Algebraization of Operator Theory

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Operator theory

Algebra

John von Neumann's dream...

... to capture abstractly the concept of an algebra of observables in quantum mechanics.

Non-commutative measure $\leftrightsquigarrow$ trace $\longrightarrow$ dimension function.

Algebraization of Operator Theory

"Von Neumann algebras are blessed with an excess of structure – algebraic, geometric, topological – so much, that one can easily obscure, through proof by overkill, what makes a particular theorem work."

"If all the functional analysis is stripped away ... what remains should (be) completely accessible through algebraic avenues".

Berberian, S. K. Baer $*$-rings; Springer-Verlag, Berlin-Heidelberg-New York, 1972.

The overkill

The overkill that Berberian is referring to:

a mosquito

a machine gun

What structure do we need?

- **With** $+$ **and** $\cdot$ $\longrightarrow$

  a ring.

- **With an involution,**

  an additive map $*$
  with $(xy)^* = y^* x^*$
  and $(x^*)^* = x$ $\longrightarrow$

  a $*$-ring.

Traditional candidate – a Baer $*$-ring

A (left) **annihilator** of a set X = set of all elements a such that $ax = 0$.

A **projection** = a self-adjoint ($p^* = p$) idempotent ($pp = p$).

A **Baer $*$-ring** = every annihilator is generated by a projection.

So **annihilator** $\leftrightsquigarrow$ **closed subspace.**

Kaplansky's dream: to axiomatize (at least part of) the theory of VNAs.

Followed similar path as von Neumann (looked at projections, idempotents, annihilators) and ended up defining **Baer $*$-rings.**

Examples and traces

Finite matrices $M_n(\mathbb{C})$	$\mathcal{B}(H)$, $\dim(H) = n$ with the usual trace
Infinite matrices	$\mathcal{B}(H)$, $\dim(H) = \infty$, usual trace, not finite
Group VNAs $\mathcal{N}(G)$	G-invariant operators on Hilbert space $l^2(G)$ i.e. $f(xg) = f(x)g$. Kaplansky trace on $l^2(G)$ $\mathrm{tr}(\sum a_g g) = a_1$ produces $\mathrm{tr}(f) = \mathrm{tr}(f(1))$.

First and thirds are examples of **finite von Neumann algebras.** Finite means

$$\boxed{xx^* = 1 \quad \text{implies} \quad x^*x = 1.}$$

Trace to a dimension

- A finite VNA $\mathcal{A}$ has a finite, normal and faithful **trace** $\mathrm{tr}_{\mathcal{A}} : \mathcal{A} \to \mathbb{C}$.
- The trace extends to matrices: $\mathrm{tr}([a_{ij}]) = \sum_{i=1}^{n} \mathrm{tr}(a_{ii})$.

[Lück] Trace $\longrightarrow$ dimension.

1. If P is a fin. gen. proj.,

$$\dim_{\mathcal{A}}(P) = \mathrm{tr}(f) \quad \in [0, \infty).$$

where $f : \mathcal{A}^n \to \mathcal{A}^n$ is a projection with image P.

2. If M is **any** module,

$$\dim_{\mathcal{A}}(M) = \sup \{\dim_{\mathcal{A}}(P) \mid P \leq M \text{ fin. gen. proj.}\}$$

$\in [0, \infty]$.

What kind of rings have this type of dimension

2005 Baer *-rings satisfying certain ~~nine~~ **eight** ([2006]) axioms.
If R is such, then $M_n(R)$ is Baer for every n.

2012 **Strongly semihereditary rings** = every fin. gen. nonsingular is projective.
If such R also has $* \Rightarrow M_n(R)$ is Baer for every n.

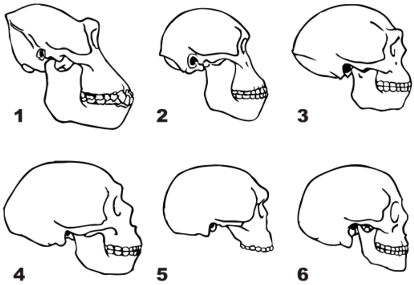
Examples: Finite AW^*-algebras ($AW^* = C^* +$ Baer).

Many ways to bridge the fields

Group Von Neumann algebras	$\leftrightsquigarrow$	Group rings
AW^*-algebras	$\leftrightsquigarrow$	Baer $*$-rings
Graph C^*-algebras	$\leftrightsquigarrow$	Leavitt Path Algebras

Graph algebra evolution

1. **1950s:** Leavitt algebras as examples of rings with $R^m \cong R^n$.
2. **1970s:** Cuntz's algebras – C^*-algebras defined by analogous identities.
3. **1980s:** Cuntz-Krieger algebras – generalization of 2.



4. **1990s:** Graph C^*-algebras.
5. **2000s:** Leavitt path algebras as algebraic analog of 4. and generalization of 1.

Recall: C^* = complete normed and $*$-algebra,
$\cdot$ and $*$ agree with $|| \;\; ||$.

Same analogy

Graph C^*-algebra: The graph encodes the structure $\rightarrow$ easy to work with and classify. Encompasses many important examples of C^*-algebras.

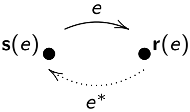
Leavitt path algebra: no operator theory. Axiomatic approach.

Graphs and paths

1. Start with **a graph**: vertices, edges, and source and range map, **s** and **r** $\mathbf{s}(e) \bullet \xrightarrow{e} \bullet \mathbf{r}(e)$
2. Form paths, multiply them by concatenation.

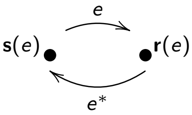
   pq is $\bullet \overset{p}{\dashrightarrow} \bullet \overset{q}{\dashrightarrow} \bullet$ if $\mathbf{r}(p) = \mathbf{s}(q)$ and 0 otherwise.

3. Add the set of **ghost edges**...



... and consider **ghost paths** too.

Leavitt path and graph C^*-algebras



Can also do this by **axioms**:

V $vv = v$ and $vw = 0$ if $v \neq w$,

E1 $e = \mathbf{s}(e)e = e\mathbf{r}(e)$

E2 $e^* = e^*\mathbf{s}(e) = \mathbf{r}(e)e^*$

Add **two more**.

CK1 $e^*e = \mathbf{r}(e)$, and $e^*f = 0$ if $e \neq f$

CK2 $v = \sum_{e \in \mathbf{s}^{-1}(v)} ee^*$ if v **regular** $(0 < |\mathbf{s}^{-1}(v)| < \infty)$.

K= field. The **Leavitt path algebra** $L_K(E)$ of E is a free K-algebra (on v, e and e^*) satisfying these axioms.

If $K = \mathbb{C}$. The **graph C*-algebra** $C^*(E)$ of E is the completion of $L_K(E)$. Universal C^*-algebra with

vertices	=	generating projections
edges	=	partial isometries

and CK1 and CK2.

Some basic properties

1. Element in a Leavitt path algebra $L_K(E)$

$\sum k_{p,q} pq^*$	p, q are paths, $r(p) = r(q)$ $k_{p,q} \in K$

2. $L_K(E)$ has **involution ***.

For involution $k \mapsto \overline{k}$ in K (can always take it to be identity), define

$$(\sum kpq^*)^* = \sum \overline{k} qp^*$$

Basic properties continued

3. If $\{\text{vertices}\}$ is finite, $L_K(E)$ is unital:

$$1 = \sum \text{ all vertices}$$

4. If $\{\text{vertices}\}$ is not finite, $L_K(E)$ has **local units**:

for every $x_i, i = 1, \ldots, n$ there is idempotent u,

$$x_i u = u x_i = x_i.$$

(u is the sum of sources of paths in representation of x_i)

Example 1

$$^{v}\bullet \xrightarrow{e} \bullet^{w}$$

Paths: v, w, e. Representation:

$$v = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad w = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \qquad e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Path algebra: triangular 2x2 matrices $T_2(K) = \begin{bmatrix} K & K \\ 0 & K \end{bmatrix}$

Ghost edge e^*. Representation:

$$e^* = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

Leavitt path algebra: all 2x2 matrices $M_2(K) = \begin{bmatrix} K & K \\ K & K \end{bmatrix}$

Graph C*-algebra: all 2x2 matrices $M_2(\mathbb{C})$

Example class 1 – Matrices

$^{u}\bullet \xrightarrow{e} \bullet^{v} \xrightarrow{f} \bullet^{w}$ Representation via 3x3 matrices.

Generalizes to *n*-line.

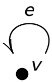
$\bullet \xrightarrow{e_1} \bullet \xrightarrow{e_2} \bullet \cdots\cdots \bullet \xrightarrow{e_{n-1}} \bullet$

Path algebra: triangular *n*x*n* matrices $T_n(K)$

Leavitt path algebra: all *n*x*n* matrices $M_n(K)$

Graph C*-algebra: all *n*x*n* matrices $M_n(\mathbb{C})$

Example 2 – Loop



Paths: $v = 1, e, e^2, e^3 \ldots$ Representation: $e = x$

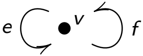
Path algebra: Polynomials $K[x]$.

Ghost edge: $e^* = x^{-1}$

Leavitt path algebra: Laurent polynomials $K[x, x^{-1}]$.

Graph C*-algebra: continuous functions on a circle $C(S^1)$.

Example 3 – Rose

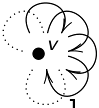


Paths: $v = 1, e, f, ef, e^2, f^2 \ldots$ Representation: $e = x$, $f = y$ **Path algebra:** $K\langle x, y\rangle$. Ghost edges e^*, f^*.

Leavitt path algebra: $L(1,2)$ (universal R with $R^2 \cong R$).

Graph C*-algebra: Cuntz algebra $\mathcal{O}_2$.

Generalizes to n-**rose.**



Path algebra: $K[x_1, \ldots, x_n]$.

Leavitt path algebra: $L(1,n)$

Graph C*-algebra: Cuntz algebra $\mathcal{O}_n$

Traces on graph algebras

1. The usual trace on $M_n(K)$.
2. Kaplansky trace on $K[x, x^{-1}]$.
   $\text{tr}(\sum k_n x^n) = k_0$.

- Traces on graph C^*algebras?
- On Leavitt path algebras?

So, let us look at a trace...

... in the most general way. Let R and T be rings. A

T-valued trace on R

is a map $\mathrm{tr} : R \to T$ which is

- **additive** and
- **central** i.e. $\boxed{\mathrm{tr}(xy) = \mathrm{tr}(yx)}$ for all $x, y \in R$

If R and T are K-algebras, we also want it to be

- K-**linear** i.e. $\mathrm{tr}(kx) = k\mathrm{tr}(x)$ for all $x \in R$ and $k \in K$.

Additional requirements if $*$ is around

x in $*$-ring is **positive** $(x \geq 0)$ if

$$x = \text{finite sum of } yy^*.$$

Comes from complex conjugation:

$$(a + ib)(a - ib) = a^2 + b^2 \geq 0.$$

R, T $*$-rings, $\mathrm{tr} : R \to T$ trace.

- tr is **positive** if $x \geq 0$ implies $\mathrm{tr}(x) \geq 0$.
- tr is **faithful** if $x > 0$ implies $\mathrm{tr}(x) > 0$.

It should all depend on the vertices...

... but not just any map on vertices agrees with CK2.

A central map tr $\boxed{\textbf{agrees with CK2}}$ iff

$$\mathrm{tr}(v) = \mathrm{tr}(\sum ee^*) = \sum \mathrm{tr}(ee^*) = \sum \mathrm{tr}(e^*e) = \sum \mathrm{tr}(\mathbf{r}(e))$$

for v regular with $e \in \mathbf{s}^{-1}(v)$.

Example.

$$\bullet^1 \longleftarrow \bullet^3 \longrightarrow \bullet^1$$

This does not agree with CK2 since $3 \neq 1 + 1$.

Graph traces

Tomforde 2002. A **graph trace** is a map t on the set of vertices such that

- $t(v) = \sum_{e \in I} t(\mathbf{r}(e))$ for $I = \mathbf{s}^{-1}(v)$, and v regular.

It is

- **positive** if (P) $t(v) \geq \sum_{e \in I} t(\mathbf{r}(e))$ for all v, and finite $I \subseteq \mathbf{s}^{-1}(v)$.
- **faithful** if positive and (F) $t(v) > 0$ for all v.

Desirable properties

1. **Graph traces $\leftrightsquigarrow$ Traces.**
2. **(P) $\Longleftrightarrow$ positive,** **(F) $\Longleftrightarrow$ faithful.**

Both fail. The $\mathbb{C}$-valued tr on $\mathbb{C}[x, x^{-1}]$ (=LPA of a loop) given by

$$\mathrm{tr}(x^n) = i^n, \mathrm{tr}(x^{-n}) = i^n$$

has (P) and (F) but is not positive since

$$\mathrm{tr}((1+x)(1+x^{-1})) = 2+2i.$$

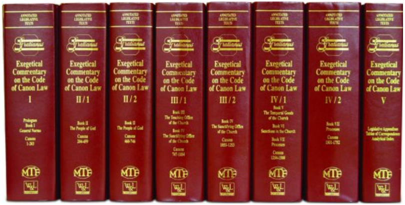
Also, the graph trace with $\mathrm{tr}(1) = 1$ extends to a different trace.

Fixing this – Canonical traces

tr = trace on $L_K(E)$, p, q = paths.

tr is **canonical** if tr("nondiagonal") = 0 and
tr("diagonal") = tr(vertex).

$$\mathrm{tr}(pq^*) = 0, \text{ for } p \neq q \text{ and } \mathrm{tr}(pp^*) = \mathrm{tr}(\mathbf{r}(p)).$$



Harmony

Theorem [2016].

canonical trace on $L_K(E)$	$\leftrightsquigarrow$	graph trace on E
canonical tr is positive	$\Longleftrightarrow$	(P) holds.
canonical tr is faithful	$\Longleftrightarrow$	(F) holds.

Instead of going over 6 pages of proof...

... let me tell you what my **driving force** was.

1. Classification of von Neumann algebras via traces.
2. Results on traces of graph C^*-algebras.

Connecting with the C^*-algebra world

Theorem [Pask-Rennie, 2006]. E row-finite and countable. All maps are $\mathbb{C}$-valued.

faithful, semifinite, lower semicontinuous gauge-invariant trace on $C^*(E)$	$\leftrightsquigarrow$	faithful graph trace on E

semifinite $= \{x \in C^*(E)^+ | \operatorname{tr}(x) < \infty\}$ is norm dense in $C^*(E)^+$.

lower semicontinuous $= \operatorname{tr}(\lim_{n\to\infty} a_n) \leq \liminf_{n\to\infty} \operatorname{tr}(a_n)$ for all $a_n \in C^*(E)^+$ norm convergent.

Bridging

Operator theory trace

Defined on the positive cone.

$\mathrm{tr}(\mathbf{x}\mathbf{x}^*) = \mathrm{tr}(\mathbf{x}^*\mathbf{x})$

Faithful if

$$\mathrm{tr}(xx^*) = 0 \Rightarrow x = 0.$$

Algebra trace

Defined everywhere.

Central.

Faithful if positive and

$$\mathrm{tr}\left(\sum xx^*\right) = 0 \Rightarrow \sum xx^* = 0.$$

Connecting the worlds

Corollary [2016]. E countable.

semifinite,				
lower semicont.,				
faithful,		faithful,		faithful
gauge-invariant		canonical		
trace	$\leftrightsquigarrow$	trace	$\leftrightsquigarrow$	graph trace
on $C^*(E)$		on $L_{\mathbb{C}}(E)$		on E

Where to next with this?

My **driving force**:

A **von Neumann** algebra is	**finite**
iff	there is a finite, normal, faithful trace.

I wandered:

A **Leavitt path** algebra $L_K(E)$ is	**finite**
iff	there is a K-valued canonical, faithful trace (?)
iff	the graph is ___________.

Recall that a $*$-ring is **finite** if

$$\boxed{xx^* = 1 \quad \text{implies} \quad x^*x = 1.}$$

Easy: the existence of a faithful trace implies finiteness.

$$xx^* = 1 \;\Rightarrow\; 1 - x^*x \geq 0 \text{ and } \mathrm{tr}(1 - xx^*) = 0 \text{ so}$$

$$\mathrm{tr}(1 - x^*x) = \mathrm{tr}(1 - xx^*) = 0 \;\Rightarrow\; 1 - x^*x = 0 \;\Rightarrow\; x^*x = 1.$$

Houston, we have a problem

finite iff $xx^* = 1 \;\Rightarrow\; x^*x = 1.$

What is "1" if E is not finite?

There are still **local units**: for every finite set of elements, there is an idempotent acting like a unit.

A $*$-ring with local units R is **finite** if for every x and an idempotent u with $xu = ux = x$,

$$xx^* = u \quad \text{implies} \quad x^*x = u.$$

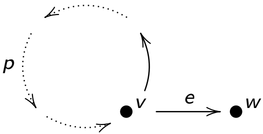
In this case u is a projection (selfadjoint idempotent).

LPAs is finite iff E has "no exits"

If a cycle p has an **exit**,

then a LPA is

not finite.



Let $x = p + w$, and $u = v + w$.

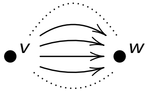
Then $ux = ux = x$ and $x^*x = u$. If $xx^* = u$, then $pp^* = v \Rightarrow 0 = pp^*e = ve = e$ contradiction.

If $v = w$, take $x = p$, $u = v$ and arrive to contradiction too.

Where will the trace take us next?

Idea of "localizing": more general than just for finiteness.

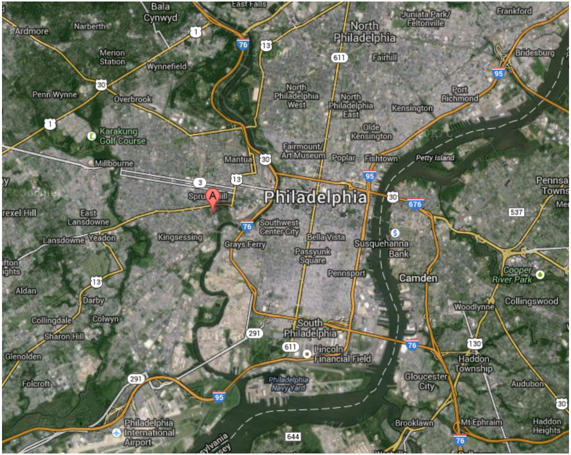
$L_K(E)$ is finite	$\not\Rightarrow$ $\Leftarrow$	$L_K(E)$ has a faithful trace
$\Updownarrow$		$\Updownarrow$
E is no-exit	$\Leftarrow$	E is no-exit and ________ ?



No exits here.

No trace since value of $\mathrm{tr}(v) \geq n\mathrm{tr}(w)$ for all n.

Localizing



http://liavas.net and arXiv.