K-theory classification of graded ultramatricial $*$-algebras

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Classification

The tamest of them all

Approximately finite-dimensional C^*-algebras

$=$ countable direct limits of finite dimensional C^*-algebras.

Some nice properties:

- K_0 classifies them completely. As a corollary,
- two AF-algebras are isomorphic **as rings** iff they are isomorphic as $*$-**algebras**.

Crossing the bridge

Operator theory

Field $K = \mathbb{C}$

Finite dimensional C^*-algebra:

finite sum of $\mathbb{M}_n(\mathbb{C})$

AF-algebra:

$$\varinjlim_n A_n$$

for A_n fin. dim.

Same as $\overline{\bigcup_n \phi_n(A_n)}$

Algebra

Any field K.

Matricial algebra over K:

finite sum of $\mathbb{M}_n(K)$

Ultramatricial algebra:

$$\varinjlim_n A_n$$

for A_n matricial.

Same as $\bigcup_n \phi_n(A_n)$

Operator Theory versus Algebra

For C^*-algebras A, B

$A \cong B$ **as algebras** iff $A \cong B$ **as** $*$**-algebras**.

For $*$-algebras over a field K, this **does not hold.**

Example: $\mathbb{C}$ with the identity involution and
$\mathbb{C}$ with the complex-conjugate involution.

So, the following question is relevant.

For which $*$-algebras A and B,
$A \cong B$ **as algebras** iff $A \cong B$ **as** $*$**-algebras**?

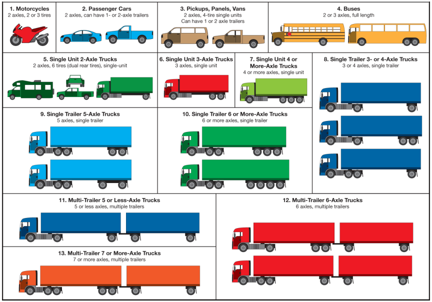
More generally...

For which class of algebras $\mathcal{C}$ and $A, B \in \mathcal{C}$

when $A \cong B$ **as algebras** iff $A \cong B$ **as rings**?

Partial answer: when K_0 is sensitive enough invariant – when

$A \cong B$ **as algebras**
iff
$K_0(A) \cong K_0(B)$
as (pointed) **groups**.



In this case, we say that K_0 **completely classifies** the algebras in $\mathcal{C}$.

Partial answer because...

Every ring homomorphism $A \to B$ induces a pointed group homomorphism $K_0(A) \to K_0(B)$.

So, if K_0 classifies the algebras and $f : A \to B$ is a ring iso, then $K_0(f)$ is an iso and so $K_0(A) \cong K_0(B)$ implies $A \cong B$ as algebras. QED

Example. Ultramatricial algebras over a field.

Operator Theory version: AF C^*-algebras.

Non-Example. $K = \mathbb{Z}_2, \quad \mathcal{C} =$ all K-algebras,

$A = \mathbb{C} \oplus \mathbb{C}$ with trivial $\mathbb{Z}_2$-action,

$B = \mathbb{C} \oplus \mathbb{C}$ with $1(a, b) = (b, a)$.

When there are stars...

... then $\mathbb{Z}_2$ acts on $K_0(A)$ for $A \in \mathcal{C}$ by

$$\boxed{[P] \mapsto [\mathrm{Hom}_A(P, A)]}$$

or, via idempotents, as

$$\boxed{[p] = [(a_{ij})] \mapsto [p^*] = [(a_{ji}^*)].}$$

So, K_0 **completely classifies** $*$**-algebras** in $\mathcal{C}$ if

$A \cong B$ as $*$-**algebras** iff $K_0(A) \cong K_0(B)$ **as** (pointed) $\mathbb{Z}[\mathbb{Z}_2]$-**modules**.

$\mathbb{Z}_2$-action on K_0 is often trivial

... like, for example, when every element of K_0 can be represented using **projections**.

Happens for K_0(field), K_0(C^*-algebra) or K_0(LPA).

If

$\mathbb{Z}_2$-action is trivial
and
K_0 completely classifies
$*$-algebras in $\mathcal{C}$

then for any $A, B \in \mathcal{C}$
the following are equivalent.

$A \cong B$ **as rings**.

$A \cong B$ **as** $*$-**rings**.

$A \cong B$ **as algebras**.

$A \cong B$ **as** $*$-**algebras**.

Why was I interested in this?

Because of the **Isomorphism Conjecture** for graph algebras stating that

$L_{\mathbb{C}}(E) \cong L_{\mathbb{C}}(F)$ **as rings** iff $C^*(E) \cong C^*(F)$ **as $*$-algebras**.

Formulated by Gene Abrams and Mark Tomforde. Note that

$L_{\mathbb{C}}(E) \cong L_{\mathbb{C}}(F)$ **as $*$-algebras** $\Rightarrow$ $C^*(E) \cong C^*(F)$ **as $*$-algebras**.

Mark

Gene

State of IC and Generalized IC

IC is known to hold for:

- Acyclic graphs (Abrams-Tomforde 2008).
- Row-finite, cofinal graphs with Condition (L) and at least one cycle (Abrams-Tomforde 2008).
- Graphs with finitely many vertices (Eilers-Restorff-Ruiz-Sørensen 2016).

Generalized IC:

$L_K(E) \cong L_K(F)$ **as rings** iff $L_K(E) \cong L_K(F)$ **as $*$-algebras**.

GIC is known to hold for:

- Acyclic graphs if the involution of K is "nice" (follows from a 1987 paper of Ara).
- Finite graphs in which cycles have no exits (Aranda-Vaš 2013).

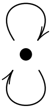
Considering K_0 is useful...

... for classification-related questions, **but $\mathbf{K_0}$ does not classify LPAs**.

$E = \bullet$

$F =$ 

$L_K(E) \ncong L_K(F)$ but $K_0(L_K(E)) = K_0(L_K(F)) = \mathbb{Z}$

Also $E =$ 

$L_K(E) \ncong 0$ but $K_0(L_K(E)) = K_0(0) = 0$

So let us consider more structure of a LPA

Leavitt path algebra is also **graded**.

If Γ is an abelian group, a ring R is Γ-**graded** if

$$R = \bigoplus_{\gamma \in \Gamma} R_\gamma \quad \text{such that} \quad R_\gamma R_\delta \subseteq R_{\gamma+\delta}.$$

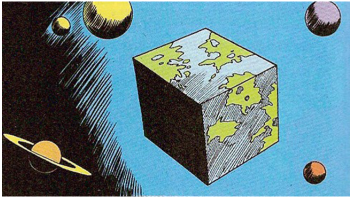
ring

graded ring

Bizarro world of graded rings

$x \in R_\gamma$ is **homogeneous**.

In the world of graded rings, "**element**" is replaced by "**homogeneous element**" in many instances.



field $=$	$\leftrightsquigarrow$	**graded field** $=$
every $x \neq 0$ has x^{-1}	$\leftrightsquigarrow$	every homog. $x \neq 0$ has x^{-1}
regular $=$	$\leftrightsquigarrow$	**graded regular** $=$
$(\forall x)\ x \in xRx$	$\leftrightsquigarrow$	$(\forall$ homog. $x)\ x \in xRx$
free $=$	$\leftrightsquigarrow$	**graded free** $=$
has basis	$\leftrightsquigarrow$	has homog. basis

Not so bizarro after all...

1. Although they appear more specific, graded rings are

more general

since every ring is graded by $\Gamma = \{0\}$.

2. Many rings are **naturally graded**: group rings, LPAs ...

For a LPA, $\Gamma = \mathbb{Z}$ and $L_K(E)_n = \text{span}\ \{pq^* \mid |p| - |q| = n\}$.

3. $K[x, x^{-1}]$ is not a field but it is a graded field (with $\mathbb{Z}$ grading $K[x, x^{-1}]_n = \{kx^n\}$).

Shifts

If $M = \bigoplus_{\gamma \in \Gamma} M_\gamma$ is a graded module and $\delta \in \Gamma$, then

$$\boxed{M(\delta) = \bigoplus_{\gamma \in \Gamma} M_{\gamma+\delta}}$$

$A = \Gamma$-graded ring, $\gamma_1, \dots, \gamma_n \in \Gamma$, $\mathbb{M}_{\mathbf{n}}(\mathbf{A})(\gamma_{\mathbf{1}}, \dots, \gamma_{\mathbf{n}})$ is $\mathbb{M}_n(A)$ graded so that

$$\boxed{(a_{ij}) \in \mathbb{M}_n(A)(\gamma_1, \dots, \gamma_n)_\delta \quad \text{iff} \quad a_{ij} \in A_{\delta+\gamma_j-\gamma_i}}$$

For LPAs, **this really helps!**

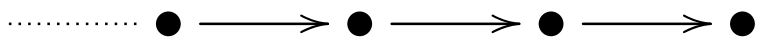
$$E = \quad \bullet \longrightarrow \bullet \longrightarrow \bullet \qquad\qquad F = \begin{tikzcd} & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

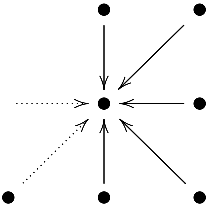
$L_K(E) \cong L_K(F) \cong \mathbb{M}_3(K)$ as algebras however

$L_K(E) \cong_{\mathrm{gr}} \mathbb{M}_3(K)(0,1,2) \not\cong_{\mathrm{gr}} L_K(F) \cong_{\mathrm{gr}} \mathbb{M}_3(K)(0,1,1)$

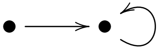
Grading helps...

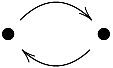
... distinguish between algebras of these pairs also.





$$\mathbb{M}_\infty(K)(0,1,2,3,\dots) \quad \not\cong_{\mathrm{gr}} \quad \mathbb{M}_\infty(K)(0,1,1,1,\dots)$$





$$\mathbb{M}_2(K[x,x^{-1}])(0,1) \quad \not\cong_{\mathrm{gr}} \quad \mathbb{M}_2(K[x^2,x^{-2}])(0,1)$$

Bizarro K_0-group

A graded module is **graded projective** iff it is a summand of graded free module.

$\mathbf{K_0^{gr}}$-group has been considered by **Roozbeh Hazrat**.

Roozbeh showed that K_0^{gr} is **more sensitive** invariant than K_0.

So, it classifies better.

Shifts induce extra structure on K_0^{gr}

K_0^{gr} of a Γ-graded ring is a $\mathbb{Z}[\boldsymbol{\Gamma}]$-**module** with

$$\gamma[P] = [P(\gamma)]$$

For LPAs, $\Gamma = \mathbb{Z}$ and we can think of $\mathbb{Z}[\mathbb{Z}]$ as $\mathbb{Z}[x, x^{-1}]$ and

$\mathbf{K_0^{gr}(L_K(E))}$ is a $\mathbb{Z}[\mathbf{x}, \mathbf{x}^{-1}]$-module.

So, K_0^{gr} classifies better.

Graded K_0 classifies better

$E = \bullet$

$F = \bullet$

$K_0^{\mathrm{gr}}(L_K(E)) = \mathbb{Z}[x, x^{-1}]$

$K_0^{\mathrm{gr}}(L_K(F)) = \mathbb{Z}$

with $xa = a$

$E = \bullet$

$K_0^{\mathrm{gr}}(L_K(E)) = \mathbb{Z}[\frac{1}{2}]$ with $xa = 2a$

But besides grading, a LPA also has an involution...

Graded and Involutive...

Graded and Involutive

A $*$-ring R graded by Γ is a **graded $*$-ring** if

$$R_\gamma^* \subseteq R_{-\gamma}.$$

In this case, $K_0^{\mathrm{gr}}(R)$ is a

$\mathbb{Z}[\mathbb{Z}_2] - \mathbb{Z}[\Gamma]$ bimodule.

Roozbeh-Lia goals:

1. Classification of graded ultramatricial $*$-algebras using K_0^{gr}...
2. ... which implies all known classifications.
3. Isomorphism Conjecture for (some?) LPAs.

Known classifications

Elliott (1976). K_0-group classification of AF C^*-algebras.

Goodearl's Regular-rings book. Ultramatricial algebras over a field.

Ara (1987). Ultramatricial $*$-algebras over a $*$-field with "nice" involution.

Hazrat (2013). Graded ultramatricial algebras over a graded field using K_0^{gr}.

Problem. Goodearl's and Roozbeh's proofs cannot be adapted to $*$-rings.

Contractive maps

If A and B are Γ-graded rings, a $\mathbb{Z}[\Gamma]$-module homomorphism

$f : K_0^{\mathrm{gr}}(A) \to K_0^{\mathrm{gr}}(B)$ is **contractive**

if

- f is **order-preserving** (i.e. $x \geq 0$ implies $f(x) \geq 0$),
- f is **generating-interval-preserving**

  (i.e. $0 \leq x \leq [A]$ implies $0 \leq f(x) \leq [B]$).

f is **unit-preserving** if $f([A]) = [B]$.

If A and B are non-unital and A^u and B^u are unitizations, $[A^u]$ and $[B^u]$ take over the role of $[A]$ and $[B]$.

Classification entails

For graded matricial $*$-algebras A and B over a Γ-graded $*$-field F show the following.

1. Fullness

For $f : K_0^{\mathrm{gr}}(A) \to K_0^{\mathrm{gr}}(B)$, there is $\phi : A \to B$ with $K_0^{\mathrm{gr}}(\phi) = f$.

2. Faithfulness

For $\phi, \psi : A \to B$,

$$K_0^{\mathrm{gr}}(\phi) = K_0^{\mathrm{gr}}(\psi) \text{ if and only if } \phi = \theta\psi$$

for some type of inner automorphism θ of B.

3. Elliott-Bratteli intertwining

If A and B are graded ultramatricial $*$-algebras

$$f : K_0^{\mathrm{gr}}(A) \cong K_0^{\mathrm{gr}}(B) \text{ iff there is } \phi : A \cong B \text{ with } K_0^{\mathrm{gr}}(\phi) = f.$$

Fullness. Non-graded Example.

$$A = \mathbb{M}_2(K) \oplus K,$$
$$B = \mathbb{M}_5(K) \oplus \mathbb{M}_4(K).$$

$$f = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathbb{M}_2(\mathbb{Z})$$

contractive $\Rightarrow a_{ij} \geq 0$ and

$$2a_{11} + 1a_{12} \leq 5$$
$$2a_{21} + 1a_{22} \leq 4$$

the dimension formulas

For example, $f = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$ induces $\phi : R \to S$ given by

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, e\right) \mapsto \left(\left(\begin{array}{cccc|c} a & 0 & b & 0 & 0 \\ 0 & a & 0 & b & 0 \\ c & 0 & d & 0 & 0 \\ 0 & c & 0 & d & 0 \\ \hline 0 & 0 & 0 & 0 & e \end{array}\right), \left(\begin{array}{ccc|c} e & 0 & 0 & 0 \\ 0 & e & 0 & 0 \\ 0 & 0 & e & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right)\right).$$

Fullness. Graded $*$-Example

$\Gamma = \mathbb{Z}_3 = \mathbb{Z}[x]/(x^3 = 1)$. K $*$-field trivially graded.

$$\begin{aligned} A &= \mathbb{M}_2(K)(1,x) && \oplus A(x) \\ B &= \mathbb{M}_5(K)(1,1,x,x,x^2) && \oplus \mathbb{M}_4(K)(1,1,x^2,x^2) \end{aligned}$$

For example, $f = \begin{pmatrix} 2 & x^2 \\ x & x+x^2 \end{pmatrix}$, induces $\phi : A \to B$

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, e\right) \mapsto \left(\begin{pmatrix} a & 0 & b & 0 & 0 \\ 0 & a & 0 & b & 0 \\ c & 0 & d & 0 & 0 \\ 0 & c & 0 & d & 0 \\ 0 & 0 & 0 & 0 & e \end{pmatrix}, \begin{pmatrix} d & 0 & c & 0 \\ 0 & e & 0 & 0 \\ b & 0 & a & 0 \\ 0 & 0 & 0 & e \end{pmatrix}\right).$$

Fullness. Dimension Formulas

In the previous example, $f \in \mathbb{M}_2(\mathbb{Z}[\Gamma])$ is contractive if

$$a_{ij} = a_{ji1} + a_{ji2}x + a_{ji3}x^2$$

with $a_{jit} \geq 0$ satisfies the **pre-dimension** formulas

$$(1) \quad a_{111}+a_{112}x+a_{113}x^2+a_{111}x^2+a_{112}+a_{113}x+a_{121}x^2+a_{122}+a_{123}x \leq 2+2x^2+x$$

$$(2) \quad a_{211}+a_{212}x+a_{213}x^2+a_{211}x^2+a_{212}+a_{213}x+a_{221}x^2+a_{222}+a_{223}x \leq 2+2x.$$

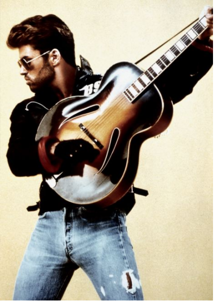
which imply the **dimension formulas**

$$\begin{array}{lll}
\text{inequality (1)} & (x^0\text{-terms}) & a_{111}+a_{112}+a_{122} \leq 2 \\
 & (x^1\text{-terms}) & a_{112}+a_{113}+a_{123} \leq 1 \\
 & (x^2\text{-terms}) & a_{111}+a_{113}+a_{121} \leq 2 \\
\text{inequality (2)} & (x^0\text{-terms}) & a_{211}+a_{212}+a_{222} \leq 2 \\
 & (x^1\text{-terms}) & a_{211}+a_{213}+a_{221} \leq 0 \\
 & (x^2\text{-terms}) & a_{213}+a_{212}+a_{223} \leq 2
\end{array}$$

Faithfulness

A, B graded matricial $*$-algebras, $\phi, \psi : A \to B$ are graded $*$-homomorphisms (not necessarily unital). TFAE.

1. $$K_0^{\mathrm{gr}}(\phi) = K_0^{\mathrm{gr}}(\psi)$$
2. $\exists u \in B_0$ unitary, $\forall a \in A$, $\phi(a) = u\psi(a)u^*$.
3. $\exists u \in B_0$ invertible, $\forall a \in A$, $\phi(a) = u\psi(a)u^{-1}$.



Assumptions

For fullness. F has **enough unitaries**

i.e. every component F_γ of F contains an unitary.

For faithfulness. F_0 is **2-proper and $*$-pythagorean**

i.e. $xx^* + yy^* = 0 \Rightarrow x = y = 0$ and
for all x, y there is z with $xx^* + yy^* = zz^*$.

Not very restrictive since

- A field K and $\mathbb{Z}$-graded $K[x, x^{-1}]$ have enough unitaries.
- C^*-algebras are 2-proper and $*$-pythagorean.
- K 2-proper and $*$-pythagorean $\Rightarrow K[x, x^{-1}]$, $\mathbb{M}_n(K)$, $\mathbb{M}_n(K[x, x^{-1}])$ are such too.

Intertwining

Classification Theorem.

- F = graded $*$-field with enough unitaries, F_0 = 2-proper and $*$-pythagorean.
- A, B = graded ultramatricial $*$-algebras.

For

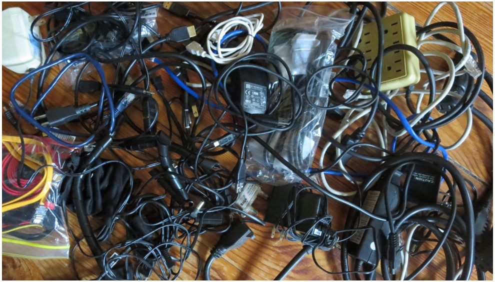
a contractive $\mathbb{Z}[\Gamma]$-iso $f : K_0^{gr}(A) \to K_0^{gr}(B)$

there is

a graded $*$-iso $\phi : A \to B$ with $K_0^{gr}(\phi) = f$.

Intertwining idea

$$
\begin{tikzcd}
A_{n(1)} \arrow[r, "\phi_{n(1)n(2)}"] \arrow[d, "\rho_1"] & A_{n(2)} \arrow[r, "\phi_{n(2)n(3)}"] \arrow[d, "\rho_2"] & A_{n(3)} \arrow[r, "\phi_{n(3)n(4)}"] \arrow[d, "\rho_3"] & \cdots \arrow[r] \arrow[d] & A \arrow[d, shift right, "\rho"'] \\
B_{m(1)} \arrow[r, "\psi_{m(1)m(2)}"'] \arrow[ur, "\sigma_1"] & B_{m(2)} \arrow[r, "\psi_{m(2)m(3)}"'] \arrow[ur, "\sigma_2"] & B_{m(3)} \arrow[r, "\psi_{m(3)m(4)}"'] \arrow[ur, "\sigma_3"] & \cdots \arrow[r] \arrow[ur] & B \arrow[u, shift right, "\sigma"']
\end{tikzcd}
$$



Corollaries

1. All known classifications.

Elliott (1976). AF C^*-algebras.

Goodearl's Regular-rings book. Ultramatricial algebras over a field.

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Hazrat (2013). Graded ultramatricial algebras over a graded field.

2. Iso Conjecture for a class of LPAs.

No-exit graphs

Abrams-Aranda-Perera-Siles (2010).
If E is row-finite, countable and

- every infinite path ends in a sink or a cycle, and
- no cycle has an exit,

then $L_K(E)$ is a direct sum of **ultramatricial algebras over K and over $K[x, x^{-1}]$.**

Roozbeh-Lia. An iso on K_0^{gr}-level maps the acyclic to the acyclic part and the comet to the comet part. Hence,

K_0^{gr} completely classifies this class of LPAs

Corollary. Graded GIC holds for these LPAs.

Wondering about...

- The assumptions for Classification Theorem for graded ultramatricial algebras. Can they be weakened?
- Classification Conjecture for LPAs.

K_0^{gr} classifies all LPAs?

- Is there C^*-analogue? Something with gauge action possibly?

References:
http://liavas.net or arXiv