Algebras of quotients of some (Leavitt) path algebras

My path to Spain

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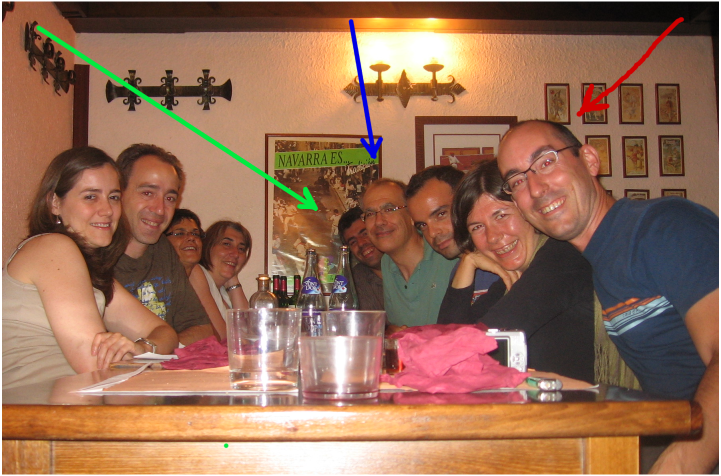
How it all started?

Barcelona/Bellaterra, July 2007.

E. Ortega (symmetric quotient rings)	P. Ara & M. Brustenga (regular alg. of path algebras)	G. Aranda Pino (Leavitt path algebras)



Building blocks of a Leavitt path algebra

A graph

$E = (E^0, E^1, r, s)$ – an oriented graph. E^0 – vertices. E^1 edges. For $e \in E$ $s(e)$= source, $r(E)$= range. $^{s(e)}\bullet \xrightarrow{e} \bullet^{r(e)}$

A path μ **in a graph** is a list of edges $\mu = e_1 \dots e_n$ with $r(e_i) = s(e_{i+i})$ for $i = 1, \dots n-1$. n= **length** of μ.

E^* = **set of all paths** (with vertices as paths of length 0).

A path algebra

K= field. A **path algebra** $P_K(E)$ is a K-algebra such that

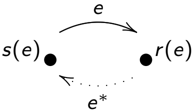
- the basis is a set of all paths E^*
- the multiplication of two paths p and q is concatenation if $r(p) = s(q)$ and it is 0 otherwise.

A path to a Leavitt path algebra

Add ghost paths.

For every edge e, add a **ghost edge** e^* such that

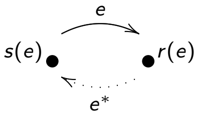
source e = **range** e^* **and**
range e = **source** e^*



Consider paths over this new graph.

Leavitt path algebra – the definition

E= graph. K= field. A **Leavitt path algebra** $L_K(E)$ is a free K-algebra with basis consisting of **vertices, edges and ghost edges** of E such that



P1 $vv = v$ and $vw = 0$ if $v \neq w$,

P2 $e = s(e)e = er(e)$

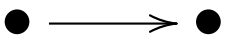
CK1 $e^*e = r(e)$, and $e^*f = 0$ if $e \neq f$

CK2 $v = \sum ee^*$ for all e's that originate from v.

In CK2, v is a vertex that emits at least one and not infinitely many edges.

P1, P2: path algebra axioms. CK1, CK2: **Cuntz - Krieger relations** (originate from graph C^* algebras).

3 famous examples: Matrices, Loop and Rose



$P_K(E) = T_2(K)$,

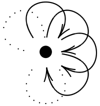
$L_K(E) = M_2(K)$

Get $n \times n$ **matrices** if $E = n - 1$ line.

Loop

$P_K(E) = K[x]$,

$L_K(E) = K[x, x^{-1}]$



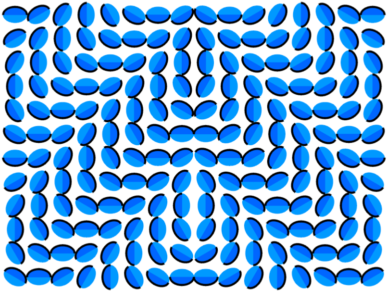
Rose

$P_K(E) = K[x_1, \dots, x_n]$,

$L_K(E) = L(1, n)$

Work of Eduard Ortega 2006

- Computed **left, right and symmetric maximal ring of quotients** of $P_K(E)$ for E directly finite acyclic.
- Defined **symmetric modules of quotients**: parallels right Gabriel filters and right modules of quotients.



In 2007, I was more interested in symmetric results than in path algebras...

The "most perfect" ring of quotients

Ring homomorphism $R \to Q$ makes Q into a **perfect right ring of quotients** if $Q \otimes_R Q \cong Q$ and Q is flat as left R-module.

The total right ring of quotients $Q^r_{\mathrm{tot}}(R)$ is the largest perfect ring of quotients in which R embeds. Introduced in '60s and '70s.

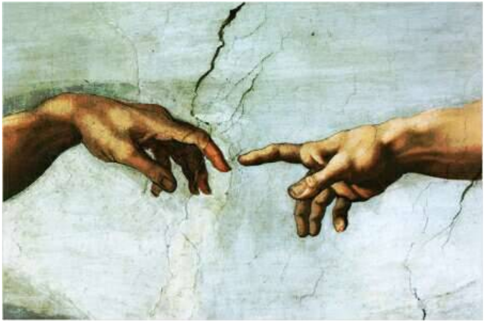
$$(Q^r_{\mathrm{cl}} \subseteq) Q^r_{\mathrm{tot}} \subseteq Q^r_{\mathrm{max}}$$

Q^r_{tot} always exists (as opposed to Q^r_{cl}).
Q^r_{max} can be too big.
Q^r_{tot} **"just right"**.

"Just right" meets symmetric

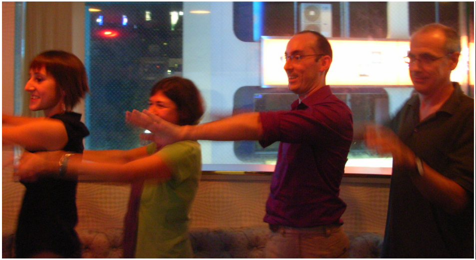
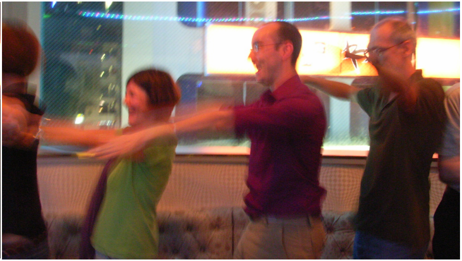
[L. V.] Total symmetric ring of quotients $Q^{\sigma}_{\mathrm{tot}}(R)$

1. Symmetric version of perfect right rings of quotients.
2. Symmetric version of perfect right filters.
3. Symmetric version of the total right ring of quotients $Q^{\sigma}_{\mathrm{tot}}$.



And then some time passed by...

Tokyo, Japan; Ankara, Turkey; Colorado Spring, CO; Washington, DC; and Lens, France...

... got me closer to Leavitt path algebras.

Work of Pere Ara and Miquel Brustenga

Defined a **regular algebra** $Q(E)$ of $P(E)$ and $L(E)$.

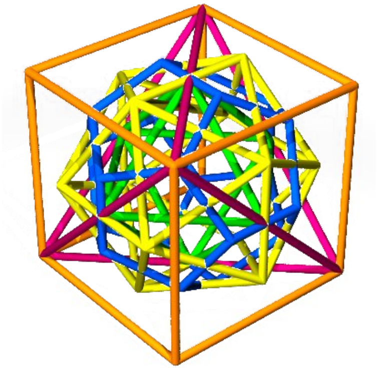
- $\Sigma =$ set of matrices with entries in $P(E)$ that become invertible in the algebra of power series $P((E))$.
- $\Sigma_1 =$ set of homomorphisms μ. For every non-sink vertex v, let $e_1, \ldots e_n$ be all the edges that v emits.

$$\begin{array}{rcl} \mu : P(E)v & \longrightarrow & P(E)r(e_1) \oplus \ldots \oplus P(E)r(e_n) \\ x & \longmapsto & (xe_1, \ldots, xe_n). \end{array}$$

$$\begin{array}{ccccc} P(E) & \xrightarrow{\Sigma^{-1}} & P_{rat}(E) & \longrightarrow & P((E)) \\ \downarrow{\scriptstyle \Sigma_1^{-1}} & & \downarrow{\scriptstyle \Sigma_1^{-1}} & & \downarrow{\scriptstyle \Sigma_1^{-1}} \\ L(E) & \xrightarrow{\Sigma^{-1}} & Q(E) & \longrightarrow & U(E) \end{array}$$

Pere Ara and Miquel Brustenga's results

1. $Q(E)$ is (von Neumann) regular.
2. $Q(E)$ satisfies (P1), (P2), (CK1), and (CK2).
3. $Q(E) = Q^l_{\mathrm{tot}}(L(E)) = Q^l_{\mathrm{tot}}(P(\overline{E}))$.
4. The monoids of fin. gen. projectives $V(Q(E))$ and $V(L(E))$ are isomorphic.



regular

The forgotten stars

$L_K(E)$ has involution *.

An **involution** $*$ is an additive map with $(xy)^* = y^*x^*$, and $(x^*)^* = x$.

For involution $k \mapsto \overline{k}$ in K (can always take it to be identity) define

$$(kpq^*)^* = \overline{k}qp^*$$

for monomials in $L_K(E)$.

Extend it to all elements so it is additive.

Why study the involution?

For $*$-rings **projections** (selfadjoint idempotents $p = p^* = p^2$) take over the role of **idempotents.**

Advantages:

- Berberian: "Projections are vastly easier to work with than idempotents."
- Left-right does not matter that much (certain dose of symmetry is present).
- Good feature of LPAs: vertices are **projections**.

*-adapted concepts:

- Baer becomes Baer *-ring,
- Rickart becomes Rickart *-ring,
- regular becomes *-regular.

This opens up some questions

1. When is $L(E)$ **Baer (Rickart)** *-ring?
2. **Does** $*$ **extend** to $Q(E)$?
3. If it does, is $Q(E)$ ***-regular**?
4. When is $Q(E)$ **unit-regular**?

$$\begin{aligned} E \text{ acyclic or } E = \text{loop} \quad &\longrightarrow Q(E) \text{ is unit-regular,} \\ E = \text{rose} \quad &\longrightarrow Q(E) \text{ is not unit-regular.} \end{aligned}$$

High hopes: if there is a case when $Q(E)$ is *-regular but not unit-regular, this will answer

Handelman Conjecture: Is every *-regular ring unit-regular?

More questions

5. Is $L(E)$ **finite** (i.e. $xx^* = 1$ implies $x^*x = 1$)?
6. When $L(E)$ is finite, is it **directly finite** (i.e. $xy = 1$ implies $yx = 1$)?
7. If $Q(E)$ is *-regular, then it is finite. Does this implies $L(E)$ finite as well?

To find all the answers...

...I needed somebody who knows Leavitt path algebras well.

Finding answers using "no exit"

A graph E has **NE condition** if no cycle has an exit. Call them: no-exit graphs.

In a no-exit graph, every path leads towards either

- to a **sink** or
- to a **cycle** (without an exit).

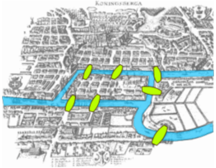
Known results

[G. Abrams, G. A. P., M. Siles Molina] E finite. TFAE:

1. E is no-exit.
2. $L(E)$ is left (right) noetherian.
3.
$$L(E) \cong \left(\bigoplus_{i=1}^{l} M_{m_i}(K[x, x^{-1}]) \right) \oplus \left(\bigoplus_{j=1}^{l'} M_{n_j}(K) \right)$$

l = no. of cycles, m_i = no. of paths ending in a vertex of a cycle c_i (not counting the cycle itself), l' = no. of sinks, n_j = no. of paths ending in a fixed sink.

$\longleftrightarrow$



Answers

Theorem [G. A. P. & L. V.] E – finite graph. The following are equivalent.

1. E is no-exit.
2. $Q(E)$ is unit-regular.
3. $L(E)$ is finite.
4. $L(E)$ is directly finite.
5. $Q(E)$ is directly finite.
6. $Q(E)$ is left and right self-injective.
7. $Q(E)$ is semisimple.
8. $L(E)$ has finite universal dimension.
9. The monoid of equivalence classes of finitely generated projectives $V(L(E)) \cong V(Q(E))$ is cancellative.

Answers - continued

Theorem [G. A. P. & L. V.] If K is positive definite, then the following are equivalent to 1–9 also.

10. The involution $*$ extends from $L(E)$ to $Q(E)$.
11. $Q(E)$ is $*$-regular (for involution inherited from $L(E)$).
12. $Q(E)$ is finite (for involution inherited from $L(E)$).
13. $Q(E) = Q^{\sigma}_{\max}(L(E))$.
14. $Q(E) = Q^{\sigma}_{\mathrm{tot}}(L(E))$.
15. $Q^{r}_{\max}(L(E)) = Q^{l}_{\mathrm{tot}}(L(E))$.
16. $Q^{r}_{\max}(L(E)) = Q^{l}_{\mathrm{tot}}(L(E)) = Q^{r}_{\mathrm{tot}}(L(E))$.
17. Every fin. gen. nonsingular $L(E)$-module is projective.
18. $M_n(L(E))$ is strongly Baer (i.e. every complemented right ideal is generated by an idempotent) for every n.

Corollaries

[G. A. P. & L. V.] K = field with positive definite involution, E = finite no-exit graph.

1. $L(E)$ is Baer ring.
2. P fin. gen. nonsingular (= fin. gen. proj.) $L(E)$-module, then $E(P) = P \otimes_{L(E)} Q(E)$ and there is a one-to-one correspondence

$$\begin{array}{rcl} \{\text{dir. sum. of } P\} & \Longleftrightarrow & \{\text{dir. sum. of } E(P)\} \\ \text{given by } K & \rightarrow & K \otimes_{L(E)} Q(E) = E(K) \\ \text{and the inverse by } K \cap P & \leftarrow & K \end{array}$$

3. The inverse of the isomorphism $\varphi : V(L(E)) \to V(Q(E))$, is induced by $P \mapsto P \cap L(E)^n$ if P is a finitely generated projective $Q(E)$-module that can be embedded in $Q(E)^n$.

Questions

1. Which of the conditions remain equivalent if E is a row-finite graph? Any graph?
2. It turns that $Q(E)$ is unit-regular exactly when it is $*$-regular - so no hope for Handelman's Conjecture using LPAs. Since it is a great question we ask again: is HC true?

Some references

- **ArXiv.** Papers by Ortega, Ara and Brustenga.
- **Google** "Gonzalo Aranda Pino"
- **Google** "Lia Vas". First google hit = http://www.usp.edu/∼lvas