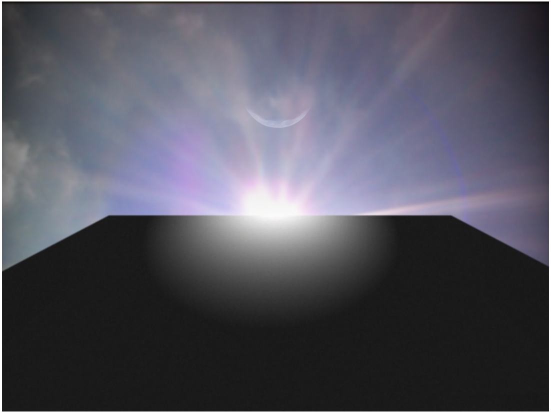
Conquering Dimensions

Lia Vaš

University of the Sciences, Philadelphia



In your math courses ...

... you get to learn about a lot of concepts.

$$\overrightarrow{a} \times \overrightarrow{b} = (a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1)$$

$$\overrightarrow{F} = (P, Q, R) \Rightarrow W = \int_C \overrightarrow{F} \cdot d\overrightarrow{r} = \int_C Pdx + Qdy + Rdz$$

$$\int_a^b f(x)dx = F(b) - F(a)$$

$$V = \int\int\int dx\, dy\, dz = \int\int\int r^2 \cos\phi dr\, d\theta\, d\phi$$

Difficult but important (1)

Math Courses

Real World (other disciplines)

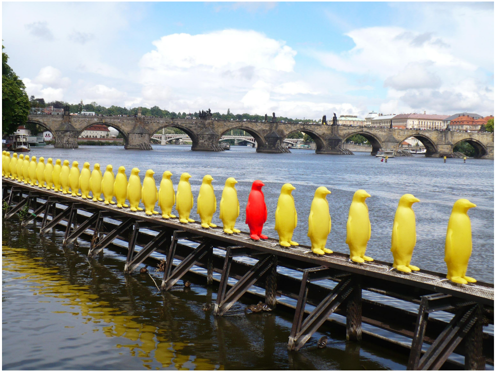
Crossing the bridge

Example: Point Groups in Chemistry

$$
\begin{tikzcd}
\boxed{\text{Geometry of a molecule}} \arrow[r] & \boxed{\text{Symmetry (point) group}} \arrow[d] \\
\boxed{\begin{array}{c}\text{Electronic and vibronic} \\ \text{states, electronic spectrum,} \\ \text{other molecular properties}\end{array}} & \boxed{\text{Group Representation}} \arrow[l]
\end{tikzcd}
$$

Difficult but important (2)

Adapting what you learn to more general set ups or more complex situations.



Example 1 – Chain Rule

y	y'
$(f(x))^n$	$n\,(f(x))^{n-1}\;\; f'(x)$
$e^{f(x)}$	$e^{f(x)}\;\; f'(x)$
$\ln(f(x))$	$\frac{1}{f(x)}\;\; f'(x)$
$\sin(f(x))$	$\cos\,(f(x))\;\; f'(x)$
$\arcsin(f(x))$	$\frac{1}{\sqrt{1-(f(x))^2}}\;\; f'(x)$

$\longrightarrow$

y	y'
$g\,(f(x))$	$g'\,(f(x))\;\; f'(x)$

y	$\frac{dy}{dt}$
$g\,(f(t))$	$\frac{dg}{df}\;\frac{df}{dt}$

$\longrightarrow$

y	$\frac{dy}{dt}$
$g(f_1(t), f_2(t), \ldots f_n(t))$	$\sum_{i=1}^{n} \frac{\partial g}{\partial f_i}\;\frac{df_i}{dt}$

Example 2

A line $\quad \mathbf{y} = \mathbf{mx} + \mathbf{b} \quad$ or $\quad \mathbf{y} = \mathbf{mt} + \mathbf{y(0)} \qquad \rightarrow$

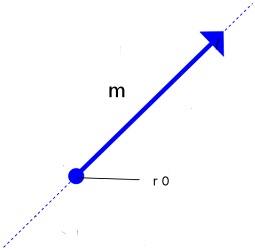
$$\boxed{\text{Vector equation of a line:} \qquad \overrightarrow{\mathbf{r}} = \overrightarrow{\mathbf{m}}\mathbf{t} + \overrightarrow{\mathbf{r_0}}}$$

If $\overrightarrow{m} = (a, b, c)$ and $\overrightarrow{r_0} = (x_0, y_0, z_0)$,

$$\overrightarrow{r} = \overrightarrow{m}t + \overrightarrow{r_0}$$

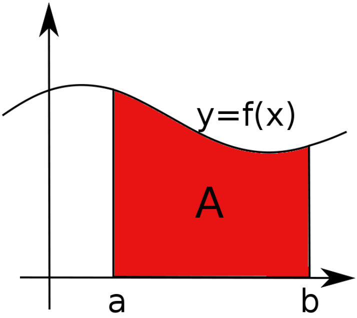
$$\downarrow$$

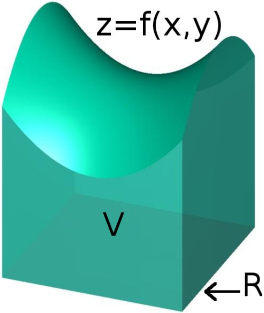
$$x = at + x_0$$
$$y = bt + y_0$$
$$z = ct + z_0$$



Example 3

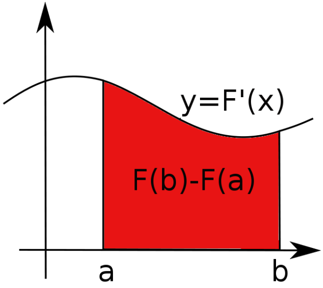
$y = f(x) \geq 0$ **Area** under $f(x)$ is $A = \int_a^b f(x)dx$	$z = f(x,y) \geq 0$ **Volume** under $f(x,y)$ is $V = \int\int_R f(x,y)dxdy$





Example 4

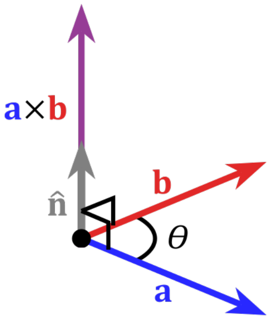
If $f(x) = F'(x)$, then $\int_a^b f(x)dx = \int_a^b F'(x)dx = = F(b) - F(a)$	If $\vec{f}(\vec{r}) = \nabla F(\vec{r})$ and $C_{\text{init.}} = \vec{r}(a)$, $C_{\text{final}} = \vec{r}(b)$, then $\int_C \vec{f}(\vec{r}(t))d\vec{r} = \int_C \nabla F d\vec{r} = = F(\vec{r}(b)) - F(\vec{r}(a))$



Work done by the force $\vec{f}$ acting along the curve C from $C_{\text{initial}} = \vec{r}(a)$ to $C_{\text{final}} = \vec{r}(b)$.

Today's agenda

To generalize the cross product from 3 to higher dimensions.



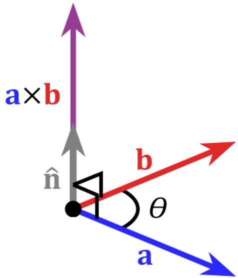
Cross product in 3-dimensions. Main features

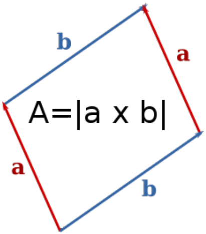
If given two vectors $\overrightarrow{a}$ and $\overrightarrow{b}$ **are not** colinear:

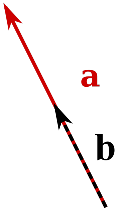
1. $\overrightarrow{a} \times \overrightarrow{b}$ is a vector **perpendicular** to both $\overrightarrow{a}$ and $\overrightarrow{b}$.
2. The length of $\overrightarrow{a} \times \overrightarrow{b}$ is the **area of parallelogram** determined by $\overrightarrow{a}$ and $\overrightarrow{b}$.

If $\overrightarrow{a}$ and $\overrightarrow{b}$ **are** colinear:

3. $\overrightarrow{a} \times \overrightarrow{b} = \overrightarrow{0}$.







Computing the cross product in 3-dimensions

If $\overrightarrow{a} = (a_1, a_2, a_3)$ and $\overrightarrow{b} = (b_1, b_2, b_3)$, then

$$\begin{aligned}
\overrightarrow{a} \times \overrightarrow{b} &= \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\
&= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \overrightarrow{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \overrightarrow{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \overrightarrow{k} \\
&= (a_2 b_3 - a_3 b_2, \quad -(a_1 b_3 - a_3 b_1), \quad a_1 b_2 - a_2 b_1).
\end{aligned}$$

Recall that $\overrightarrow{i} = (1, 0, 0)$, $\overrightarrow{j} = (0, 1, 0)$ and $\overrightarrow{k} = (0, 0, 1)$.

How would you check the property 1?

Recall: we need to check that

$$\overrightarrow{a} \times \overrightarrow{b} \text{ is perpendicular to } \overrightarrow{a},$$

and that

$$\overrightarrow{a} \times \overrightarrow{b} \text{ is perpendicular to } \overrightarrow{b}.$$

Checking the three properties. Property 1.

To check that the two vectors are perpendicular you want to

check that their dot product is zero.

$$(\overrightarrow{a} \times \overrightarrow{b}) \cdot \overrightarrow{a} =$$

$$= (a_2 b_3 - a_3 b_2,\ -(a_1 b_3 - a_3 b_1),\ a_1 b_2 - a_2 b_1) \cdot (a_1, a_2, a_3) =$$

$$= \underline{a_1 a_2 b_3} - \overline{a_1 a_3 b_2} - \underline{a_1 a_2 b_3} + \underbrace{a_2 a_3 b_1} + \overline{a_1 a_3 b_2} - \underbrace{a_2 a_3 b_1} = 0$$

Note that $(\overrightarrow{a} \times \overrightarrow{b}) \cdot \overrightarrow{a} = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = 0$

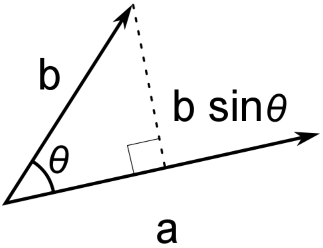
Checking the three properties. Property 2.

Use the formula

$$\boxed{|\overrightarrow{a} \times \overrightarrow{b}| = |\overrightarrow{a}||\overrightarrow{b}| \sin\theta}$$

Area of parallelogram = base times the height.

Base $= |\overrightarrow{a}|$, height $= |\overrightarrow{b}| \sin\theta$



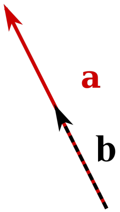
Checking the three properties. Property 3.

If nonzero, $\overrightarrow{a}$ and $\overrightarrow{b}$ are colinear if (and only if) there is a constant k such that

$$\boxed{\overrightarrow{b} = k\overrightarrow{a} \quad \text{that is} \quad b_1 = ka_1,\ b_2 = ka_2,\ b_3 = ka_3.}$$

Calculate that

$$\begin{vmatrix} a_2 & a_3 \\ ka_2 & ka_3 \end{vmatrix} = 0, \quad \begin{vmatrix} a_1 & a_3 \\ ka_1 & ka_3 \end{vmatrix} = 0, \quad \begin{vmatrix} a_1 & a_2 \\ ka_1 & ka_2 \end{vmatrix} = 0.$$



Another way to check property 3

Use the formula

$$\boxed{|\overrightarrow{a} \times \overrightarrow{b}| = |\overrightarrow{a}||\overrightarrow{b}|\sin\theta}$$

$\overrightarrow{a} \times \overrightarrow{b} = \overrightarrow{0} \to |\overrightarrow{a} \times \overrightarrow{b}| = 0 \to |\overrightarrow{a}||\overrightarrow{b}|\sin\theta = 0 \to$
$\sin\theta = 0 \to \theta = 0$ or 180°.

Conversely,
$\overrightarrow{a}$ and $\overrightarrow{b}$ are colinear $\to \theta = 0$ or $180^\circ \to \sin\theta = 0 \to$
$|\overrightarrow{a} \times \overrightarrow{b}| = |\overrightarrow{a}||\overrightarrow{b}|\sin\theta = 0 \to \overrightarrow{a} \times \overrightarrow{b} = \overrightarrow{0}$.

Tim was not convinced...

Timothy P. Enright, chem. major at the time, in 2008 calc. 3 class: wanted to "see" why $|\overrightarrow{a} \times \overrightarrow{b}|$ is the area of the parallelogram.

Tim began to investigate projections of $\overrightarrow{a}$ and $\overrightarrow{b}$ onto the different coordinate planes. For xy-plane, he denoted:

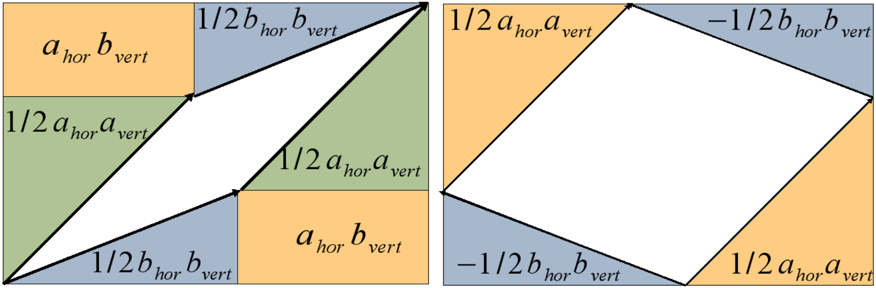
$a_{hor} =$ **length of proj. of $\overrightarrow{a}$ on x-axis**

and

$a_{ver} =$ **length of proj. of $\overrightarrow{a}$ on y-axis.**

And obtain the following images...

Tim's projections



On the first figure,

$$(a_{hor}+b_{hor})(a_{ver}+b_{ver})-2\cdot\frac{1}{2}a_{hor}a_{ver}-2\cdot\frac{1}{2}b_{hor}b_{ver}-2\cdot a_{hor}b_{ver} =$$

$$= a_{ver}b_{hor} - a_{hor}b_{ver}$$

It works

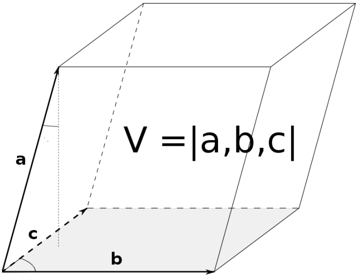
Tim concluded that the three terms compute the **areas of three projected parallelograms**.

$$\begin{aligned}
\overrightarrow{a} \times \overrightarrow{b} &= \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\
&= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \overrightarrow{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \overrightarrow{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \overrightarrow{k} \\
&= (a_2 b_3 - a_3 b_2, \quad -(a_1 b_3 - a_3 b_1), \quad a_1 b_2 - a_2 b_1).
\end{aligned}$$

From 3 to 4 dimensions

Cross product in

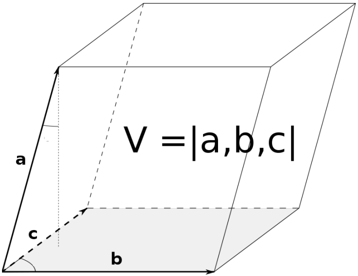
	three dimensions	**four dimensions**
projections	parallelograms	parallelepipeds
i-th coordinate computed by	area of parallelogram 2×2 determinant	volume of parallelepiped 3×3 determinant



Volume of the parallelepiped

Volume spanned by $\overrightarrow{a}$, $\overrightarrow{b}$ and $\overrightarrow{c}$ is

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$



Thus, the formula for $\overrightarrow{a} \times \overrightarrow{b}$ ...

$$
\begin{aligned}
\overrightarrow{a} \times \overrightarrow{b} &= \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\
&= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \overrightarrow{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \overrightarrow{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \overrightarrow{k} \\
&= (a_2 b_3 - a_3 b_2, \quad -(a_1 b_3 - a_3 b_1), \quad a_1 b_2 - a_2 b_1).
\end{aligned}
$$

... generalizes to...

Product in 4 dimensions

$$\overrightarrow{a} \times \overrightarrow{b} \times \overrightarrow{c} = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} & \overrightarrow{l} \\ a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{vmatrix}$$

$$\begin{vmatrix} a_2 & a_3 & a_4 \\ b_2 & b_3 & b_4 \\ c_2 & c_3 & c_4 \end{vmatrix} \overrightarrow{i} - \begin{vmatrix} a_1 & a_3 & a_4 \\ b_1 & b_3 & b_4 \\ c_1 & c_3 & c_4 \end{vmatrix} \overrightarrow{j} + \begin{vmatrix} a_1 & a_2 & a_4 \\ b_1 & b_2 & b_4 \\ c_1 & c_2 & c_4 \end{vmatrix} \overrightarrow{k} - \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \overrightarrow{l} =$$

$$\left(\begin{vmatrix} a_2 & a_3 & a_4 \\ b_2 & b_3 & b_4 \\ c_2 & c_3 & c_4 \end{vmatrix}, - \begin{vmatrix} a_1 & a_3 & a_4 \\ b_1 & b_3 & b_4 \\ c_1 & c_3 & c_4 \end{vmatrix}, \begin{vmatrix} a_1 & a_2 & a_4 \\ b_1 & b_2 & b_4 \\ c_1 & c_2 & c_4 \end{vmatrix}, - \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \right)$$

The three properties continue to hold. Property 1.

1. $\overrightarrow{a} \times \overrightarrow{b} \underline{\times \overrightarrow{c}}$ is a vector **perpendicular** to $\overrightarrow{a}$, $\overrightarrow{b}$ $\underline{\text{ and } \overrightarrow{c}.}$

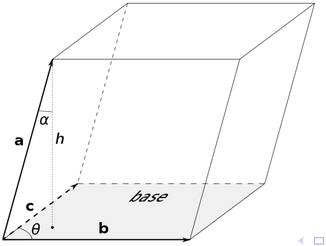
Two (x_1, x_2, x_3, x_4) and (y_1, y_2, y_3, y_4) are orthogonal if their dot product $x_1 y_1 + x_2 y_2 + x_3 y_3 + x_4 y_4$ is zero.

Property 2.

2. The length of $\overrightarrow{a} \times \overrightarrow{b} \times \overrightarrow{c}$ is the ~~**area**~~ **volume of** ~~**parallelogram**~~ **parallelepiped** determined by $\overrightarrow{a}$, $\overrightarrow{b}$ and $\overrightarrow{c}$.

$$\begin{aligned} \text{Volume} &= \text{hight times (area of the base)} \\ &= |\overrightarrow{a}| \cos\alpha \text{ times } |\overrightarrow{b} \times \overrightarrow{c}| \\ &= |\overrightarrow{a}||\overrightarrow{b}||\overrightarrow{c}| \cos\alpha \sin\theta \end{aligned}$$

$$|\overrightarrow{a} \times \overrightarrow{b} \times \overrightarrow{c}| = |\overrightarrow{a}||\overrightarrow{b}||\overrightarrow{c}| \cos\alpha \sin\theta$$



Property 3.

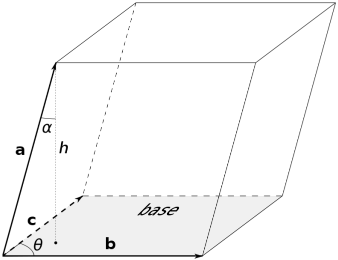
If $\overrightarrow{a}$, $\overrightarrow{b}$ <u>and $\overrightarrow{c}$</u> are ~~colinear~~ **coplanar** :

3. $\overrightarrow{a} \times \underline{\overrightarrow{b} \times \overrightarrow{c}} = \overrightarrow{0}$.

Use the formula

$$|\overrightarrow{a} \times \overrightarrow{b} \times \overrightarrow{c}| = |\overrightarrow{a}|\,|\overrightarrow{b}|\,|\overrightarrow{c}| \cos\alpha \sin\theta$$

The vectors are in the same plane iff $\alpha = 90^\circ$ iff $|\overrightarrow{a} \times \overrightarrow{b} \times \overrightarrow{c}|$=volume =0 iff $\overrightarrow{a} \times \overrightarrow{b} \times \overrightarrow{c} = \overrightarrow{0}$.



Generalize to higher dimensions

Get the **wedge (or exterior) product**. In higher dimensions wedge $\wedge$ is used instead of cross $\times$.

- Start with $n-1$ n-dimensional vectors $\overrightarrow{a_i} = \langle a_{i1}, a_{i2}, \ldots, a_{in} \rangle$, $i = 1, \ldots, n-1$.
- The result $\overrightarrow{a_1} \wedge \overrightarrow{a_2} \wedge \ldots \wedge \overrightarrow{a_{n-1}}$ is an n-dimensional vector with the i-th coordinate equal to:

delete

$$A_i = (-1)^{1+i} \begin{vmatrix} a_{11} & \cdots & a_{1\,i-1} & \boxed{a_{1i}} & a_{1\,i+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2\,i-1} & \boxed{a_{2\,i}} & a_{2\,i+1} & \cdots & a_{2n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n-1\,1} & \cdots & a_{n-1\,i-1} & \boxed{a_{n-1\,i}} & a_{n-1\,i+1} & \cdots & a_{n-1\,n} \end{vmatrix}$$

If you took linear algebra...

Let $\overrightarrow{e_1} = (1, 0, \ldots, 0)$, $\overrightarrow{e_2} = (0, 1, \ldots, 0)$, ...
$\overrightarrow{e_n} = (0, 0, \ldots, 1)$.

$$\overrightarrow{a_1} \wedge \overrightarrow{a_2} \wedge \ldots \wedge \overrightarrow{a_{n-1}} = \begin{vmatrix} \overrightarrow{e_1} & \overrightarrow{e_2} & \ldots & \overrightarrow{e_n} \\ a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{i1} & a_{i2} & \ldots & a_{in} \end{vmatrix} =$$

$$\text{the first row determinant expansion } = A_1\overrightarrow{e_1} + A_2\overrightarrow{e_2} + \ldots + A_n\overrightarrow{e_n} =$$

$$= (A_1, A_2, \ldots, A_n).$$

The three properties still hold

Property 1.

1. $\overrightarrow{a_1} \wedge \overrightarrow{a_2} \wedge \ldots \wedge \overrightarrow{a_{n-1}}$ is **perpendicular** to all of $\overrightarrow{a_1}, \ldots \overrightarrow{a_{n-1}}$.

Note that

$$\overrightarrow{a_1} \cdot (\overrightarrow{a_1} \wedge \overrightarrow{a_2} \wedge \ldots \wedge \overrightarrow{a_{n-1}}) = \begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{i1} & a_{i2} & \ldots & a_{in} \end{vmatrix} = 0$$

Property 2

The generalization of a parallelepiped
in n-dimensions is called an n-**parallelotope.**

Property 2.

2. The length of $\overrightarrow{a_1} \wedge \overrightarrow{a_2} \wedge \ldots \wedge \overrightarrow{a_{n-1}}$ is the **volume of** n-**parallelotope** determined by $\overrightarrow{a_1}$, ... $\overrightarrow{a_{n-1}}$.

$$V = |\overrightarrow{a_1} \wedge \overrightarrow{a_2} \wedge \ldots \wedge \overrightarrow{a_{n-1}}|$$

Property 3

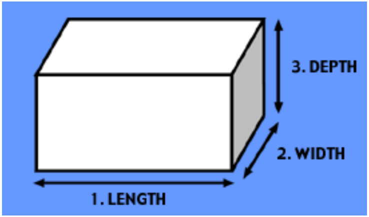
If $\overrightarrow{a_1}$, ... $\overrightarrow{a_{n-1}}$ are ~~coplanar~~ in an $n-1$-dimensional plane:

3. $\overrightarrow{a_1} \wedge \overrightarrow{a_2} \wedge \ldots \wedge \overrightarrow{a_{n-1}} = \overrightarrow{0}$.

1-dim. plane = line 2-dim. plane = plane ... n-dim. plane	$ax + by = c$ $ax + by + cz = d$ ... $a_1x_1 + a_2x_2 + \ldots + a_{n+1}x_{n+1} = b$

In the same way you can...

1. Define n-dimensional "surfaces" (called **manifolds** then).
2. Define derivatives on manifolds and tangent n-plane at a point.
3. Define n-tuple integrals and use to compute n-volumes.



...

In this case, you are doing **differential geometry**.

References

On cross product:

- L. Vas, T. P. Enright, Generalization of Cross Product to Higher Dimensions Using Geometric Approach, For the Learning of Mathematics, 30 (2), (2010) 24 – 25.

Further material on exterior product and algebras:

- N. Bourbaki, Elements of mathematics, Algebra I, Springer-Verlag, 1989.
- S. MacLane, G. Birkhoff, Algebra, AMS Chelsea, 1999.
- Wikipedia.

Preprint of Tim and my paper is available on

http://www.usciences.edu/∼lvas.