Von-Neumann-algebra-like rings and the answer to a S. K. Berberian's question

A drama in 3 acts

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Main Characters

A damsel in distress:
von Neumann Algebra

A knight in shinning armor:
certain Baer $*$-Ring

The story of von Neumann Algebra begins...

John von Neumann's dream – to capture abstractly the concept of an algebra of observables in quantum mechanics. He constructed

- non-commutative generalization of Hilbert space/ probability theory.

Von Neumann Algebra – the powerful

Overcomes the limits of classical Hilbert space/probability theory.	Suitable for quantum mechanics.
Yields all the types of non-commutative measures that occur in classical theory	Still capable of describing large (infinite in size or in degrees of freedom) quantum systems.
Dimension function	Corresponds to normalized measure in classical probability space.

Von Neumann Algebra – in distress

"Von Neumann algebras are blessed with an excess of structure – algebraic, geometric, topological – so much, that one can easily obscure, through proof by overkill, what makes a particular theorem work."

"If all the functional analysis is stripped away ... what remains should (be) completely accessible through algebraic avenues".

Berberian, S. K. Baer $*$-rings; Springer-Verlag, Berlin-Heidelberg-New York, 1972.

The overkill

The overkill that Berberian is referring to:

a mosquito

a machine gun

VNA Character Expanded

H – Hilbert space

$\mathcal{B}(H)$ – bounded operators.

A **von Neumann algebra** $\mathcal{A}$ is a

1) $*$-closed unital subalgebra of $\mathcal{B}(H)$,

2a) equal to its double commutant $\mathcal{A}''$ (where $\mathcal{A}' = \{x \in \mathcal{B}(H) \mid ax = xa$ for all $a \in \mathcal{A}\}$)

equivalently

2b) weakly closed in $\mathcal{B}(H)$.

Five Types

finite, discrete	$I_n,\ n \in \omega$	μ on $\{1, 2, \ldots, n\}$
infinite, discrete	I_∞	μ on $\{1, 2, \ldots\}$
finite, continuous	II_1	μ on [0,1]
infinite, continuous	II_∞	μ on $\mathbb{R}$
very infinite	III	μ on $\{0, \infty\}$

Examples

I_n	$\mathcal{B}(H)$, $\dim(H) = n$ "finite matrices"
I_∞	$\mathcal{B}(H)$, $\dim(H) = \infty$ "infinite matrices"
II_1	group VNA for G "very infinite and nonabelian" G-invariant operators on Hilbert space $l^2(G)$ i.e. $f(xg) = f(x)g$
II_∞	"infinite matrices" over type II_1

Types I_n and II_1 are called **finite von Neumann algebras.**

Exposition – Plot Device

- A finite VNA $\mathcal{A}$ has a normal and faithful linear **trace** $\mathrm{tr}_{\mathcal{A}} : \mathcal{A} \to \mathbb{C}$.

  Examples. 1. Usual trace on $M_n(\mathbb{C})$. "Kaplansky trace" on group VNAs: $\mathrm{tr}(f) = a_1$ for $f(1) = \sum a_g g \in l^2(G)$.

- The trace extends to matrices: $\mathrm{tr}([a_{ij}]) = \sum_{i=1}^{n} \mathrm{tr}(a_{ii})$.

- [Lück] This defines **dimension**. For fin. gen. proj. module P

  $$\boxed{\dim_{\mathcal{A}}(P) = \mathrm{tr}(f)} \quad \in [0, \infty).$$

  where $f : \mathcal{A}^n \to \mathcal{A}^n$ is a projection with image P.

- [Lück] The dimension extends to **all $\mathcal{A}$-modules**.

  $$\boxed{\dim_{\mathcal{A}}(M) = \sup\{\dim_{\mathcal{A}}(P) | P \leq M, P \text{ fin. gen. proj.}\}}$$
  $$\in [0, \infty].$$

Law and Order – Enter the Rings

Von Neumann: studied lattice of projections. Led him to von Neumann regular rings.

Kaplansky's dream: to axiomatize (at least part of) the theory of VNAs. Followed similar path as von Neumann (looked at projections, idempotents, annihilators) – ended up defining Baer $*$-rings and AW^*-algebras.

The Knight – Baer $*$-Ring

Baer ring – every right annihilator is generated by an idempotent.

Baer $*$-ring – every right annihilator is generated by a projection.

AW*-algebra – Baer $*$-ring that is also a C^*-algebra.

AW* generalizes VNA's; Baer $*$ generalizes AW*.

Act 1 – Berberian sees trouble...

Berberian: "Baer $*$-rings are a compromise between operator algebras and lattice theory. Both the operator-theorist

("but this is too general!")

and the lattice-theorist

("but this can be generalized!")

will be unhappy..."

... but finds hope.

Berberian continues: "…. but uncommitted algebraists may find them enjoyable. (...) The test that counts is the test of intrinsic appeal. The subject will flourish if and only if students find its achievements exciting and its problems provocative."

Shining armor – Seven Axioms

- A1 A Baer $*$-ring R is **finite** if $x^*x = 1$ implies $xx^* = 1$ for all $x \in R$.
- A2 R satisfies **existence of projections** and **unique positive square root** axioms.
- A3 Partial isometries are addable.
- A4 R is **symmetric**: for all $x \in R$, $1 + x^*x$ is invertible.
- A5 There is a central element $i \in R$ such that $i^2 = -1$ and $i^* = -i$.
- A6 R satisfies the **unitary spectral** axiom (if unitary u is such that $\mathrm{ann}_r(1-u)$ is sufficiently small, then $1-u$ is locally invertible in a sequence of subrings that converge to R).
- A7 R satisfies the **positive sum** -axiom (certain positive elements have convergent countable sums).

A ring with A1 – A7 is von-Neumann-algebra-like

1. Berberian: R can be embedded in a **regular ring** Q satisfying A1–A7, having the same projections as R.
2. V.: R is **Ore** and $Q_{\mathrm{cl}}(R) = Q = Q_{\max}(R)$.
3. Berberian: There is **dimension function**: $\{$ projection over R $\} \longrightarrow$ continuous functions on a nice space with values in $[0, \infty)$ that is
   - Same on equivalent projections;
   - Fixes central projections;
   - Faithful;
   - Additive on orthogonal projections.

Culmination – Berberian's Question

If R is Baer when is $M_n(R)$ also Baer?

- For infinite types, the question is not interesting because $R \cong M_n(R)$.
- For finite types, axioms A1 – A7 do not seem to be enough!

Act 2 – Two “unwelcome guests”

A8 $M_n(R)$ satisfies the **parallelogram law.**
$(p \vee q - q \sim p - p \wedge q)$

$$
\begin{tikzcd}
 & p \vee q \arrow[dl, dash] \arrow[dr, dash, "p \vee q - q"] & \\
p \arrow[dr, dash, "p - p \wedge q"] & & q \arrow[dl, dash] \\
 & p \wedge q &
\end{tikzcd}
$$

A9 Every sequence of orthogonal projections in $M_n(R)$ has a supremum.

What do A8 and A9 bring?

- Berberian: $M_n(R)$ **is a Baer $*$-ring** with the dimension function on projections. $M_n(Q)$ is its regular ring.
- V.: $M_n(R)$ is **semihereditary.**
- V.: The **dimension** for every module can be defined.
  1. **dim(P)** is dimension of projection onto P for P f.g.p. Values in $C_{[0,\infty)}(X)$.
  2. For any M, **dim(M)** is supremum of $\dim(P)$ for $P \leq M$ f.g.p. Values in $C_{[0,\infty)}(X) \cup \{\infty\}$.
- V.: Every finitely generated R-module **splits** as

  torsion $\oplus$ finitely generated projective.
- V.: The dimension faithfully measures the torsion-free part.

Really von-Neumann-algebra-like!

Theorem [V.] dim has all the properties as the dimension of a finite von Neumann algebra.

1. **Extension**: two steps agree.
2. **Additivity** for short exact sequences.
3. **Cofinality**: dimension of directed union is supremum of dimensions.
4. **Continuity**: closure and dimension agree.
5. The dimension is **uniquely** determined by 1 – 4.

Outline of the proof: Work over regular Q. Can go back to R since the projections are the same. Prove continuity using the monotony of f.g.p. modules.

Act 3 – Another Berberian's Question

Berberian: A8 and A9 give us $M_n(R)$ is Baer but they are rather strong.

Question: Can we get rid of them?

Happy End

Yes!!!

Theorem [V.]

- We can get rid of A9 (i.e. A1 – A7 imply A9).
- For $M_n(R)$ Baer we do not need A8. So, **A1 – A7 are enough for $M_n(R)$ Baer.**

Proof follows from

1. Vaš: R is semihereditary and $Q_{\mathrm{cl}}(R) = Q = Q_{\mathrm{max}}(R)$..
2. M. W. Evans: The following are equivalent
   i) R is right semihereditary and $Q_{\mathrm{max}}(R)$ is the left and right flat epimorphic hull of R.
   ii) $M_n(R)$ is a right strongly Baer ring for all n.

A sequel?

A8 is used:

- In [Be] to show that $M_n(R)$ is finite.
- In [Va2] to show that the dimension function can be extended from projections in R to projections in $M_n(R)$.

Can we get rid of A8 also? Berberian believed so. Problem is still open.

Closing Credits

References.

[Be] Berberian, S. K. Baer $*$-rings; Springer-Verlag, Berlin-Heidelberg-New York, 1972.

[Lu] Lück, W. L^2-invariants: Theory and Applications to Geometry and K-theory, Springer-Verlag, Berlin, 2002.

[Va1] L. Vaš, Dimension and Torsion Theories for a Class of Baer $*$-Rings, Journal of Algebra 289 (2005) no. 2, 614–639.

[Va2] L. Vaš, Class of Baer $*$-rings Defined by a Relaxed Set of Axioms, Journal of Algebra, 297 (2006), no. 2, 470 - 473.

Preprints of my papers are available on

http://www.usp.edu/$\sim$lvas

and on arXiv.