $\begin{tikzcd} \bullet_{u_0} & \bullet_{v_0} & \bullet_{w_0} \\ \bullet_{u_1} \arrow[u] \arrow[r, "e"] & \bullet_{v_1} \arrow[u] \arrow[r, "g"] & \bullet_{w_1} \arrow[u] \end{tikzcd}$ and let $H = \{w_0, w_1\}$ and $G = H \cup \{v_0, v_1\}$. Then, G/H is $\begin{tikzcd} & \bullet_{v_0} \\ \bullet \arrow[r, "f^e"] & \bullet_{v_1} \arrow[u] \end{tikzcd}$.

The porcupine graph of H is $\begin{tikzcd} & & \bullet_{w_0} \\ \bullet \arrow[r, "f^{eg}"] & \bullet \arrow[r, "f^g"] & \bullet_{w_1} \arrow[u] \end{tikzcd}$ and the quotient graph E/G is $\begin{tikzcd} \bullet_{u_0} \\ \bullet_{u_1} \arrow[u] \end{tikzcd}$. The chain $\emptyset \leq H \leq G \leq E^0$ has cofinal porcupine-quotients.

Example 2

Let E be $\begin{tikzcd} \bullet_v \arrow[loop left, "e"] \arrow[r, bend left] \arrow[r, shift left] \arrow[r] \arrow[r, shift right] \arrow[r, bend right, dotted, no head] & \bullet_w \end{tikzcd}$ If $H = \{w\}$, then $B_H = \{v\}$ and we have

$$(\emptyset, \emptyset) \leq (H, \emptyset) \leq (H, \{v\}) \leq (E^0, \emptyset)$$

$(H, \{v\})/(H, \emptyset)$ is $\begin{tikzcd} \cdots\cdots \bullet \arrow[r, "f^{eee}"] & \bullet \arrow[r, "f^{ee}"] & \bullet \arrow[r, "f^{e}"] & \bullet_v \end{tikzcd}$. The porcupine graph of $(H, \emptyset)$ is

$$\begin{tikzcd}
\cdots\cdots \bullet \arrow[r, "f^{eeg_1}"] & \bullet \arrow[r, "f^{eg_1}"] & \bullet \arrow[dr, "f^{g_1}"] & \\
\cdots\cdots \bullet \arrow[r, "f^{eeg_2}"] & \bullet \arrow[r, "f^{eg_2}"] & \bullet \arrow[r, "f^{g_2}"] & \bullet_w \\
\cdots\cdots \bullet \arrow[r, "f^{eeg_3}"] & \bullet \arrow[r, "f^{eg_3}"] & \bullet \arrow[ur, "f^{g_3}"] & \\
 & & \cdots\cdots \bullet \arrow[uur, dotted, no head] &
\end{tikzcd}.$$

The quotient $E/(H, \{v\})$ is $\begin{tikzcd} \bullet_v \arrow[loop left, "e"] \end{tikzcd}$.

Everything agrees

If $(H, S) = (\emptyset, \emptyset)$,

porcupine-quotient is porcupine

If $G = E^0$ (so $T = \emptyset$),

porcupine-quotient is quotient

The graded iso

$$L_K((G,T)/(H,S)) \cong_{\text{gr}} I(G,T)/I(H,S)$$

$$\begin{array}{lllll}
v & \mapsto & v & +I & \text{if } v \in (G-H) - B_H^G \cup (G \cap S), \\
v & \mapsto & v^{G-H} & +I & \text{if } v \in ((G \cup T) - S)) \cap B_H^G \\
v & \mapsto & v^G & +I & \text{if } v \in (T-S) - B_H^G, \\
w^p & \mapsto & pp^* & +I & \text{if } p \in F_1(G-H, T-S), \\
w^p & \mapsto & \text{pr}(p)^G p^* & +I & \text{if } p \in F_2(G-H, T-S), \\
v' & \mapsto & v - v^{G-H} & +I & \text{if } v \in (G-S) \cap B_H^G, \\
v' & \mapsto & v^G - v^{G-H} & +I & \text{if } v \in (T-S) \cap B_H^G.
\end{array}$$

In the last two lines, the image is $v^H + I$.

The best part

For graph monoids and pre-ordered monoid maps:

$$\boxed{M_{(G,T)/(H,S)} \cong J(G,T)/J(H,S)}$$

For talented monoids and pre-ordered Γ-monoid maps:

$$\boxed{M^{\Gamma}_{(G,T)/(H,S)} \cong J^{\Gamma}(G,T)/J^{\Gamma}(H,S).}$$

So, the requirements that a **composition series** of any of these exists are equivalent:

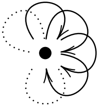
- admissible pairs of E,
- graded ideals of $L_K(E)$,
- order-ideals of M_E,
- Γ-order-ideals of M^{Γ}_E.

Graded simple LPAs

$L_K(E)$ is graded simple iff
no nontrivial and proper admissible pairs

i.e. composition series for E (equiv. $L)K(E)$) have **length 1**.

Three basic examples $\leftrightsquigarrow$ three primary colors.



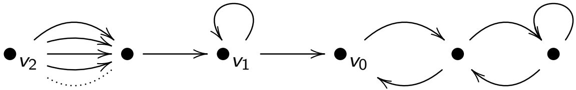
Three types of "terminal" vertices

1. A **sink** connects to no other vertex in the graph except, trivially, to itself.
2. The vertices on a **cycle without exits** do not connect to any vertices outside of the cycle.
3. An **extreme cycle** is a cycle such that the range of every exit from the cycle connects back to a vertex in the cycle. The vertices in such a cycle c connect only to the vertices on cycles in the same "cluster" as c.



Example

For

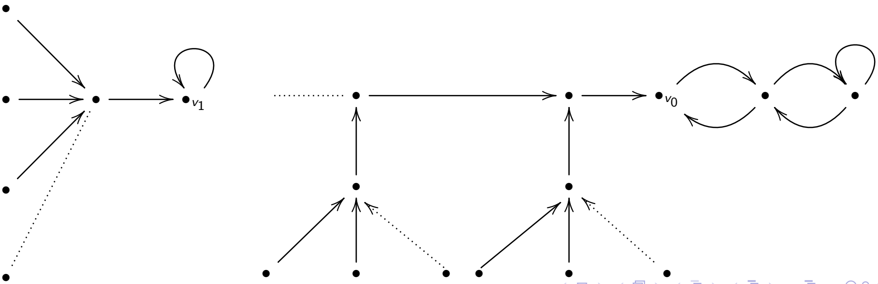


v_0 is on an extreme cycle and $\overline{\{v_0\}}$ is a cluster.

Composition factors:

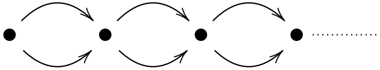
$$\emptyset \leq \overline{\{v_0\}} \leq \overline{\{v_1\}} \leq \overline{\{v_2\}} = E^0.$$

Four porcupine-quotients: $\bullet_{v_2}$



The fourth type

The algebra of



is graded simple but E has no sinks, no-exit nor extreme c-s.

$T(V)$ = **tree** of V, vertices to which $v \in V$ emits paths,
$R(V)$ = **root** of V, vertices from which $v \in V$ receives paths.

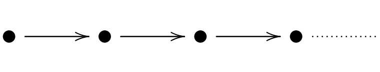
An infinite path α of E is **terminal** if

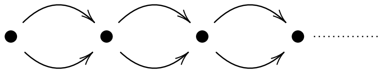
- no element of $T(\alpha^0)$ is an infinite emitter
- or on a cycle and
- for every infinite path β with $s(\beta) \in \alpha^0$, $T(\beta^0) \subseteq R(\beta^0)$.

Everything that originates in α comes back to α.

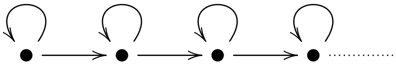
Examples

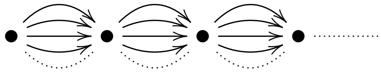
Each infinite path is terminal:

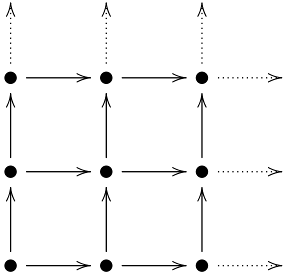




None of the infinite paths is terminal:







Formal definition of a "cluster"

A vertex $v \in E^0$ is **terminal** if it is a sink, on a cycle without exits, on an extreme cycle, or on a terminal path.

The **cluster** of a terminal vertex v is
$\{w \in E^0 \mid R(p^0) = R(q^0) \text{ for all } p, q \in E^{\leq\infty}\ v \in p^0, w \in q^0\}$

$E^{\leq\infty}$ = infinite paths or finite paths with range not regular.

<u>**Exactly four examples:**</u>

1. The cluster of a sink v is $\{v\} = T(v)$.
2. The cluster of $v \in c^0$ for c no-exit is $c^0 = T(c^0)$.
3. The cluster of $v \in c^0$ for c extreme is $T(c^0)$.
4. The cluster of $v \in \alpha^0$ for α terminal is $\bigcup T(\beta^0)$ where β is terminal with $R(\alpha^0) = R(\beta^0)$.

Four color characterization of graded simplicity

$L_K(E)$ is **graded simple** (E is cofinal) iff exactly one of the following holds.

1. $E^0 = \overline{\{v\}}$ for v a sink. If so, E is row-finite and acyclic.
2. $E^0 = \overline{c^0}$ for a cycle c without exits. If so, E is row-finite and c is the only cycle.
3. $E^0 = \overline{c^0}$ for an extreme cycle c. If so, every cycle is extreme and every inf. emitter is on a cycle.
4. $E^0 = \overline{\alpha^0}$ for a terminal path α. If so, E is row-finite and acyclic.

Corollaries. $L_K(E)$ is **simple** iff 1, 3, or 4 hold.

$L_K(E)$ is **(graded) purely infinite simple** iff 3 holds.

Comparison with Trichotomy Principle

Trichotomy Principle. $L_K(E)$ is **graded simple** iff exactly one of the following holds.

1. $L_K(E)$ is locally matricial.
2. $L_K(E) \cong \mathbb{M}_\kappa(K[x, x^{-1}])$ for a cardinal κ.
3. $L_K(E)$ is purely-infinite simple.

The four-color characterization does not contradicts this. It

- refines 1 and
- gives a graph, not algebra, characterization.

Necessary cond. for having a composition series

Let

$$\mathrm{Ter}(E) = \overline{\text{terminal vertices}} =$$

$$\overline{\text{sinks}} \cup \overline{\text{no-exits}} \cup \overline{\text{extremes}} \cup \overline{\text{terminal paths}}.$$

If E^0 is finite, $I(\mathrm{Ter}(E))$ has been known as I_{lce}.

If a graph E has a composition series, the following hold.

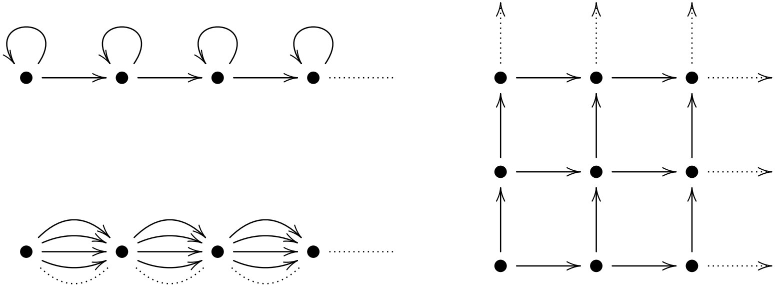
(a) $\mathrm{Ter}(E)$ is **nonempty**.

(b) The set of terminal vertices of E contains **finitely many clusters.**

(c) The set of breaking vertices of $\mathrm{Ter}(E)$ is **finite**.

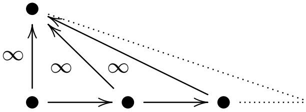
3 types of graphs not having a composition series

Ter(E) is empty:



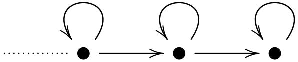
Infinitely many clusters: 

Infinitely many breaking vertices of Ter(E) :



Characterization of having a composition series

One more type of obstruction



Define the **composition quotients** F_n of E. Let $F_0 = E$.
If $\mathrm{Ter}(F_n) \subsetneq F_n^0$, let

$$F_{n+1} = F_n / (\mathrm{Ter}(F_n), B_{\mathrm{Ter}(F_n)}).$$

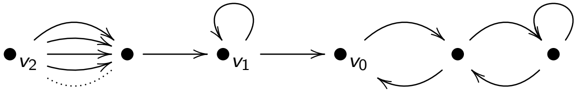
If $\mathrm{Ter}(F_n) = F_n^0$, let $F_{n+1} = F_{n+2} = \ldots = \emptyset$.

The graph E **has a composition series** iff

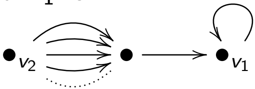
1. Conditions (a), (b), and (c) hold for F_n for each n for which $F_n \neq \emptyset$.
2. There is $n \geq 0$ such that $F_{n+1} = \emptyset$ and $F_n \neq \emptyset$.

Example

If E is



then $\mathrm{Ter}(E) = \overline{\{v_0\}}$ so F_1 is



$\mathrm{Ter}(F_1) = \overline{\{v_1\}}$ so F_2 is

This gives us the composition series

$$\emptyset \leq \overline{\{v_0\}} \leq \overline{\{v_1\}} \leq \overline{\{v_2\}} = E^0.$$

The proof is constructive. A corollary.

If $C_1, \ldots, C_n$ are clusters in $\mathrm{Ter}(E)$, start by

$$(\emptyset, \emptyset) \leq (\overline{C_1}, \emptyset) \leq (\overline{C_1} \cup \overline{C_2}, \emptyset) \leq \ldots (\mathrm{Ter}(E), \emptyset).$$

If $v_1, \ldots, v_m$ are breaking vertices of $\mathrm{Ter}(E)$, continue with

$$(\mathrm{Ter}(E), \emptyset) \leq (\mathrm{Ter}(E), \{v_1\}) \leq (\mathrm{Ter}(E), \{v_1, v_2\}) \leq \ldots$$

$$(\mathrm{Ter}(E), B_{\mathrm{Ter}(E)}).$$

"Extend" a series for F_1 by $(\mathrm{Ter}(E), B_{\mathrm{Ter}(E)})$ and append it to this.

Corollary.

Every **unital** $L_K(E)$ has a graded composition series.

Surprising? **Yes** – LPAs appear to be "wilder" than this. **No** – there are fin many vertices and we are cutting nonzero many in each step.

Talented monoid

For $v \in E^0$, $\langle [v] \rangle$ is **minimal** iff v is **terminal**.

If v is a terminal vertex,

M_E^Γ		E
$[v]$ periodic	$\leftrightsquigarrow$	v in $\overline{\text{no-exits}}$
$[v]$ aperiodic	$\leftrightsquigarrow$	v in $\overline{\text{extremes}}$
$[v]$ incomparable	$\leftrightsquigarrow$	v in $\overline{\text{sink}}$ or in $\overline{\text{terminal paths}}$

Also, characterization when all factors of M_E^Γ are of **only one** or **only two** types. E.g.

- $M^\Gamma_{(G,T)/(H,S)}$ is periodic or incomparable iff the cycle of E are disjoint.
- $M^\Gamma_{(G,T)/(H,S)}$ is incomparable iff E is acyclic.

Questions

- What other useful corollaries we get for LPAs with finite composition series?
- How to realize the idea from the beginning of the talk?
- Does the porcupine-quotient construction has a groupoid generalization?

Full papers on porcupine and porcupine-quotients are on arXiv and at **http://liavas.net**